CS 240 – Data Structures and Data Management

Module 6: Dictionaries for special keys

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Based on lecture notes by many previous cs240 instructors

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References: Sedgewick 12.4, 15.2-15.4
Goodrich & Tamassia 23.5.1-23.5.2

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Outline

1. Lower bound
2. Interpolation Search
3. Tries
   - Standard Tries
   - Variations of Tries
   - Compressed Tries

Lower bound for search

The fastest realizations of *ADT Dictionary* require $\Theta(\log n)$ time to search among n items. Is this the best possible?

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Theorem: In the comparison model (on the keys), $\Omega(\log n)$ comparisons are required to search a size-n dictionary.

Proof:

But can we beat the lower bound for special keys?

Binary Search

Recall the run-times in a *sorted array*:

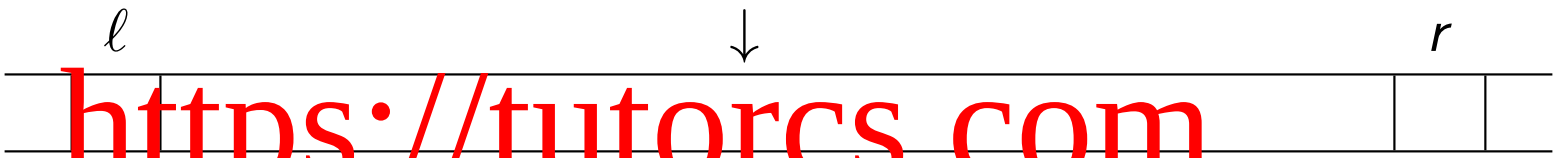
- *insert, delete*: $\Theta(n)$
- *search*: $\Theta(\log n)$

Binary-search(A, n, k)
A: Sorted array of size n, k: key

1. $\ell \leftarrow 0$
2. $r \leftarrow n-1$
3. **while** $(\ell < r)$
4. $m \leftarrow \lfloor \frac{\ell+r}{2} \rfloor$
5. **if** $(A[m] < k)$ **then** $\ell = m+1$
6. **else if** $(k < A[m])$ **then** $r = m-1$
7. **else return** m
8. **if** $(k = A[\ell])$ **return** ℓ
9. **else return** "not found, but would be between $\ell-1$ and ℓ"

Interpolation Search: Motivation

binary-search$(A[\ell, r], k)$: Compare at index $\lfloor \frac{\ell+r}{2} \rfloor = \ell + \lfloor \frac{1}{2}(r-\ell) \rfloor$



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Question: If keys are numbers, where would you expect key $k = 100$?

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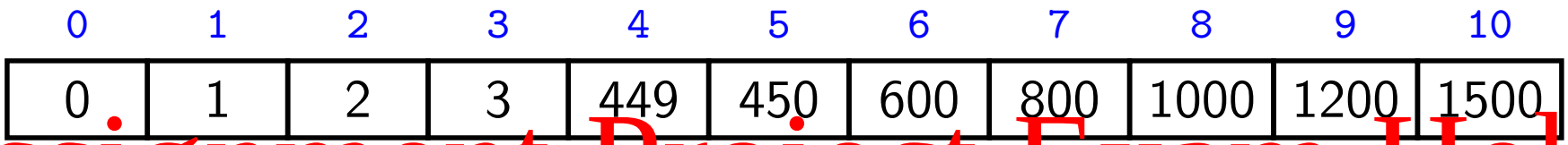
Question: If keys are numbers, where would you expect key $k = 100$?

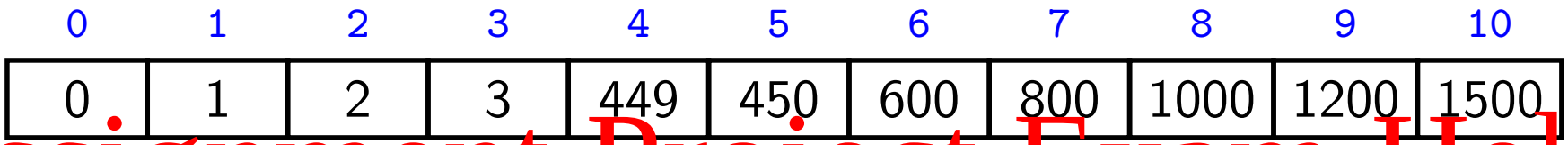
interpolation-search$(A[\ell, r], k)$: Compare at index $\ell + \left\lfloor \frac{k-A[\ell]}{A[r]-A[\ell]}(r-\ell) \right\rfloor$

Interpolation Search Example



interpolation-search(A[0..10],449):

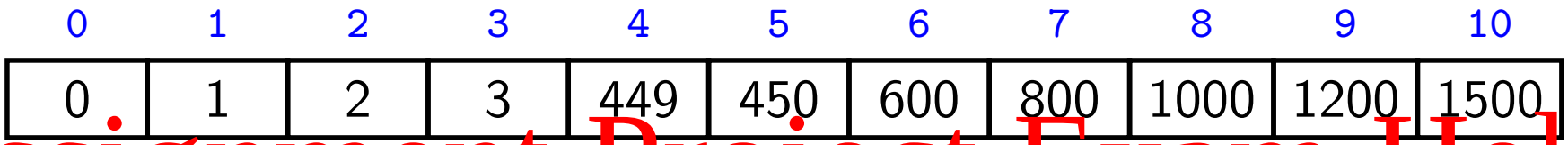
Interpolation Search Example



interpolation-search(A[0..10],449):

- Initially $\ell = 0$, $r = n - 1 = 10$, $m = \ell + \lfloor \frac{449-0}{1500-0}(10-0) \rfloor = \ell + 2 = 2$

Interpolation Search Example



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- Initially $\ell = 0$, $r = n - 1 = 10$, $m = \ell + \lfloor \frac{449-0}{1500-0}(10-0) \rfloor = \ell + 2 = 2$
- $\ell = 3$, $r = 10$, $m = \ell + \lfloor \frac{449-3}{1500-3}(10-3) \rfloor = \ell + 2 = 5$

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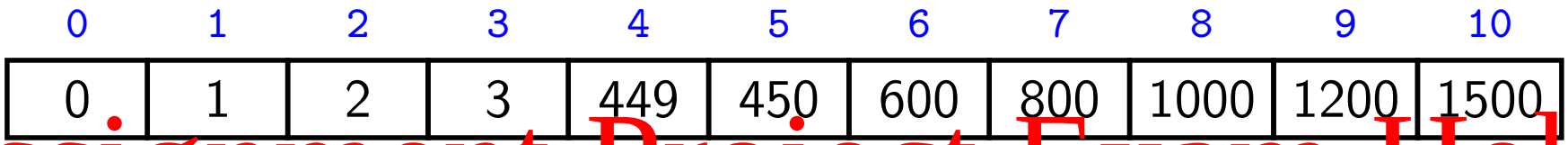

ℓ, r

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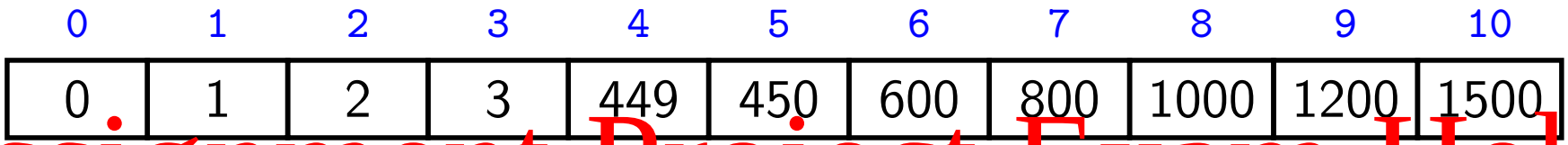
- Initially $\ell = 0$, $r = n - 1 = 10$, $m = \ell + \lfloor \frac{449-0}{1500-0}(10-0) \rfloor = \ell + 2 = 2$
- $\ell = 3$, $r = 10$, $m = \ell + \lfloor \frac{449-3}{1500-3}(10-3) \rfloor = \ell + 2 = 5$
- $\ell = 3$, $r = 4$, $m = \ell + \lfloor \frac{449-3}{449-3}(4-3) \rfloor = \ell + 1 = 4$, found at $A[4]$

Interpolation Search Example



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- Initially $\ell = 0$, $r = n - 1 = 10$, $m = \ell + \lfloor \frac{449-0}{1500-0}(10-0) \rfloor = \ell + 2 = 2$
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Works well if keys are *uniformly* distributed:

- Can show: the array in which we recurse into has size $\sqrt{n}$ on average.
- Recurrence relation is $T^{(\text{avg})}(n) = T^{(\text{avg})}(\sqrt{n}) + \Theta(1)$.
- This resolves to $T^{(\text{avg})}(n) \in \Theta(\log \log n)$.

But: Worst case performance $\Theta(n)$

Interpolation Search

- Code very similar to binary search, but compare at interpolated index
- Need a few extra tests to avoid crash due to $A[\ell] = A[r]$

interpolation-search(A, n, k)
A: Sorted array of size n, k: key

1. $\ell \leftarrow 0$
2. $r \leftarrow n-1$
3. **while** $(\ell < r)\&\&(A[r] \neq A[\ell])\&\&(k \geq A[\ell])\&\&(k \leq A[r]))$
4. $\quad m \leftarrow \ell + \lfloor \frac{k-A[\ell])}{A[r]-A[\ell]} \cdot (r-\ell) \rfloor$
5. $\quad$ **if** $(A[m] < k)$ **then** $\ell = m+1$
6. $\quad$ **else if** $(k < A[m])$ **then** $r = m-1$
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1. Lower bound
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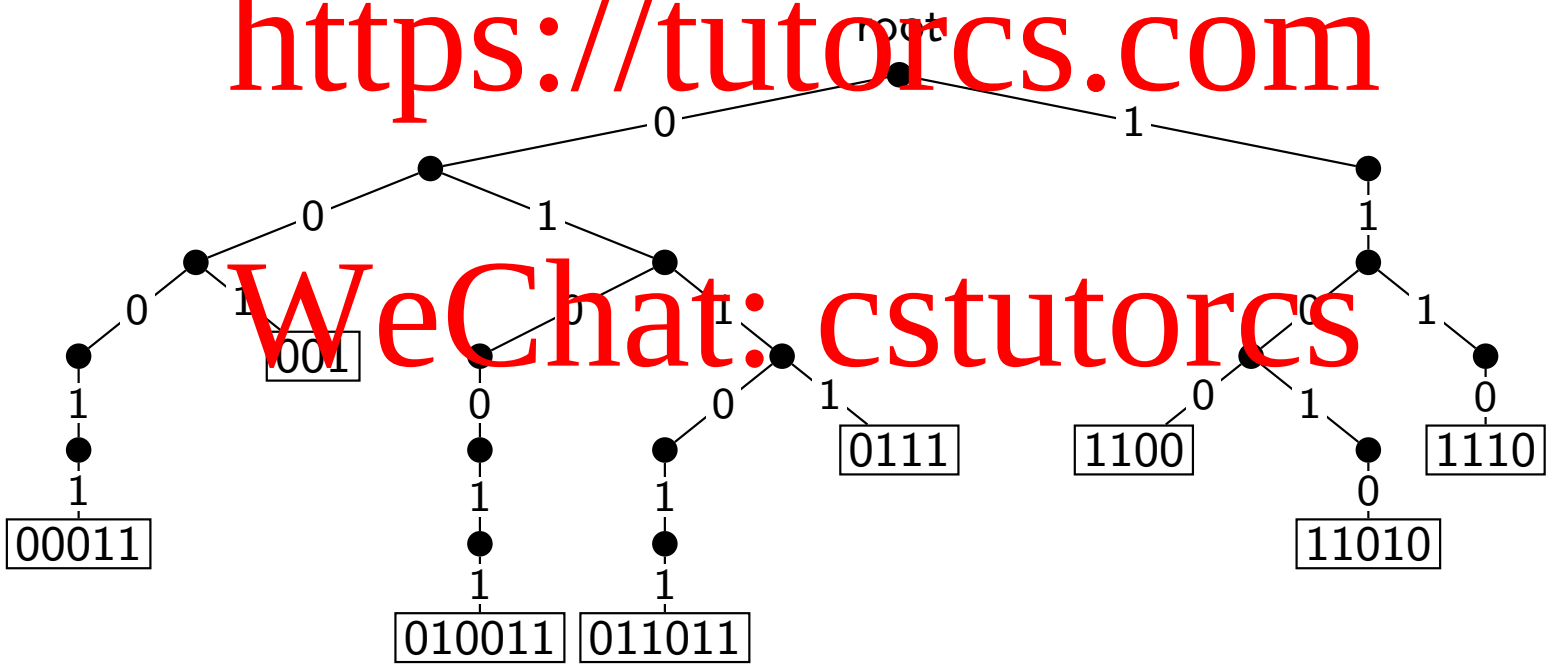
3. Tries
   - Standard Tries
   - Variations of Tries
   - Compressed Tries

Tries: Introduction

Trie (also know as **radix tree**): A dictionary for bitstrings

- Comes from retrieval, but pronounced "try"
- A tree based on *bitwise comparisons*: Edge labelled with corresponding bit
- Similar to *radix sort*: use individual bits, not the whole key



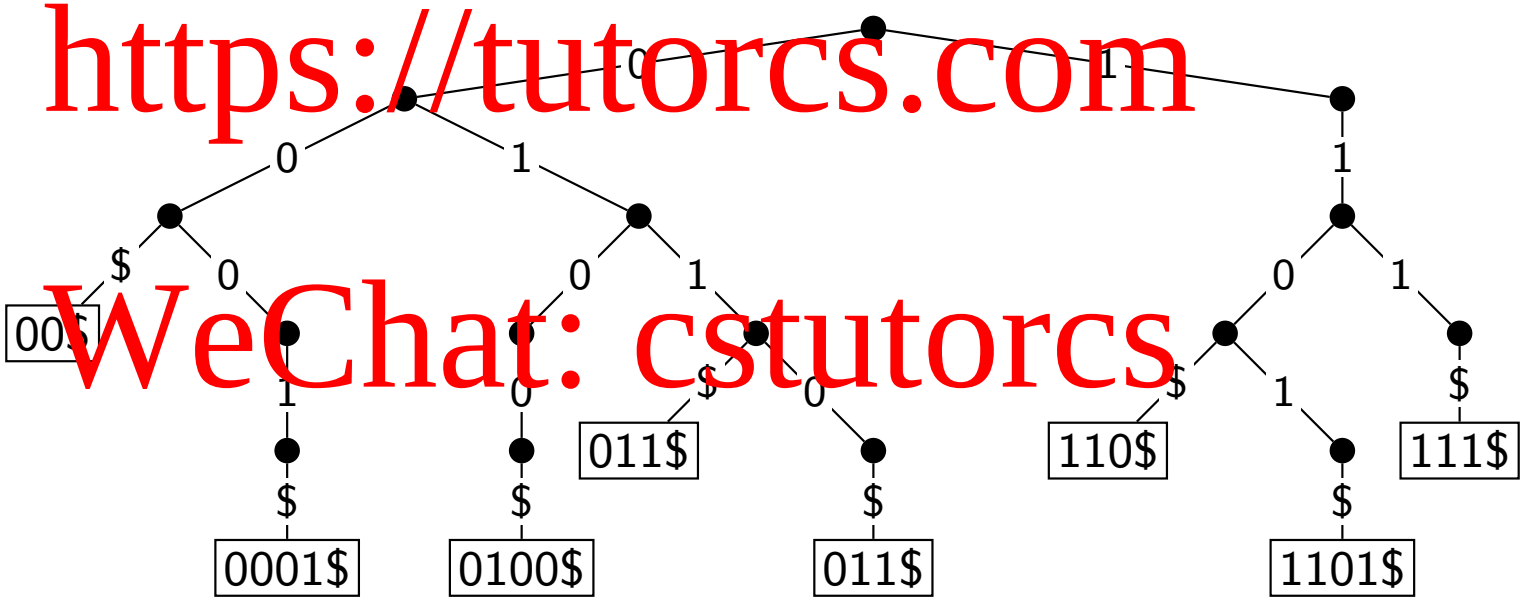
More on tries

Assumption: Dictionary is **prefix-free**: no string is a prefix of another

(A **prefix** of a string $S[0..n-1]$ is a substring $S[0..i-1]$ for some $0 \leq i \leq n$.)

- Assumption satisfied if all strings have the same length.
- Assumption satisfied if all strings end with "end-of-word" character \$.

Example: A trie for $\{00\$, 0001\$, 0100\$, 011\$, 0110\$, 110\$, 1101\$, 111\$\}$



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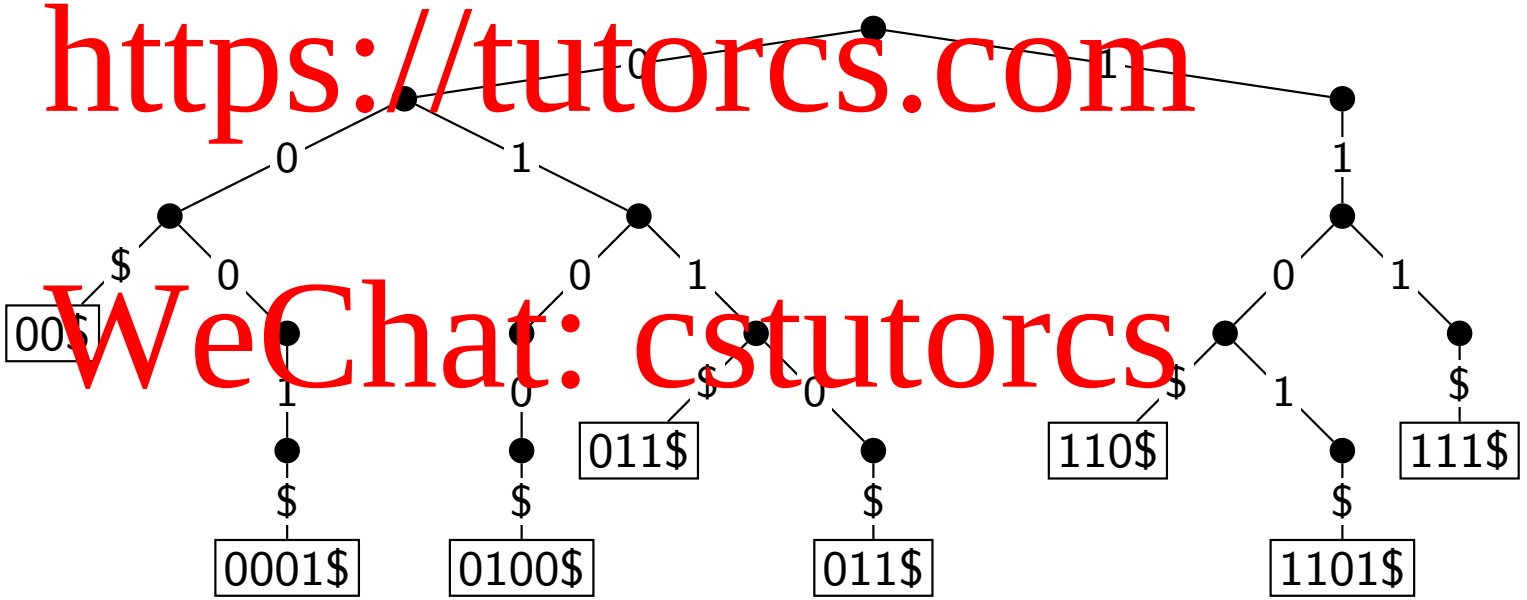


Then items (keys) are stored *only* in the leaf nodes

Tries: Search

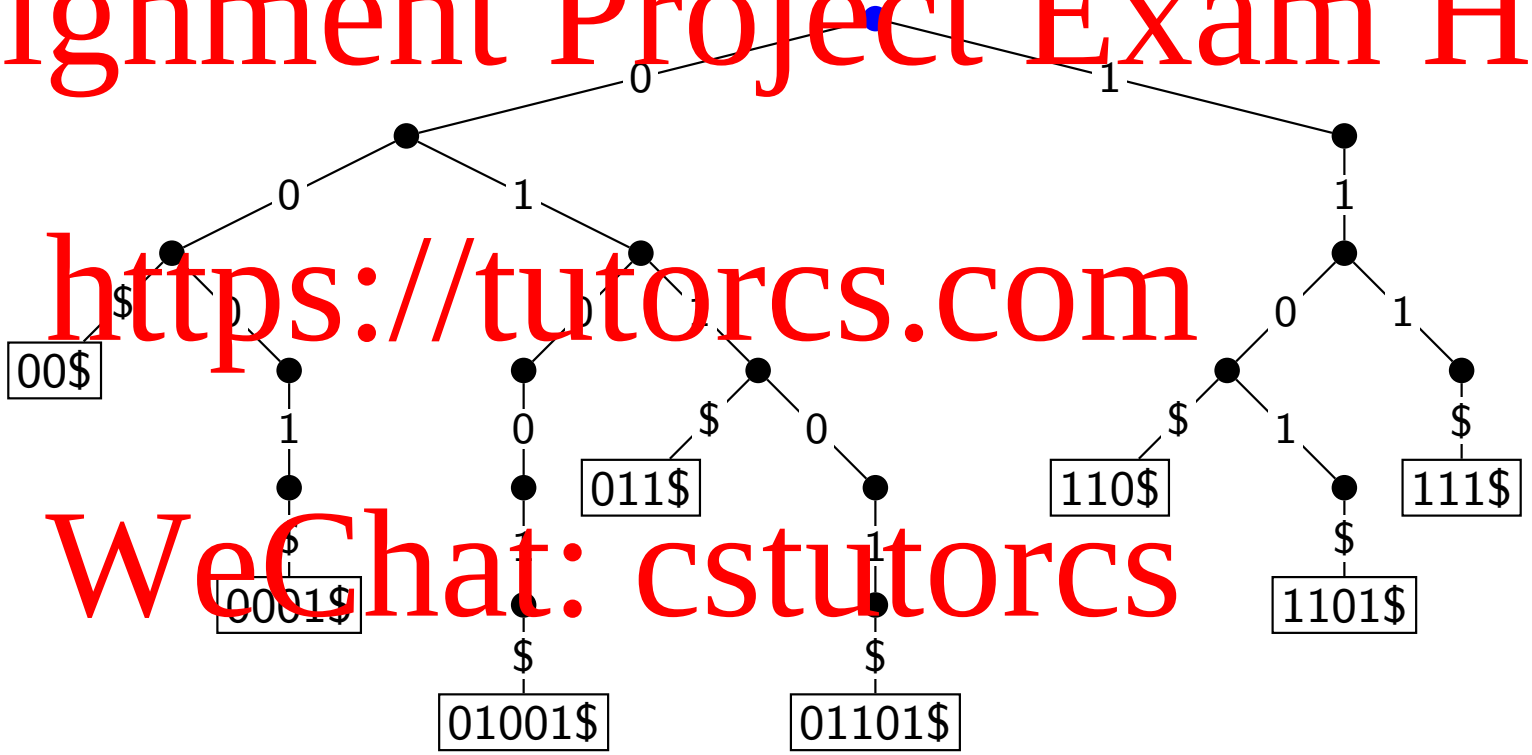
- start from the root and the most significant bit of x
- follow the link that corresponds to the current bit in x;
- return failure if the link is missing
- return success if we reach a leaf (it must store x)
- else recurse on the new node and the next bit of x

Trie::search$(v \leftarrow$ root$, d \leftarrow 0, x)$
v: node of trie; d: level of v, x: word stored as array of chars

```
1.    if v is a leaf
2.        return v
3.    else
4.        let c be child of v labelled with x[d]
5.        if there is no such child
6.            return "not found"
7.        else Trie::search(c, d + 1, x)
```

Tries: Search Example

Example: Trie::search(011$)



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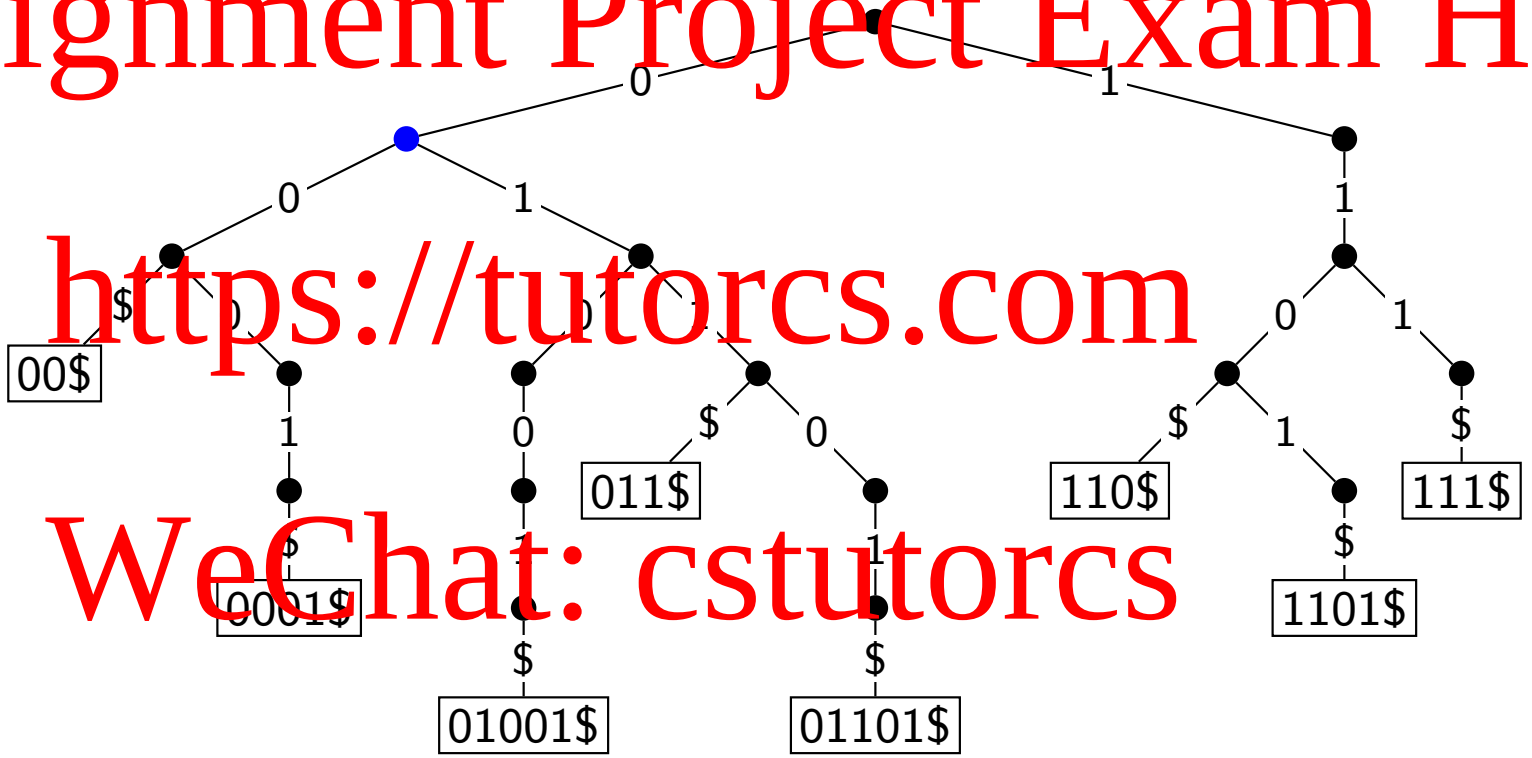

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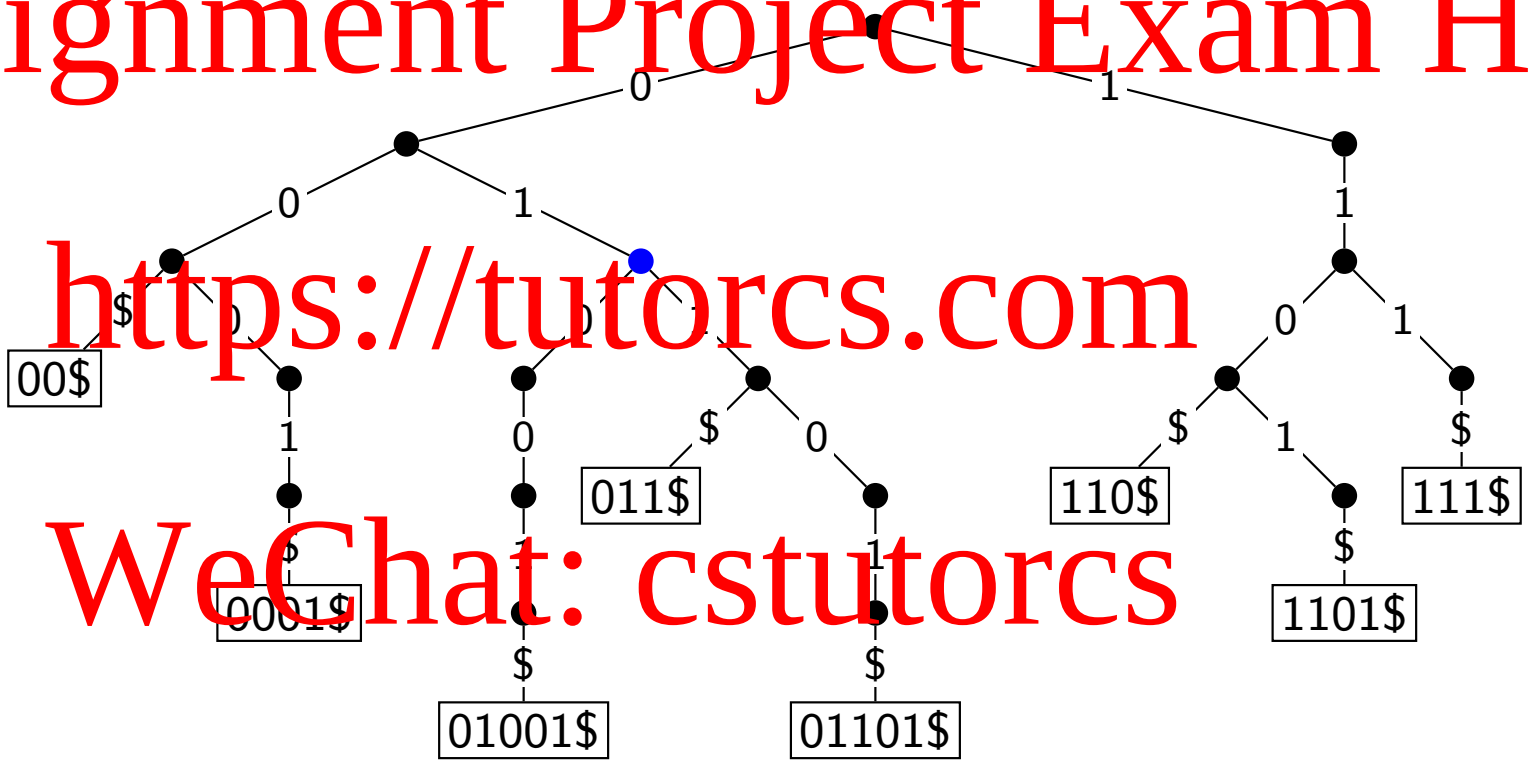

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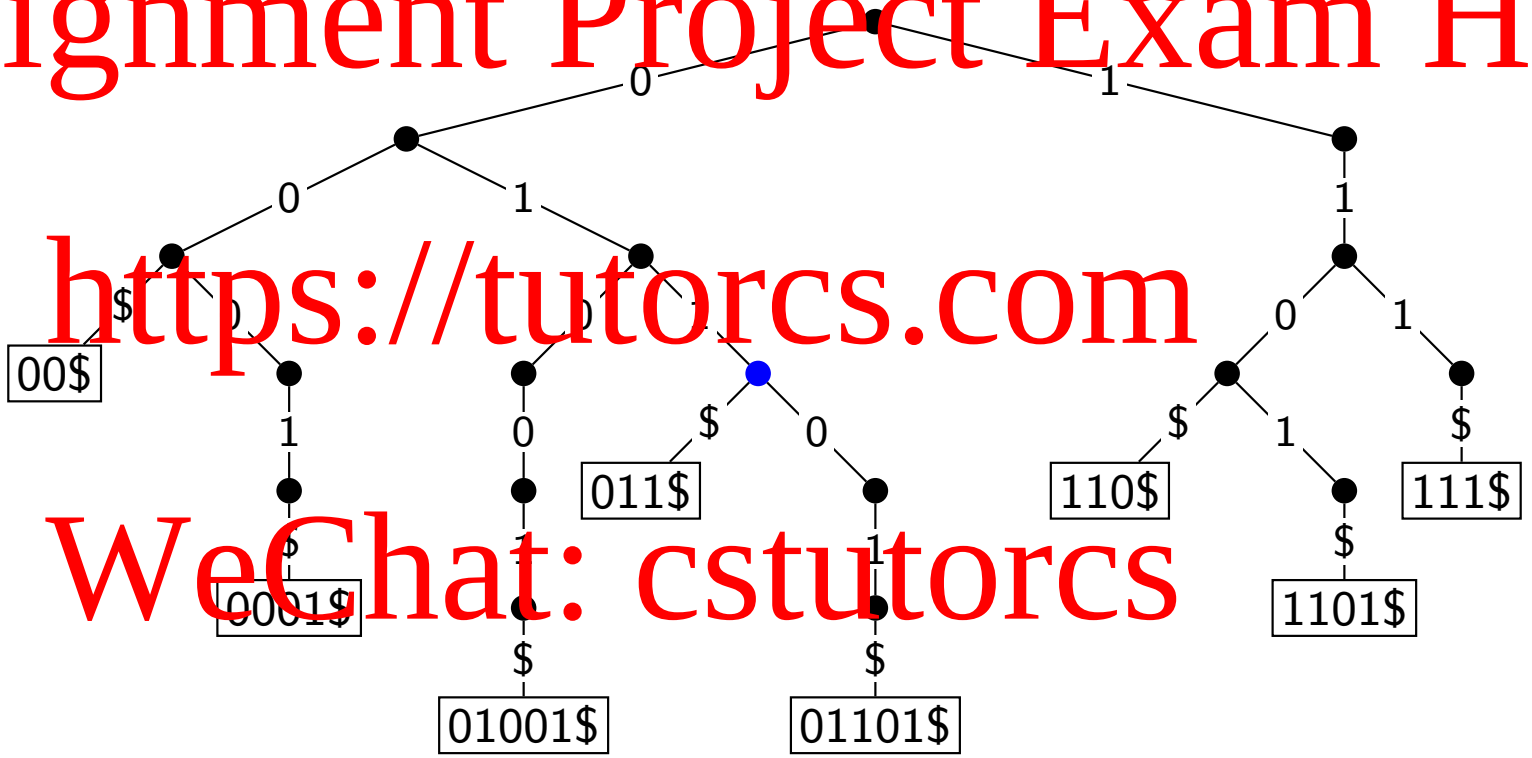
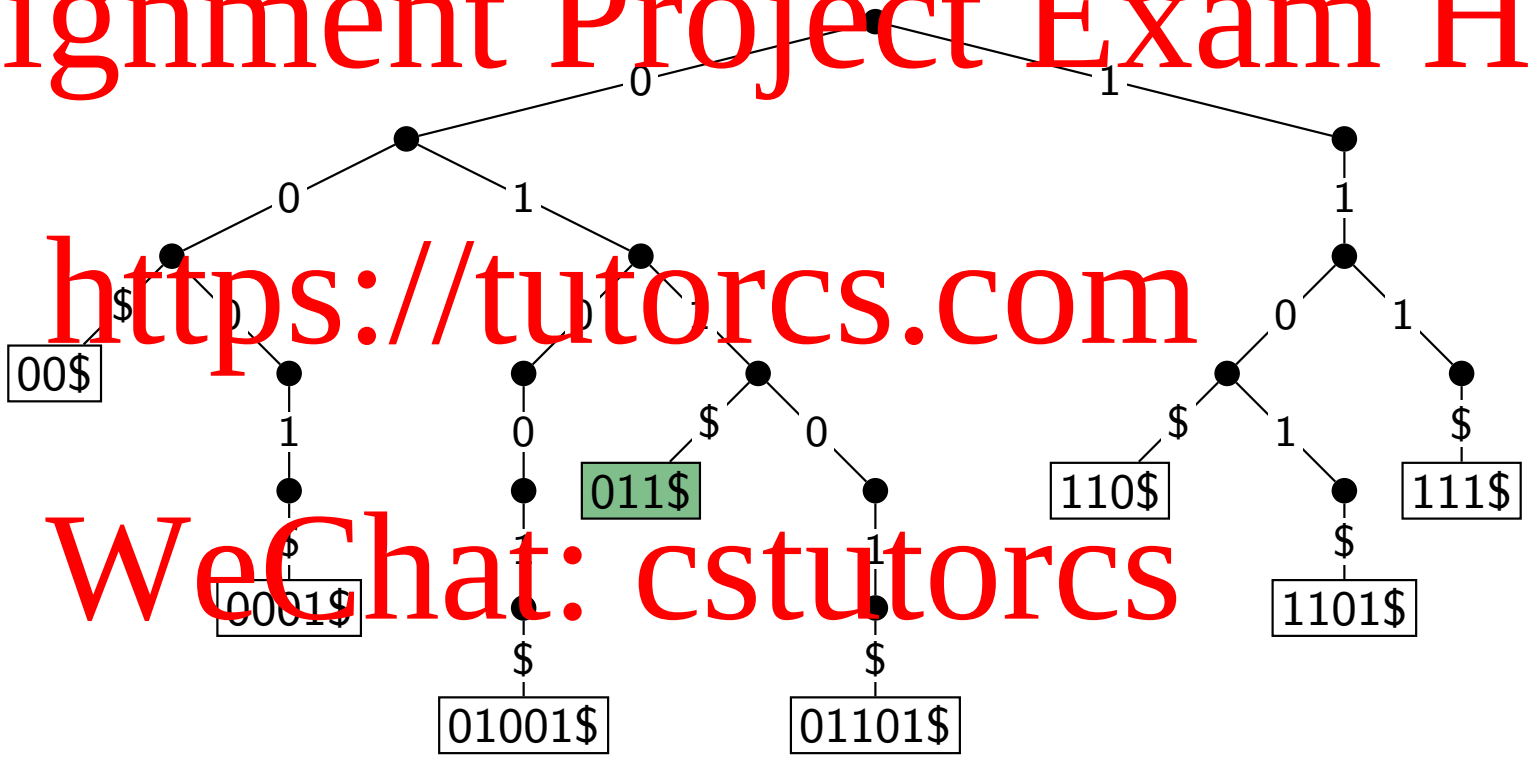


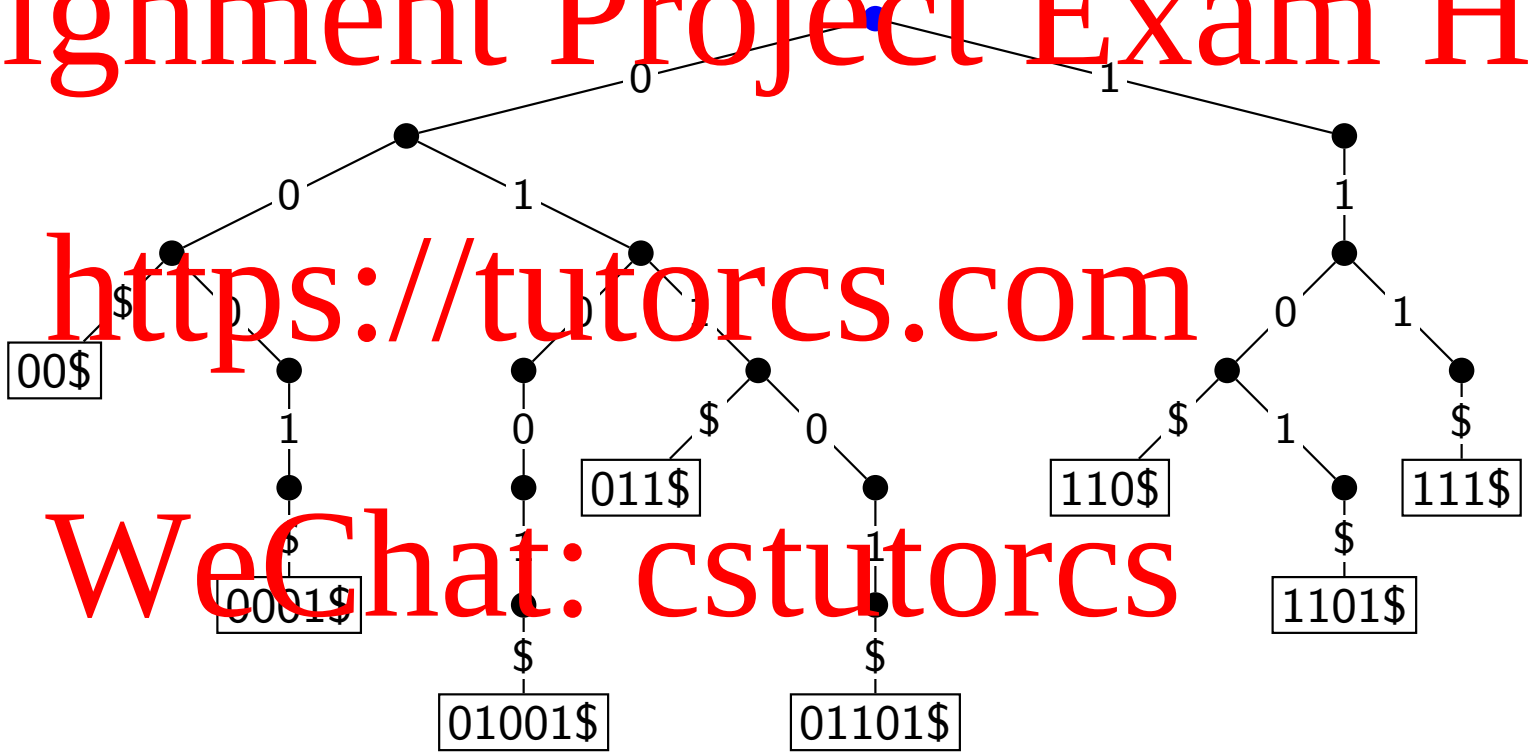
Tries: Search Example

Example: Trie::search(011$) successful



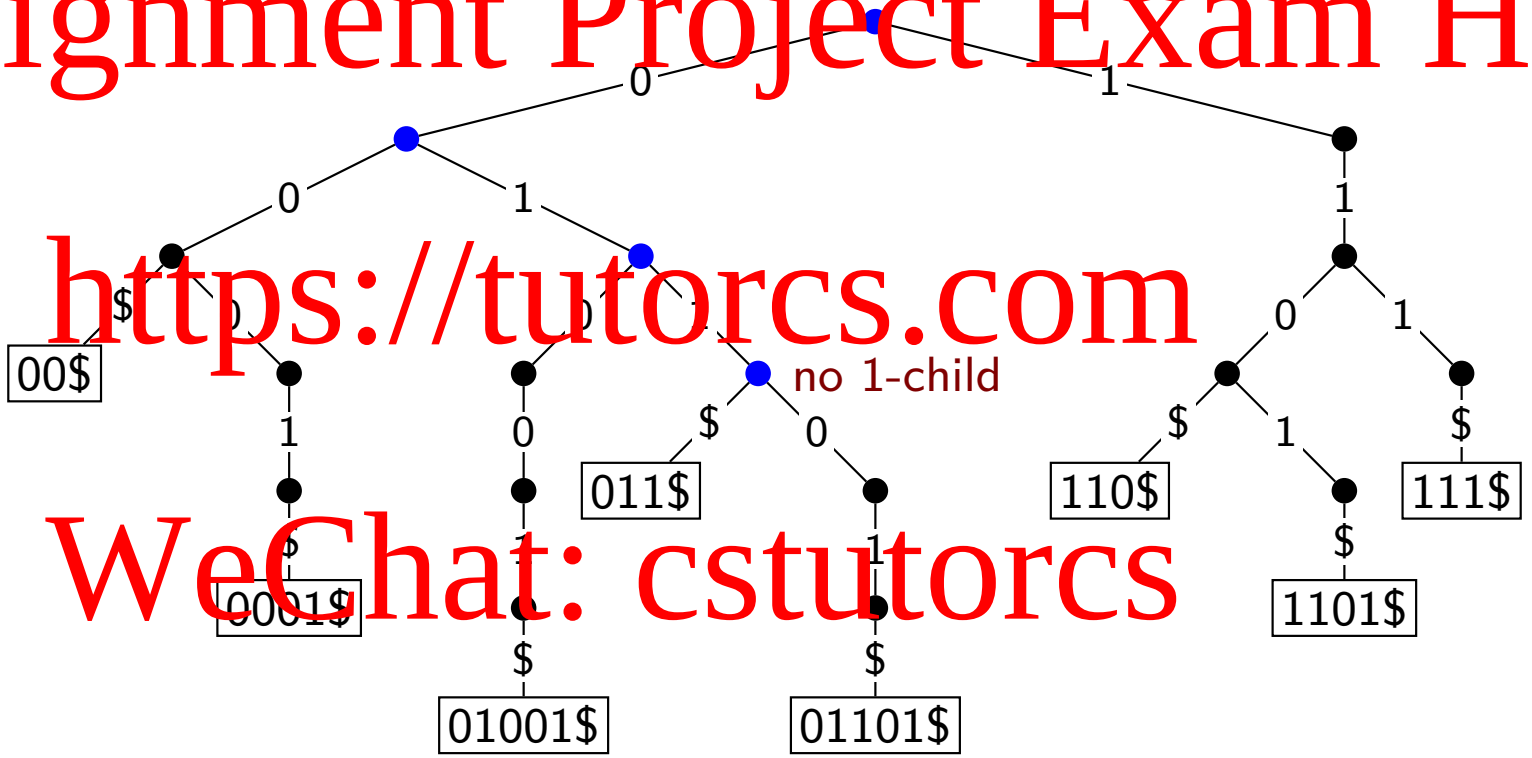
Tries: Search Example

Example: Trie::search(0111$)



Tries: Search Example

Example: Trie::search(0111$) unsuccessful



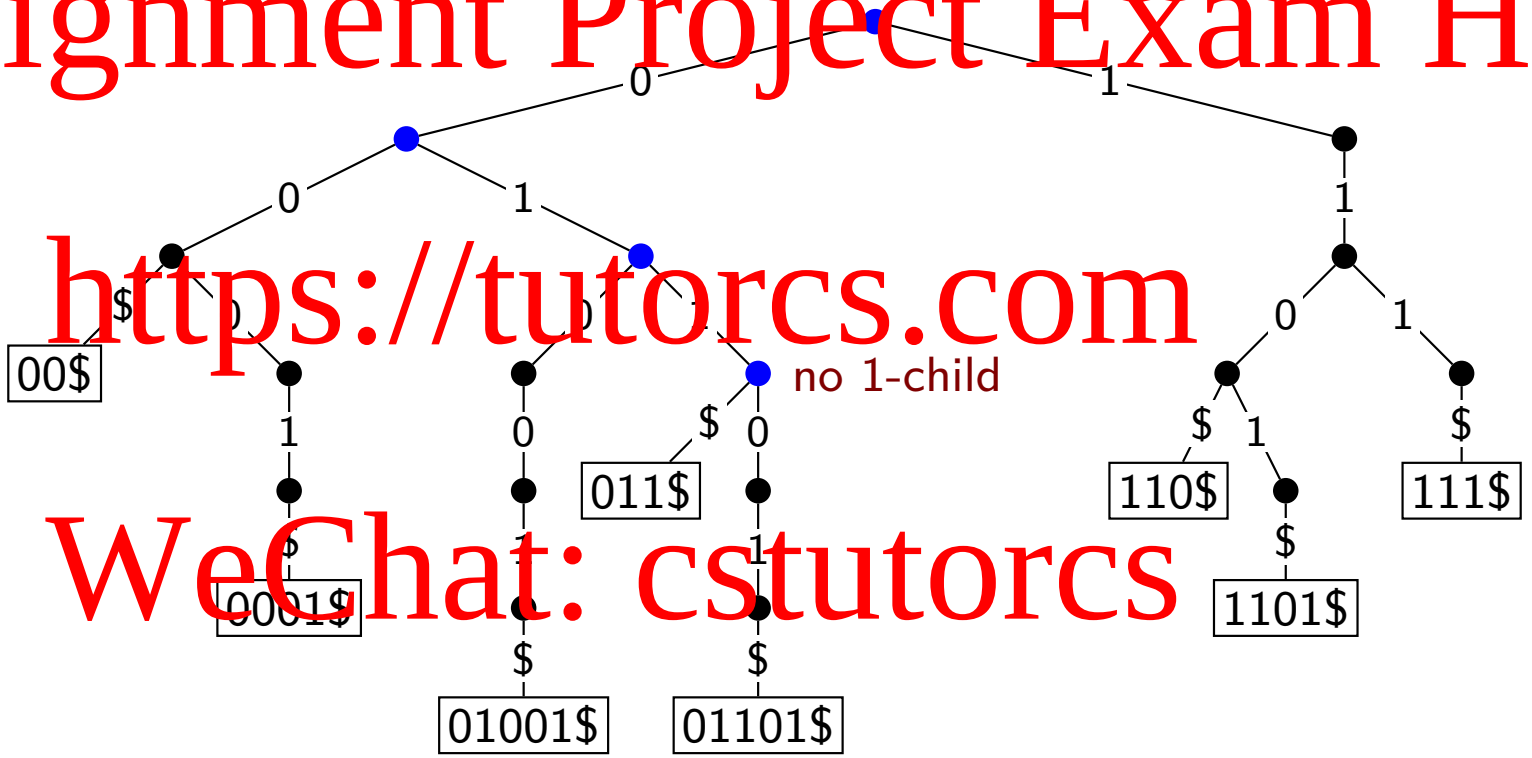
Tries: Insert & Delete

- *Trie::insert(x)*
  - Search for x, this should be unsuccessful
  - Suppose we finish at a node v that is missing a suitable child. Note: x has extra bits left.
  - Expand the trie from the node v by adding necessary nodes that correspond to extra bits of x.
- *Trie::delete(x)*
  - Search for x
  - let v be the leaf where x is found
  - delete v and all ancestors of v until we reach an ancestor that has two children.
- **Time Complexity** of all operations: $\Theta(|x|)$

  $|x|$: length of binary string x, i.e., the number of bits in x

Tries: Insert Example

Example: *Trie::insert*(0111$)



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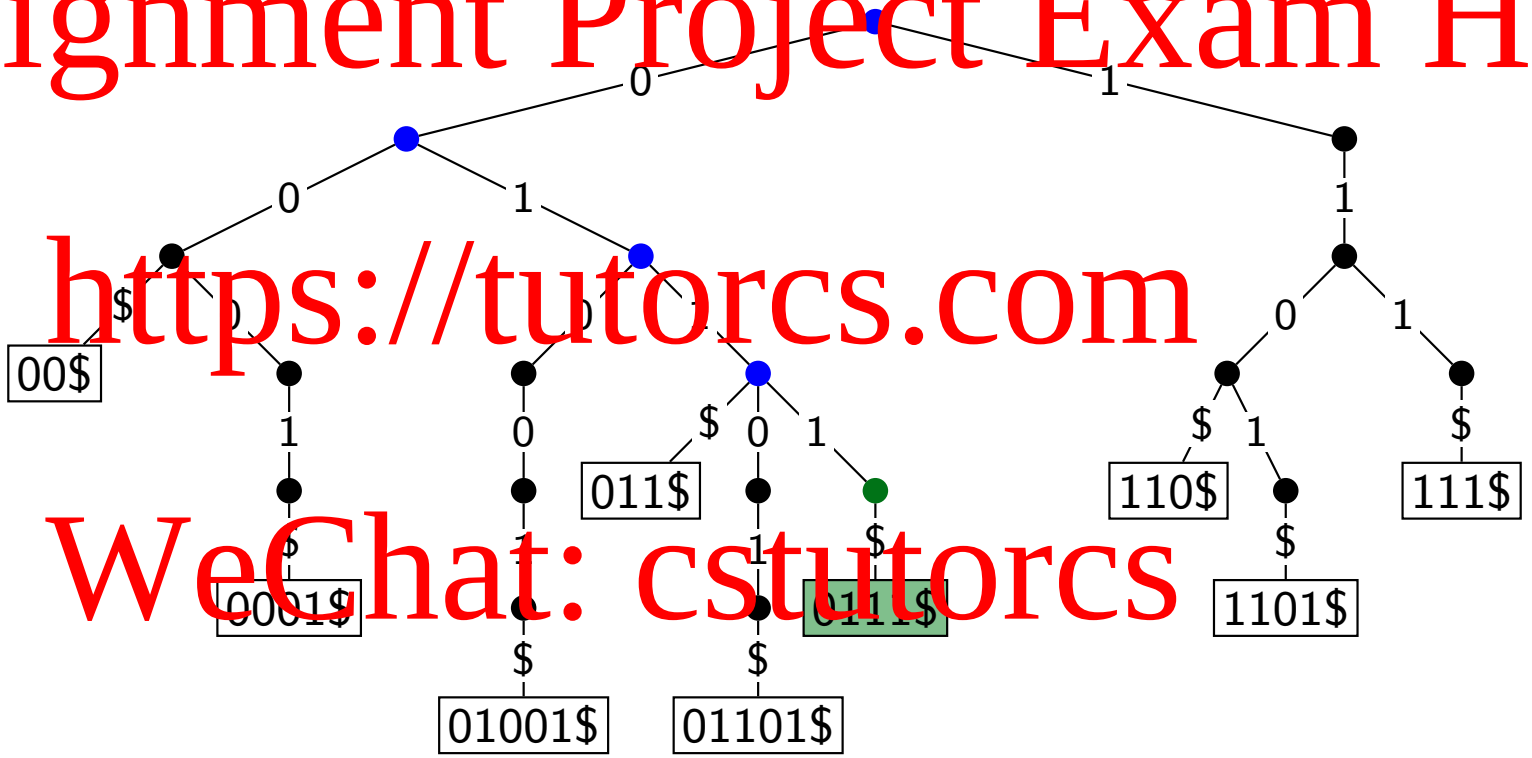
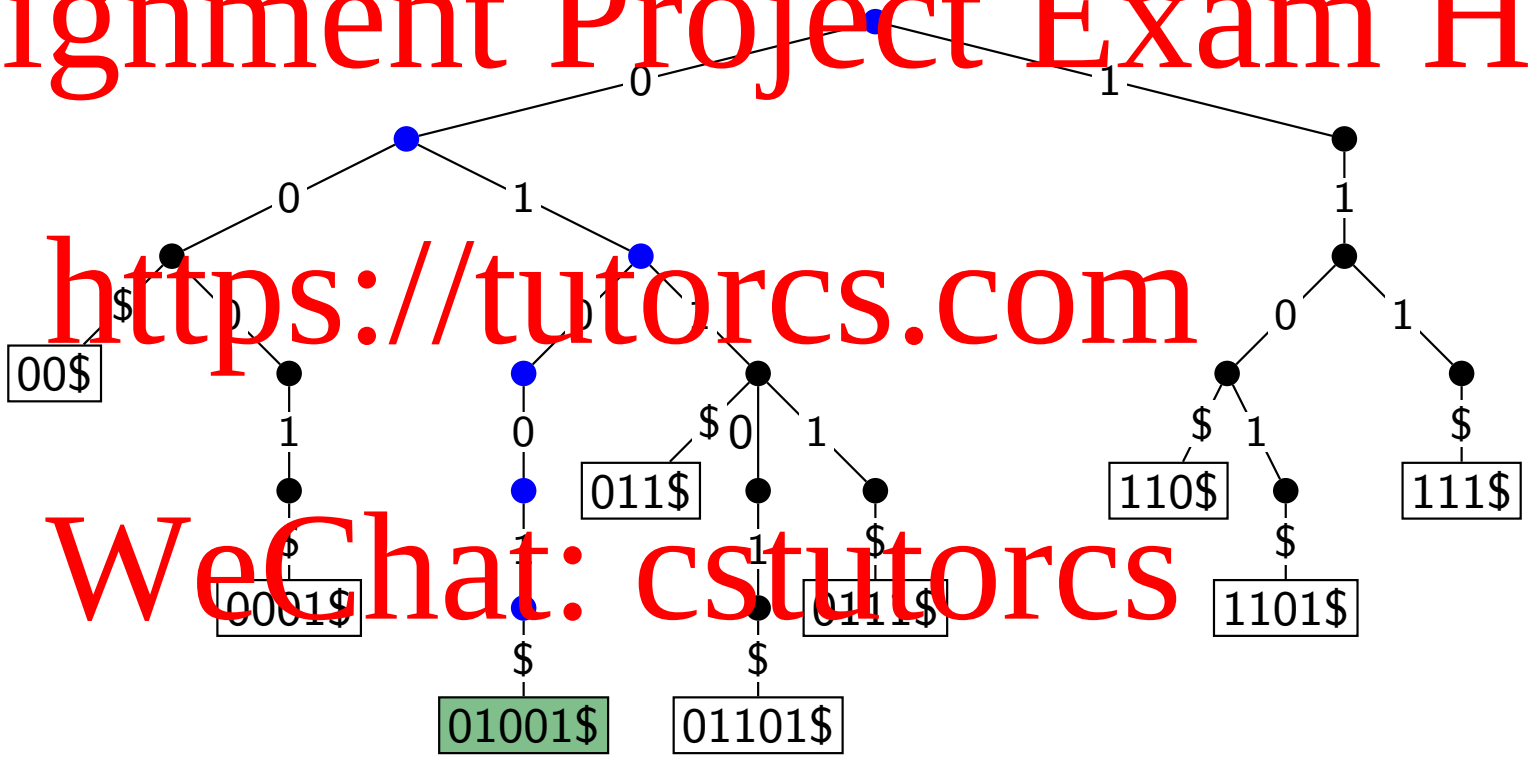


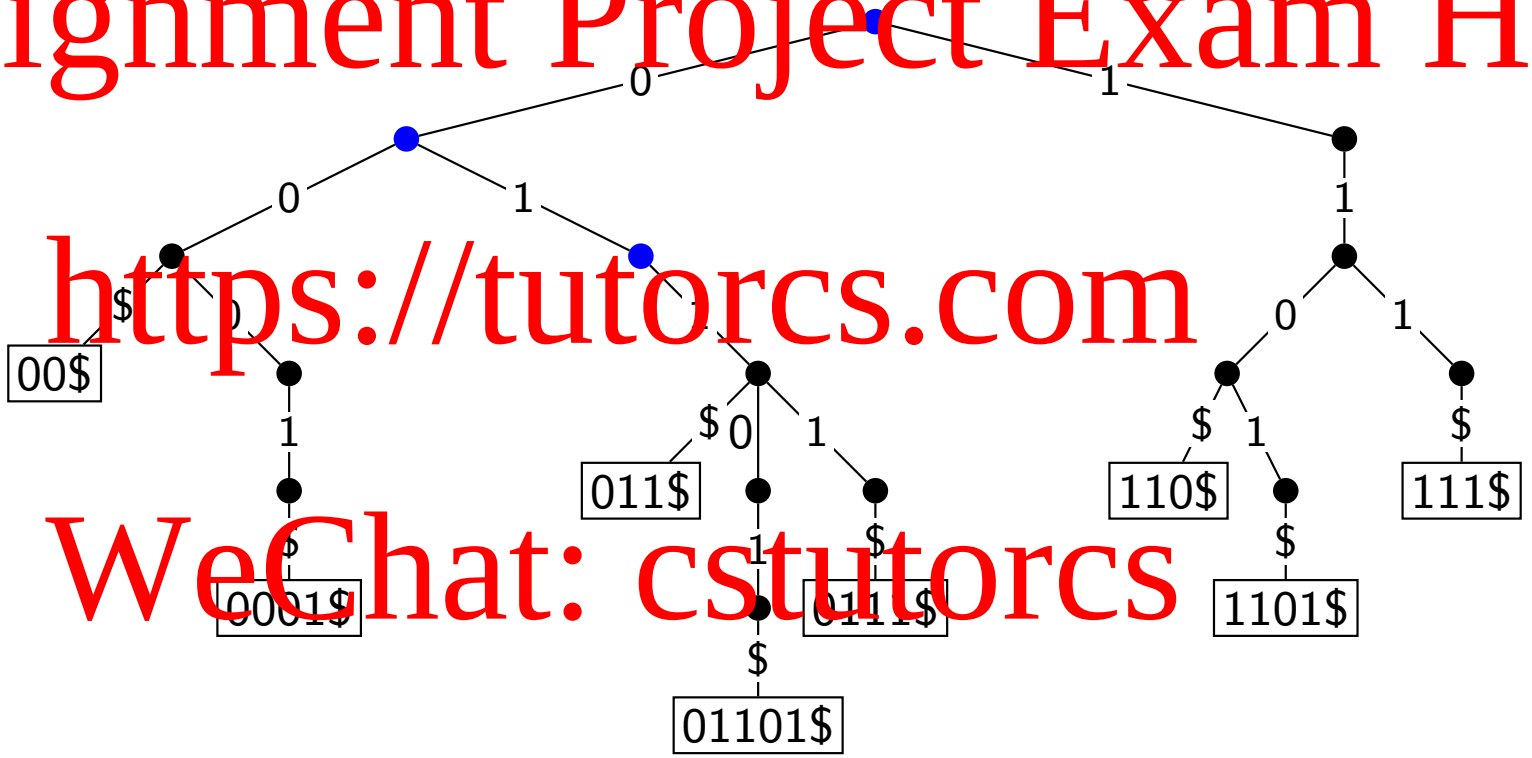
Tries: Delete Example

Example: *Trie::delete*(01001$)



Tries: Delete Example

Example: *Trie::delete*(01001$)



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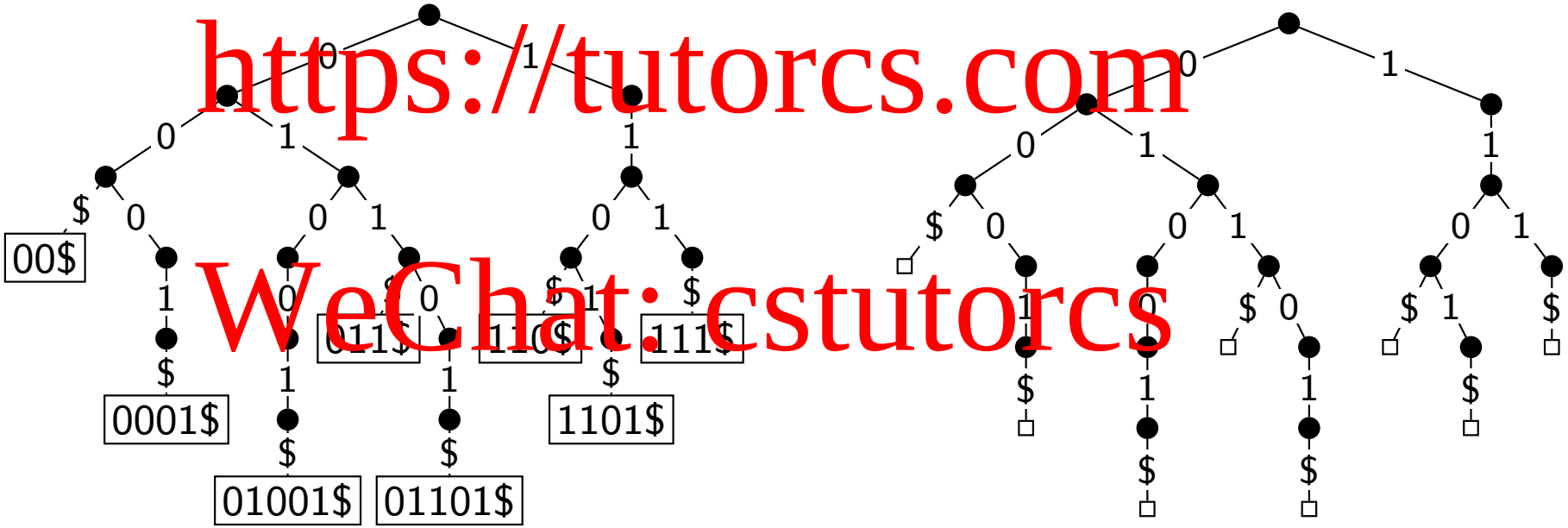
Variation 1 of Tries: No leaf labels

Do not store actual keys at the leaves.

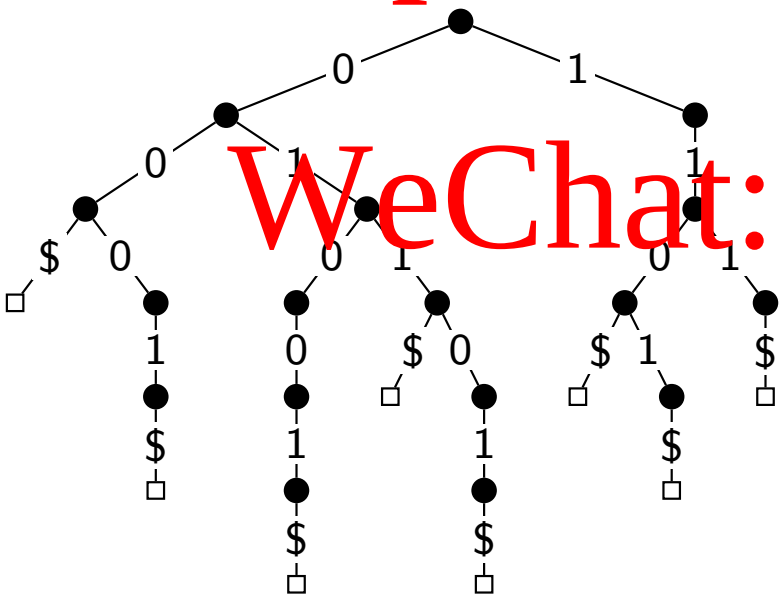
- The key is stored implicitly through the characters along the path to the leaf. It therefore need not be stored again.
- This halves the amount of space needed.

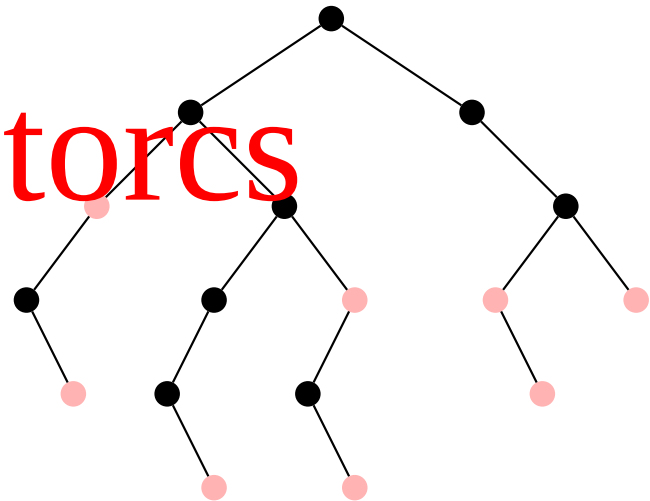


Variation 2 of Tries: Allow Proper Prefixes

Allow prefixes to be in dictionary.

- Internal nodes may now also represent keys.
  Use a *flag* to indicate such nodes.
- No need for end-of-word character \$
- Now a trie of bitstrings is a binary tree. Can express 0-child and 1-child implicitly via left and right child.
- More space-efficient.

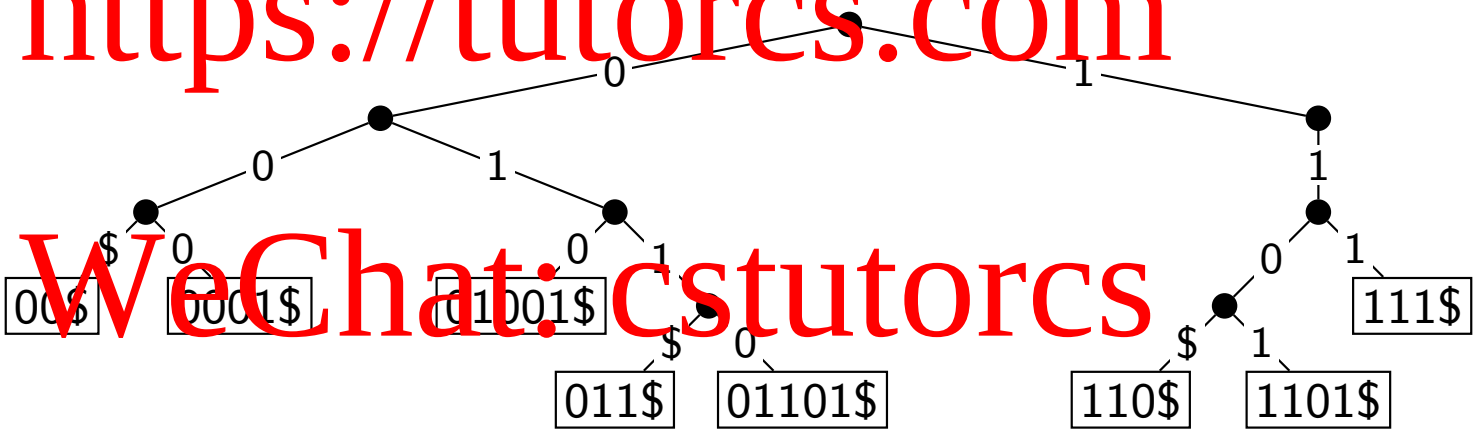




Variations 3 of Tries

Pruned Trie: Stop adding nodes to trie as soon as the key is unique.

- A node has a child only if it has at least two descendants.
- Note that now we *must* store the full keys (why?)
- Saves space if there are only few bitstrings that are long.
- Could even store infinite bitstrings (e.g. real numbers)



This is in practice the most efficient version of tries, but the operations get a bit more complicated.

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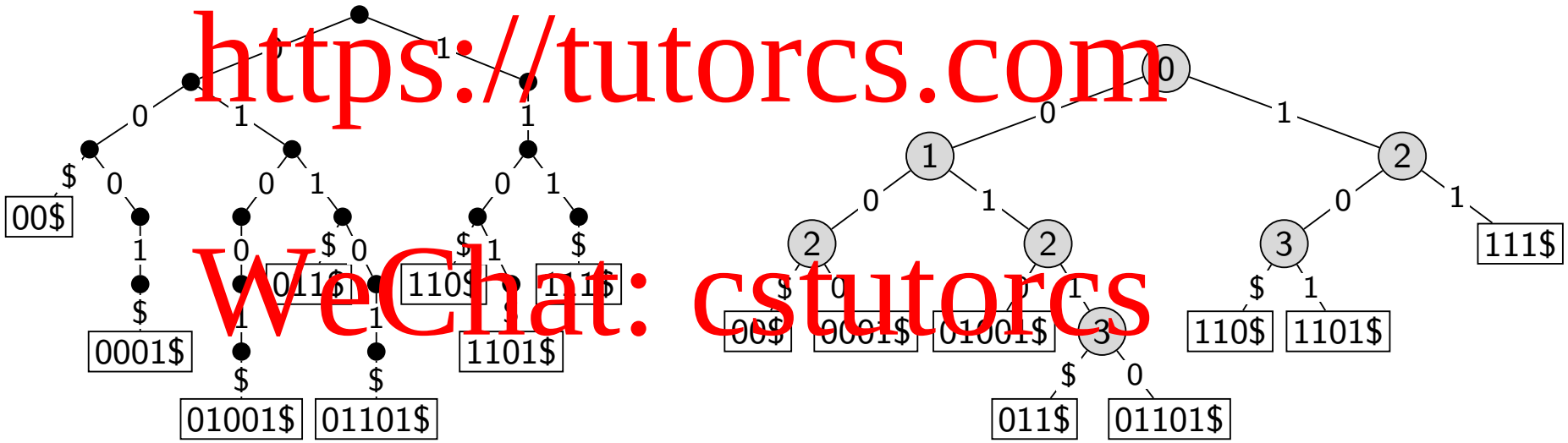
Variation 4 of Tries

Compressed Trie: compress paths of nodes with only one child

- Each node stores an *index*, corresponding to the depth in the uncompressed trie.
  - This gives the next bit to be tested during a search
- A compressed trie with n keys has at most $n - 1$ internal nodes



Also known as **Patricia-Tries**:
Practical Algorithm to Retrieve Information Coded in Alphanumeric

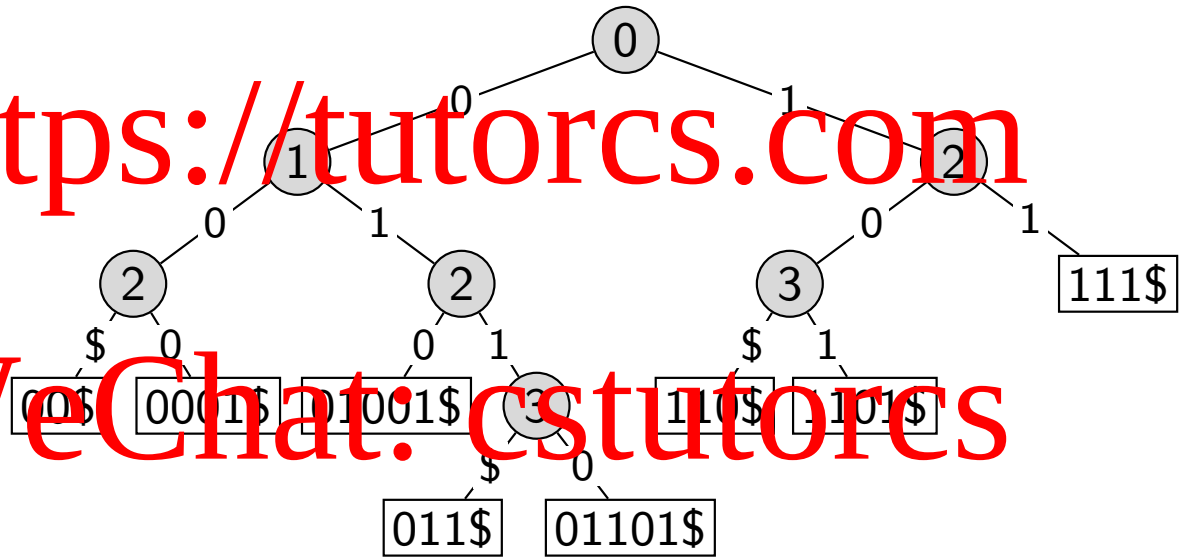
Compressed Tries: Search

- start from the root and the bit indicated at that node
- follow the link that corresponds to the current bit in x; return failure if the link is missing
- if we reach a leaf, explicitly check whether word stored at leaf is x
- else recurse on the new node and the next bit of x

```
CompressedTrie::search(v ← root, x)
v: node of trie; x: word
1.    if v is a leaf
2.        return strcmp(x, v.key)
3.    else
4.        d ← index stored at v
5.        c ← child of v labelled with x[d]
6.        if there is no such child
7.            return "not found"
8.        else CompressedTrie::search(c, x)
```

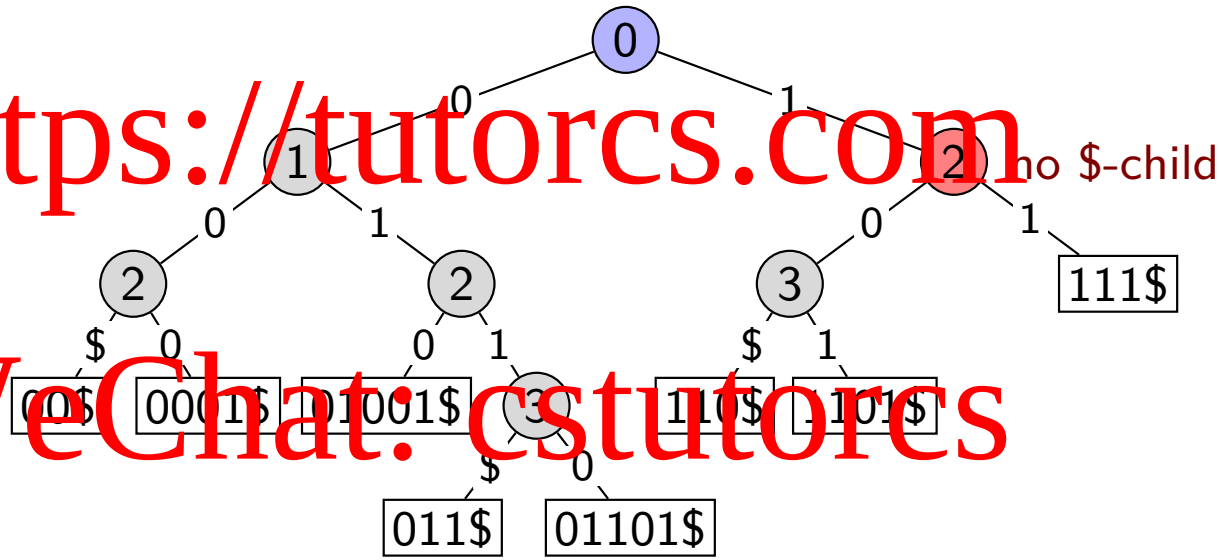
Compressed Tries: Search Example

Example: CompressedTrie::search(10$)



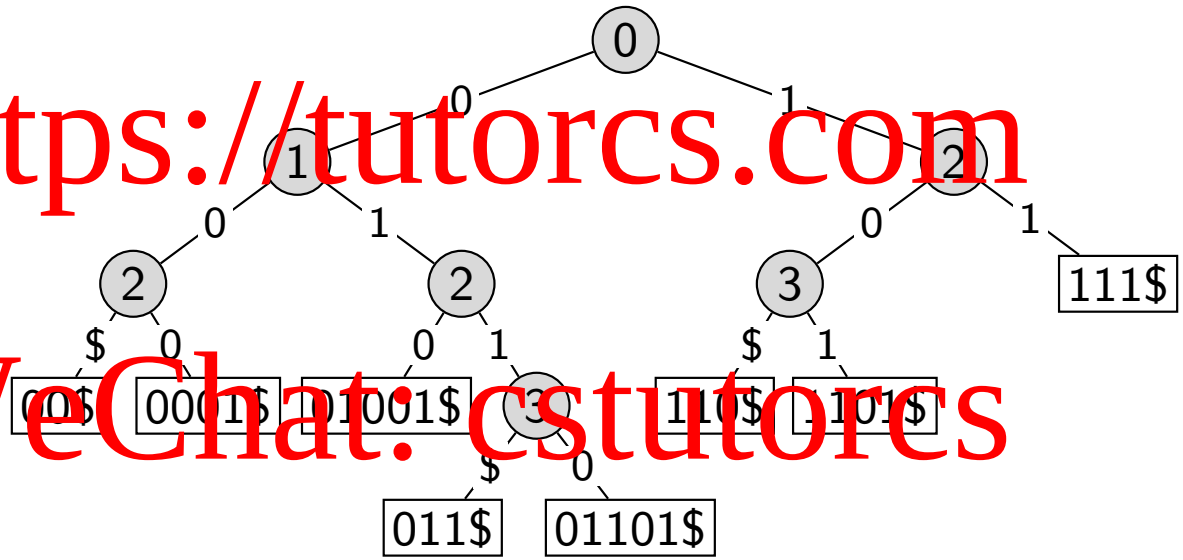
Compressed Tries: Search Example

Example: CompressedTrie::search(10$) unsuccessful



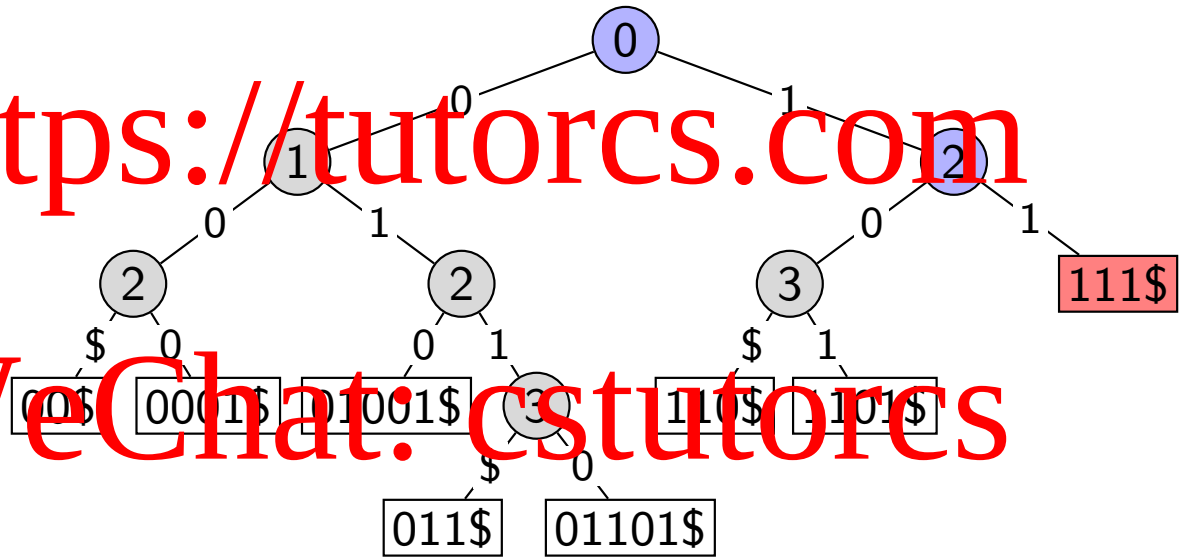
Compressed Tries: Search Example

Example: CompressedTrie::search(101$)



Compressed Tries: Search Example

Example: CompressedTrie::search(101$) unsuccessful



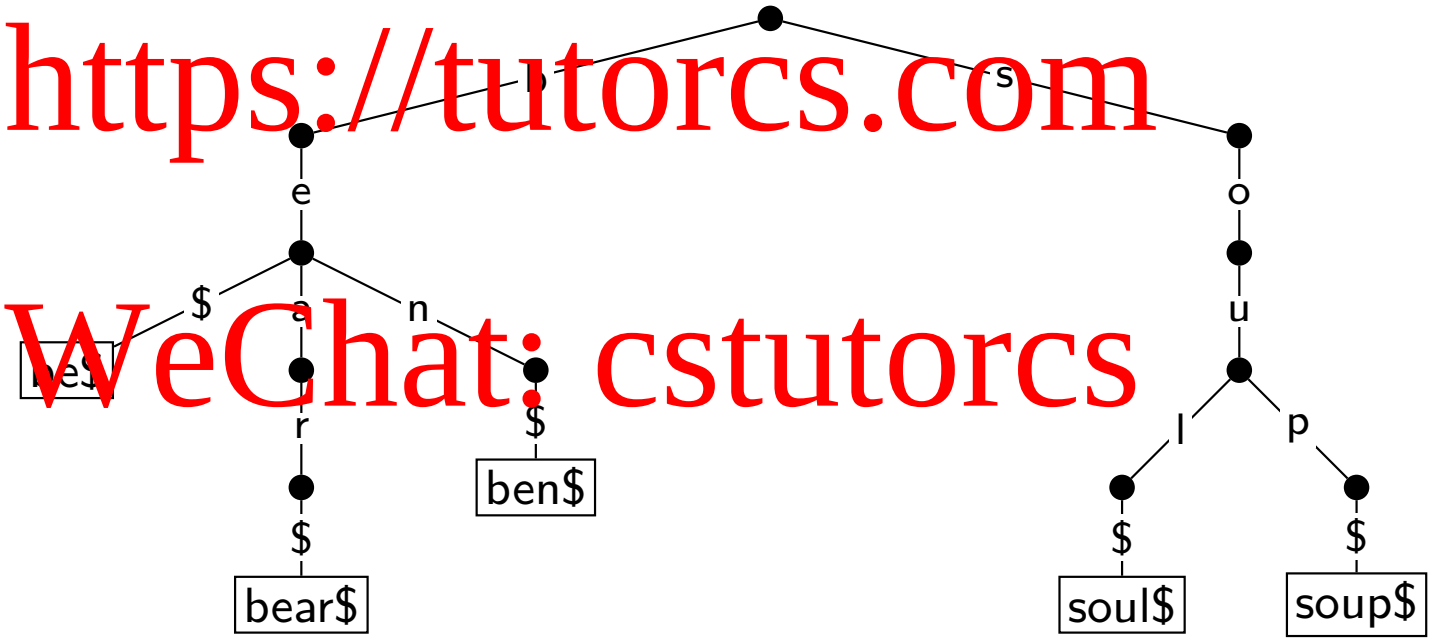
Compressed Tries: Insert & Delete

- *CompressedTrie::delete*(x):
  - Perform *search*(x).
  - Remove the node v that stored x.
  - Compress along path to v whenever possible.
- *CompressedTrie::insert*(x):
  - Perform *search*(x).
  - Let v be the node where the search ended.
  - Conceptually simplest approach:
    - ★ Uncompress path from root to v.
    - ★ Insert x as in an uncompressed trie.
    - ★ Compress paths from root to v and from root to x.

    But it can also be done by only adding those nodes that are needed, see the textbook for details.
- All operations take $O(|x|)$ time.

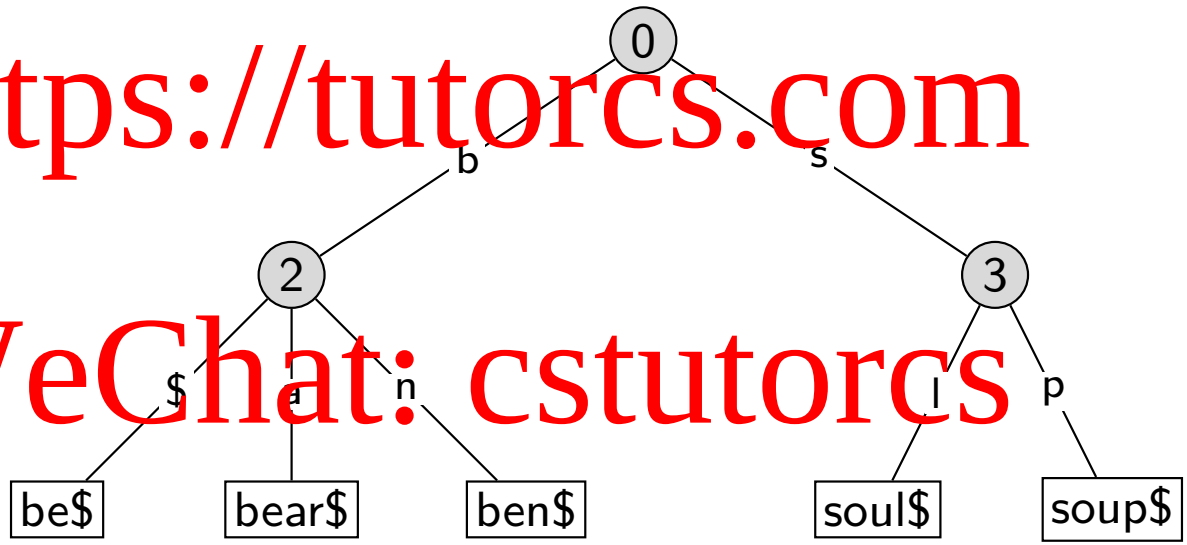
Multiway Tries: Larger Alphabet

- To represent *strings* over any *fixed alphabet* Σ
- Any node will have at most $|\Sigma| + 1$ children (one child for the end-of-word character \$)
- Example: A trie holding strings {bear$, ben$, be$, soul$, soup$}



Compressed Multiway Tries

- **Variation**: Compressed multi-way tries: compress paths as before
- Example: A compressed trie holding strings {bear$, ben$, be$, soul$, soup$}



Multiway Tries: Summary

- Operations *search*(x), *insert*(x) and *delete*(x) are exactly as for tries for bitstrings.
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- Run-time $O(|x| \cdot$ (time to find the appropriate child))

Each node now has up to $|\Sigma| + 1$ children. How should they be stored?

Solution 1: Array of size $|\Sigma| + 1$ for each node.
Complexity: $O(1)$ time to find child, $O(|\Sigma|n)$ space.

Solution 2: List of children for each node.
Complexity: $O(|\Sigma|)$ time to find child, $O(\#\text{children})$ space.

Solution 3: Dictionary (AVL-tree?) of children for each node.
Complexity: $O(\log(\#\text{children}))$ time, $O(\#\text{children})$ space.

Best in theory, but not worth it in practice unless $|\Sigma|$ is huge.

In practice, use *hashing* (keys are in (typically small) range Σ).