Objectives and Outcomes

Due: Wednesday, 9

Submission is throu ... ıld be typed, not handwritten.

Discussion of assig ... hers is permitted, but the work submitted *must* be your own in line with the University's plagiarism policy.

Problem 1 (20 marks)

Let $R \subseteq S \times S$ be any binary relation on a set S. Consider the sequence of relations $R^0, R^1, R^2, \ldots$, defined as follows:

$$\begin{aligned} R^0 &:= I = \{(x,x) : x \in S\}, \text{ and} \\ R^{n+1} &:= R^n \cup (R; R^n) \text{ for } n \geq 0 \end{aligned}$$

(a) Prove that for all $i, j \in \mathbb{N}$, if $i \leq j$ then $R^i \subseteq R^j$. *Hint: Let $P_i(j)$ be the proposition that $R^i \subseteq R^j$ and prove that $P_i(j)$ holds for all $j \geq i$.* 4 marks

(b) Let $P(n)$ be the proposition that for all $m \in \mathbb{N}$: $R^n; R^m = R^{n+m}$. Prove that $P(n)$ holds for all $n \in \mathbb{N}$. *Hint: Use results from Assignment 1* 4 marks

(c) Prove that if there exists $i \in \mathbb{N}$ such that $R^i = R^{i+1}$, then $R^j = R^i$ for all $j \geq i$. 4 marks

(d) If $|S| = k$, explain why $R^{k^2} = R^{k^2+1}$. 2 marks

(e) If $|S| = k$, show that R^{k^2} is transitive. 2 marks

(f) If $|S| = k$ show that R^{k^2} is the minimum (with respect to $\subseteq$) of all reflexive and transitive relations that contain R. 4 marks

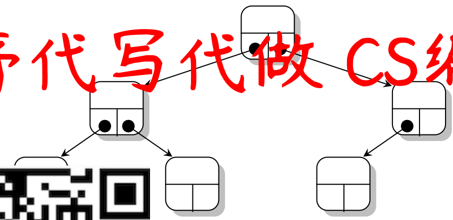
Remark

The relation at the limit[a] as n tends to infinity, $R^* = \lim_{n \to \infty} R^i$, is known as the **reflexive, transitive closure of** R, and is closely connected to the Kleene star operator.

[a] Because $R^j \subseteq R^i \subseteq S \times S$ for all $j \leq i$, the Knaster-Tarski theorem ensures this limit always exists, even for infinite S.

Problem 2 (20 marks)

A *binary tree* is a data structure where each node is linked to at most two successor nodes:



If we include empty bi[illegible]o nodes) as part of the definition, then we can simplify the description of the data [illegible] saying a node has 0, 1, or 2 successor nodes, we can instead say that a node has exa[illegible]e a child is a binary tree. That is, we can abstractly define the structure of a binar[illegible]

- (B): An empty tre[illegible]
- (R): An ordered pair $(T_{\text{left}}, T_{\text{right}})$ where T_{left} and T_{right} are trees.

So, for example, the above tree would be defined as the tree T where:

$$T = (T_1, T_2), \text{ where}$$
$$T_1 = (T_3, T_4) \text{ and } T_2 = (T_5, \tau), \text{ where}$$
$$T_3 = T_4 = T_5 = (\tau, \tau)$$

That is,

$$T = \Big(\big((\tau,\tau), (\tau,\tau) \big), \big((\tau,\tau), \tau \big) \Big)$$

A *leaf* in a binary tree is a node that has no successors (i.e. it is of the form (τ, τ)). A *fully-internal* node in a binary tree is a node that has exactly two successors (i.e. it is of the form (T_1, T_2) where $T_1, T_2 \neq \tau$). The example above has 3 leaves (T_3, T_4, and T_5) and 2 fully-internal nodes (T and T_1). For technical reasons (that will become apparent) we assume that an empty tree has 0 leaves and -1 fully-internal nodes.

(a) Based on the recursive definition above, recursively define a function $\mathsf{count}(\mathsf{T})$ that counts the number of nodes in a binary tree T. 4 marks

(b) Based on the recursive definition above, recursively define a function $\mathsf{leaves}(\mathsf{T})$ that counts the number of leaves in a binary tree T. 4 marks

(c) Based on the recursive definition above, recursively define a function $\mathsf{internal}(\mathsf{T})$ that counts the number of fully-internal nodes in a binary tree T. 4 marks

(d) If T is a binary tree, let $P(\mathsf{T})$ be the proposition that $\mathsf{leaves}(\mathsf{T}) = \mathsf{internal}(\mathsf{T}) + 1$. Prove that $P(\mathsf{T})$ holds for all binary trees T. Your proof should be based on your answers given in (b) and (c). 8 marks

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Problem 3 (12 marks)

Consider the following two algorithms that naïvely compute the sum and product of two $n \times n$ matrices.

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sum(A,B):
  for i ∈ [0,n):
    for j ∈ [0,n):
      C[i,j] = A[i,j] + B[i,j]
    end for
  end for
  return C
```

```
product(A,B):
  for i ∈ [0,n):
    for j ∈ [0,n):
      C[i,j] = add{A[i,k] * B[k,j] : k ∈ [0,n)}
    end for
  end for
  return C
```

Assuming that adding and multiplying matrix elements can be carried out in $O(1)$ time, and add will add the elements of a set S in $O(|S|)$ time.

(a) Give an asymptotic upper bound, in terms of n, for the running time of sum. (3 marks)

(b) Give an asymptotic [illegible] of n, for the running time of product. (3 marks)

When n is even, we car[illegible]cedure for multiplying two $n \times n$ matrices as follows. First, break the matrices into [illegible]

$$[illegible] \quad T \quad V \qquad B = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$$

where $S, T, U, V, W, X,$ [illegible]. Then it is possible to show:

$$AB = \begin{pmatrix} SW + TY & SX + TZ \\ UW + VY & UX + VZ \end{pmatrix}$$

where $SW + TY$, $SX + TZ$, etc. are sums of products of the smaller matrices. If n is a power of 2, each smaller product (SW, TY, etc) can be computed recursively, until the product of 1×1 matrices needs to be computed – which is nothing more than a simple multiplication, taking $O(1)$ time.

Assume n is a power of 2, and let $T(n)$ be the worst-case running time for computing the product of two $n \times n$ matrices using this method.

(c) With justification, give a recurrence equation for $T(n)$. (4 marks)

(d) Find an asymptotic upper bound for $T(n)$. (2 marks)

Problem 4 (18 marks)

Recall from Assignment 1 the neighbourhood of eight houses:

As before, each house n wi-fi network, but the wireless networks of neighbouring houses – that is, hous to each other (ignoring trees) or over the road from one another (directly opposite) – can interfere, and must therefore be on different channels. Houses that are sufficiently far away may use the same wi-fi channel. Again we would like to solve the problem of finding the minimum number of channels needed, but this time we will solve it using techniques from logic and from probability. Rather than directly asking for the minimum number of channels required, we ask if it is possible to solve it with just 2 channels. So suppose each wi-fi network can either be on channel hi or on channel lo. Is it possible to assign channels to networks so that there is no interference?

(a) Formulate this problem as a problem in propositional logic. Specifically:

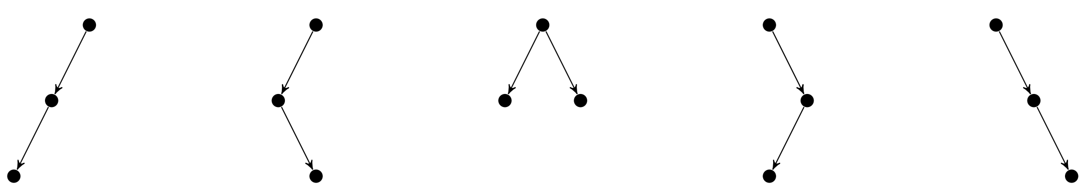
- (i) Define your propositional variables (4 marks)
- (ii) Define any propositional formulas that are appropriate and indicate what propositions they represent. (4 marks)
- (iii) Indicate how you would solve the problem (or show that it cannot be done) using propositional logic. It is sufficient to explain the method, you do not need to provide a solution. (2 marks)
- (iv)* Explain how to modify your answer(s) to (i) and (ii) if the goal was to see if it is possible to solve with 3 channels rather than 2. (4 marks)

(b) Suppose each house chooses, uniformly at random, one of the two network channels. What is the probability that there will be no interference? (4 marks)

Problem 5 (16 marks)

Recall from Problem 2 the definition of a binary tree data structure: either an empty tree, or a node with two children that are trees.

Let $T(n)$ denote the number of binary trees with n nodes. For example $T(3) = 5$ because there are five binary trees with three nodes:



(a) Using the recursive definition of a binary tree structure, or otherwise, derive a recurrence equation for $T(n)$. (6 marks)

A **full binary tree** is a non-empty binary tree where every node has either two non-empty children (i.e. is a fully-internal node) or two empty children (i.e. is a leaf).

(b) Using observations from Assignment 2, or otherwise, explain why a full binary tree must have an odd number of nodes. (2 marks)

(c) Let $B(n)$ denote the [illegible] trees with n nodes. Derive an expression for $B(n)$, involving $T(n')$ where $n' \leq n$ [illegible]*l nodes of a full binary tree to* $T(n)$. (4 marks)

A well-formed formula [illegible] **form** if it consists of just $\wedge$, $\vee$, and literals (i.e. propositional variables or negations o[illegible]es). For example, $(p \vee (\neg q \wedge \neg r))$ is in negated normal form; but $(p \vee \neg(q \vee r))$ is no[illegible]

Let $F(n)$ denote the nu[illegible]negated normal form formulas[1] there are that use precisely n propositional variables exactly one time each. So $F(1) = 2$, $F(2) = 16$, and $F(4) = 15360$.

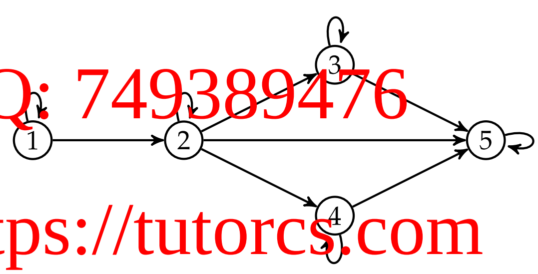
(d) Using your answer for part (c), give an expression for $F(n)$. (4 marks)

Remark

The $T(n)$ are known as the Catalan numbers. As this question demonstrates they are very useful for counting various tree-like structures.

Problem 6 (12 marks)

Consider the following directed graph:



and consider the following process:

- Initially, start at 1.
- At each time step, choose one of the outgoing edges from your current location uniformly at random, and follow it to the next location. For example, if your current location was 2, then with probability $\frac{1}{4}$ you would move to 3; with probability $\frac{1}{4}$ you would move to 4; with probability $\frac{1}{4}$ you would move to 5; and with probability $\frac{1}{4}$ you would stay at 2.

Let $p_1(n)$, $p_2(n)$, $p_3(n)$, $p_4(n)$, $p_5(n)$ be the probability your location after n time steps is 1, 2, 3, 4, or 5 respectively. So $p_1(0) = 1$ and $p_2(0) = p_3(0) = p_4(0) = p_5(0) = 0$.

(a) Express $p_1(n+1)$, $p_2(n+1)$, $p_3(n+1)$, $p_4(n+1)$, and $p_5(n+1)$ in terms of $p_1(n)$, $p_2(n)$, $p_3(n)$, $p_4(n)$, and $p_5(n)$. (5 marks)

[1]Note: we do not assume $\wedge$ and $\vee$ are associative

(b) Prove ONE of the following:

(i) For all $n \in \mathbb{N}$: $p_1(n) = \frac{1}{2^n}$ (4 marks)

(ii) For all $n \in \mathbb{N}$: $p_2(n) = 2\left(\frac{1}{2^n} - \frac{1}{4^n}\right)$ (5 marks)

(iii) For all $n \in \mathbb{N}$: [illegible] $2)\frac{1}{2^n} + \frac{2}{4^n}$ (6 marks)

(iv) For all $n \in \mathbb{N}$: [illegible] $- \frac{2}{4^n}$ (7 marks)

Note

Clearly state w[illegible] proving. A maximum of 7 marks is available for this question and m[illegible]ased on level of technical ability demonstrated. You may assume the ide[illegible]pt proving.

Remark

This is an example of a Markov chain – a very useful model for stochastic processes.

Advice on how to do the assignment

Collaboration is encouraged, but all submitted work must be done individually without consulting someone else's solutions in accordance with the University's "Academic Dishonesty and Plagiarism" policies.

- Assignments are [illegible] pera.
- When giving ans[illegible] always would like you to prove/explain/motivate your answers. You are be[illegible] nderstanding and ability.
- Be careful with g[illegible] native answers. If you give multiple answers, then we will give you marks o[illegible] ver, as this indicates how well you understood the question.
- Some of the ques[illegible] (with the help of external resources). You may make use of external material provided it is properly referenced[2] – however, answers that depend too heavily on external resources may not receive full marks if you have not adequately demonstrated ability/understanding.

- Questions have been given an indicative difficulty level:

  PASS CREDIT DISTINCTION HIGH DISTINCTION

  This should be taken as a *guide* only. Partial marks are available in all questions, and achievable by students of all abilities.

[2] Proper referencing means sufficient information for a marker to access the material. Results from the lectures or textbook can be used without proof, but should still be referenced.