CS 240 – Data Structures and Data Management

Module 9: String Matching

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Based on lecture notes by many previous cs240 instructors

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References: Goodrich & Tamassia 23

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Outline

1. String Matching
   - Introduction
   - Karp-Rabin Algorithm
   - String Matching with Finite Automata
   - Knuth-Morris-Pratt algorithm
   - Boyer-Moore Algorithm
   - Suffix Trees
   - Conclusion

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Pattern Matching Definition [1]

- Search for a string (pattern) in a large body of text
- $T[0..n-1]$ – The text (or haystack) being searched within
- $P[0..m-1]$ – The pattern (or needle) being searched for
- Strings over alphabet Σ
- Return the first i such that

$$P[j] = T[i+j] \quad \text{for} \quad 0 \leq j \leq m-1$$

- This is the first occurrence of P in T
- If P does not occur in T, return FAIL
- Applications:
  - Information Retrieval (text editors, search engines)
  - Bioinformatics
  - Data Mining

Pattern Matching Definition [2]

Example:

- $T =$ "Where is he?"
- $P_1 =$ "he"
- $P_2 =$ "who"

Definitions:

- Substring $T[i..j]$ $0 \le i \le j < n$: a string of length $j - i + 1$ which consists of characters $T[i], \ldots T[j]$ in order
- A prefix of T:
  a substring $T[0..i]$ of T for some $0 \le i < n$
- A suffix of T:
  a substring $T[i..n-1]$ of T for some $0 \le i \le n - 1$

Pattern matching algorithms consist of guesses and checks:

- A **guess** or **shift** is a position i such that P might start at $T[i]$. Valid guesses (initially) are $0 \leq i \leq n - m$.
- A **check** of a guess is a single position j with $0 \leq j < m$ where we compare $T[i + j]$ to $P[j]$. We must perform m checks of a single correct guess, but may make (many) fewer checks of an incorrect guess.

We will diagram a single run of any pattern matching algorithm by a matrix of checks, where each row represents a single guess.

Brute-force Algorithm

Idea: Check every possible guess.


```
BruteforcePM(T[0..n − 1], P[0..m − 1])
T: String of length n (text), P: String of length m (pattern)
    for i ← 0 to n − m do
        if strcmp(T[i..i+m−1], P) = 0
            return "found at guess i"
    return FAIL
```


Note: *strcmp* takes $\Theta(m)$ time.


```
strcmp(T[i..i+m−1], P[0..m − 1])
    for j ← 0 to m − 1 do
        if T[i + j] is before P[j] in Σ then return -1
        if T[i + j] is after P[j] in Σ then return 1
    return 0
```

Brute-Force Example

- Example: $T = \texttt{abbbababbab}$, $P = \texttt{abba}$

a	b	b	b	a	b	a	b	b	a	b
a	b	b	a							
	a									
		a								
			a							
				a	b	b				
					a					
						a	b	b	a	

- What is the worst possible input?
  $P = a^{m-1}b$, $T = a^n$
- Worst case performance $\Theta((n-m+1)m)$
- This is $\Theta(mn)$ e.g. if $m = n/2$.

How to improve?

More sophisticated algorithms

- Do extra preprocessing on the pattern P
  - **Karp-Rabin**
  - **Boyer-Moore**
  - Deterministic finite automata (**DFA**), **KMP**
  - We eliminate guesses based on completed matches and mismatches.
- Do extra preprocessing on the text T
  - **Suffix-trees**
  - We create a data structure to find matches easily.

Outline

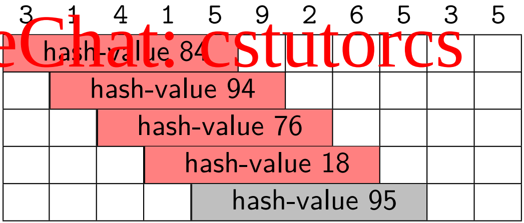
Karp-Rabin Fingerprint Algorithm – Idea

Idea: use hashing to eliminate guesses

- Compute hash function for each guess, compare with pattern hash
- If values are unequal, then the guess cannot be an occurrence
- Example: $P = 5\ 9\ 2\ 6\ 5$, $\quad T = 3\ 1\ 4\ 1\ 5\ 9\ 2\ 6\ 5\ 3\ 5$
  - Use standard hash-function: flattening + modular (radix $R = 10$):

$$h(x_0 \ldots x_4) = \big(x_0 x_1 x_2 x_3 x_4\big)_{10} \bmod 97$$

  - $h(P) = 59265 \bmod 97 = 95$.



Karp-Rabin Fingerprint Algorithm – First Attempt


```
Karp-Rabin-Simple(T, P)
    h_P ← h(P[0..m−1)])
    for i ← 0 to n − m
        h_T ← h(T[i..i+m−1])
        if h_T = h_P
            if strcmp(T[i..i+m−1], P) = 0
                return "found at guess i"
    return FAIL
```


- Never misses a match: $h(T[i..i+m-1]) \neq h(P) \Rightarrow$ guess i is not P
- $h(T[i..i+m-1])$ depends on m characters, so naive computation takes $\Theta(m)$ time per guess
- Running time is $\Theta(mn)$ if P not in T (how can we improve this?)

Karp-Rabin Fingerprint Algorithm – Fast Rehash

The initial hashes are called **fingerprints**.

Crucial insight: We can update these fingerprints in constant time.

- Use previous hash to compute next hash
- $O(1)$ time per hash, except first one

Example:

- Pre-compute: $10000 \bmod 97 = 9$
- Previous hash: $41592 \bmod 97 = 76$
- Next hash: $15926 \bmod 97 = ??$

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Crucial insight: We can update these fingerprints in constant time.

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Example:

- Pre-compute: $10000 \bmod 97 = 9$
- Previous hash: $41592 \bmod 97 = 76$
- Next hash: $15926 \bmod 97 = ??$

Observe: $15926 = (41592 - 4 \cdot 10000) \cdot 10 + 6$

$$15926 \bmod 97 = \Big((\underbrace{41592 \bmod 97}_{76 \text{ (previous hash)}} - 4 \cdot \underbrace{10000 \bmod 97}_{9 \text{ (pre-computed)}} \Big) \cdot 10 + 6 \Big) \bmod 97$$

$$= \Big((76 - 4 \cdot 9) \cdot 10 + 6 \Big) \bmod 97 = 18$$

Karp-Rabin Fingerprint Algorithm – Conclusion

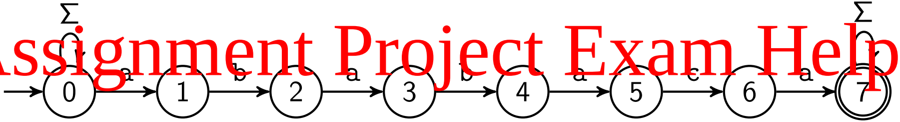
```
Karp-Rabin-RollingHash(T, P)
    h_P ← h(P[0..m−1)])
    p ← suitable prime number
    s ← 10^(m−1) mod p
    h_T ← h(T[0..m−1)])
    for i ← 0 to n − m
        if i > 0 // compute hash-value for next guess
            h_T ← ((h_T − T[i]·s)·10 + T[i+m]) mod p
        if h_T = h_P
            if strcmp(T[i..i+m−1], P) = 0
                return "found at guess i"
    return "FAIL"
```


- Choose "table size" p at random to be huge prime
- Expected running time is $O(m + n)$
- $\Theta(mn)$ worst-case, but this is (unbelievably) unlikely

Outline

String Matching with Finite Automata

Example: Automaton for the pattern $P =$ `ababaca`

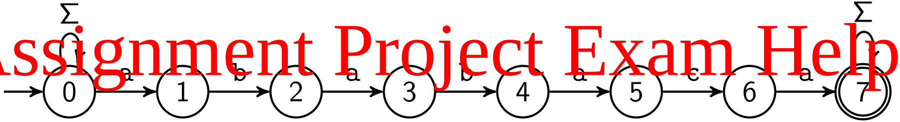


You should be familiar with:

- finite automaton, DFA, NFA, converting NFA to DFA
- transition function δ, states Q, accepting states F

String Matching with Finite Automata

Example: Automaton for the pattern $P = \texttt{ababaca}$



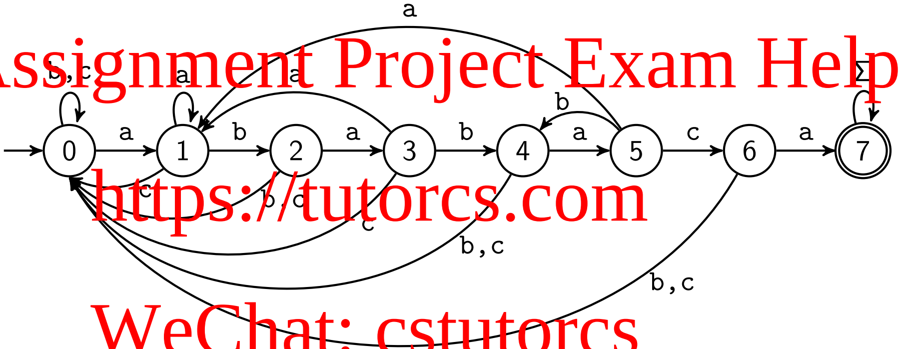
You should be familiar with:

- finite automaton, DFA, NFA, converting NFA to DFA
- transition function δ, states Q, accepting states F

- The above finite automation is an **NFA**
- State q expresses "we have seen $P[0..q-1]$"
  - NFA accepts T if and only if T contains $\texttt{ababaca}$
  - But evaluating NFAs is very slow.

String matching with DFA

Can show: There exists an equivalent small DFA.



- Easy to test whether P is in T.
- But how do we find the arcs?
- We will not give the details of this since there is an even better automaton.

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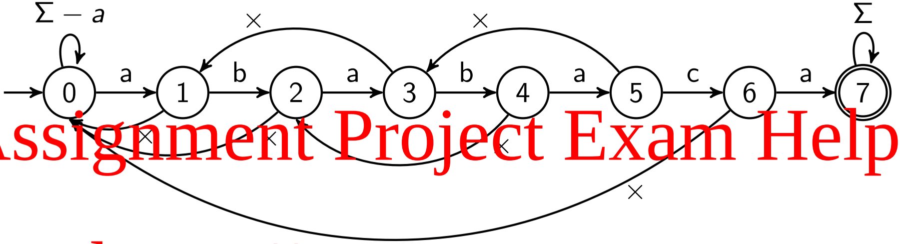
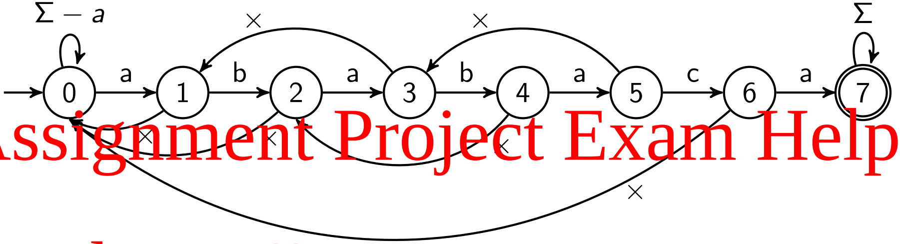
Knuth-Morris-Pratt Motivation



- Use a new type of transition × (*"failure"*)
  - Use this transition only if no other fits.
  - Does not consume a character.
  - With these rules, computations of the automaton are deterministic. (But it is formally not a valid DFA.)

Knuth-Morris-Pratt Motivation



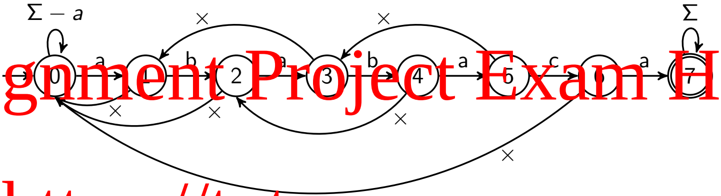
- Use a new type of transition × (*"failure"*)
  - Use this transition only if no other fits.
  - Does not consume a character.
  - With these rules, computations of the automaton are deterministic. (But it is formally not a valid DFA.)
- Can store failure-function in an array $F[0..m-1]$
  - The failure arc from state j leads to $F[j-1]$
- Given the failure-array, we can easily test whether P is in T: Automaton accepts T if and only if T contains `ababaca`

Knuth-Morris-Pratt Algorithm

```
KMP(T, P)
    F ← failureArray(P)
    i ← 0 // current character of T to parse
    j ← 0 // current state that we are in
    while i < n do
        if P[j] = T[i]
            if j = m − 1
                return "found at guess i − m + 1"
            else
                i ← i + 1
                j ← j + 1
        else // i.e. P[j] ≠ T[i]
            if j > 0
                j ← F[j − 1]
            else
                i ← i + 1
    return FAIL
```

String matching with KMP – Example

Example: T = `abababababaca`, P = `ababaca`



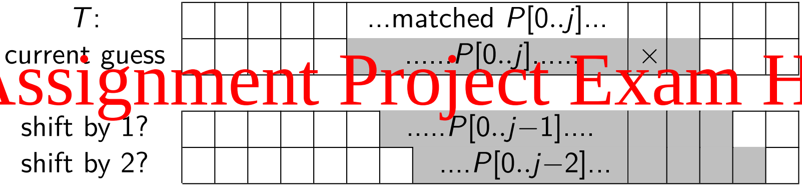
T:	a	b	a	b	a	b	a	b	a	c	a	
P:	a	b	a	b	a	×						to state 3
			(a)	(b)	(a)	b	a	×				to state 3
					(a)	(b)	(a)	b	a	c	a	

q:	1	2	3	4	5	3, 4	5	3, 4	5	6	7

(after reading this character)

String matching with KMP – Failure-function

Assume we reach state $j+1$ and now have mismatch.



- Can eliminate "shift by 1" if $P[1..j] \neq P[0..j-1]$.
- Can eliminate "shift by 2" if $P[1..j]$ does not end with $P[0..j-2]$.
- Generally eliminate guess if that prefix of P is not a suffix of $P[1..j]$.
- So want longest prefix $P[0..\ell-1]$ that is a suffix of $P[1..j]$.
- The ℓ characters of this prefix are matched, so go to state ℓ.

$$
\begin{aligned}
F[j] &= \text{head of failure-arc from state } j+1 \\
&= \text{length of the longest prefix of } P \text{ that is a suffix of } P[1..j].
\end{aligned}
$$

KMP Failure Array – Example

$F[j]$ is the length of the longest prefix of P that is a suffix of $P[1..j]$.

Consider P = ababaca

j	$P[1..j]$	Prefixes of P	longest	$F[j]$
0	Λ	Λ, a, ab, aba, abab, ababa, ...	Λ	0
1	b	Λ, a, ab, aba, abab, ababa, ...	Λ	0
2	ba	Λ, a, ab, aba, abab, ababa, ...	a	1
3	bab	Λ, a, ab, aba, abab, ababa, ...	ab	2
4	baba	Λ, a, ab, aba, abab, ababa, ...	aba	3
5	babac	Λ, a, ab, aba, abab, ababa, ...	Λ	0
6	babaca	Λ, a, ab, aba, abab, ababa, ...	a	1

This can clearly be computed in $O(m^3)$ time, but we can do better!

Computing the Failure Array

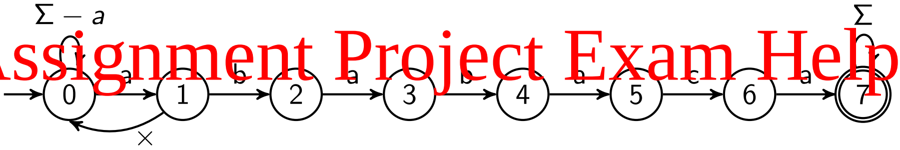
failureArray(P)
P: String of length m (pattern)

```
F[0] ← 0
i ← 1
j ← 0
while i < m do
    if P[i] = P[j]
        j ← j + 1
        F[i] ← j
        i ← i + 1
    else if j > 0
        j ← F[j − 1]
    else
        F[i] ← 0
        i ← i + 1
```


Correctness-idea: $F[\cdot]$ is defined via pattern matching of $P[1..j]$ in P. So KMP uses itself! Already-built parts of $F[\cdot]$ are used to expand it.

KMP failure function – fast computation

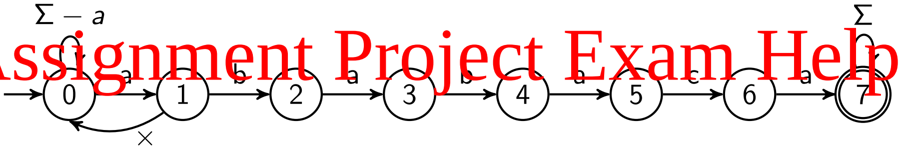
$P =$ `ababaca`



Parse $P[1..m-1] =$ `babaca` while adding failure arcs:

KMP failure function – fast computation

$P =$ ababaca

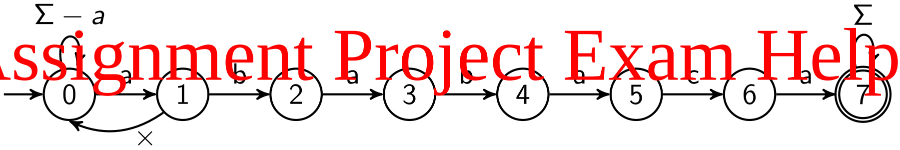


Parse $P[1..m-1] =$ babaca while adding failure arcs:

i	1																
$P[i]$																	
$P[j]$																	
j	0																
$F[i]$																	

KMP failure function – fast computation

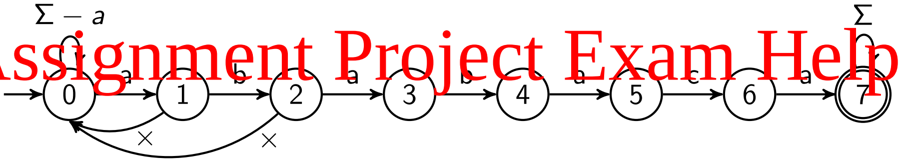
$P =$ `ababaca`



Parse $P[1..m-1]$ = babaca while adding failure arcs:

i	1	1															
$P[i]$		b															
$P[j]$		a															
j	0	$\xrightarrow{b}$															
$F[i]$																	

$P = \texttt{abababaca}$



Parse $P[1..m-1] = \texttt{bababaca}$ while adding failure arcs:

i	1	1	1														
$P[i]$		b															
$P[j]$		a															
j	0	$\xrightarrow{b}$	0														
$F[i]$			0														

KMP failure function – fast computation

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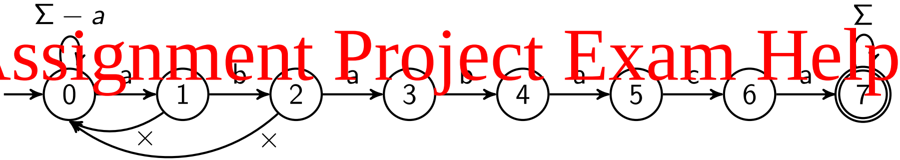
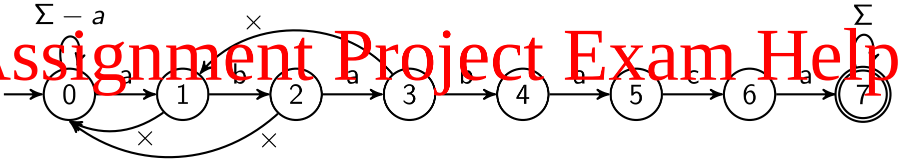


Parse $P[1..m-1] =$ `babaca` while adding failure arcs:

i	1	1	1	2													
$P[i]$		b		a													
$P[j]$		a		a													
j	0	$\xrightarrow{b}$	0														
$F[i]$			0														

KMP failure function – fast computation

$P =$ `ababaca`

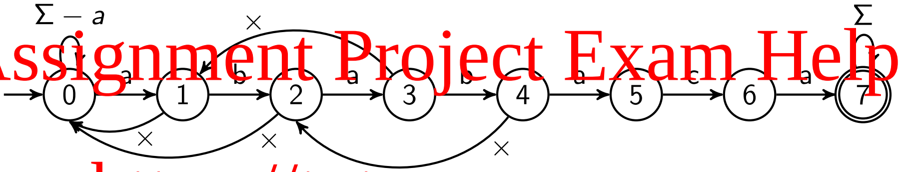


Parse $P[1..m-1] =$ babaca while adding failure arcs:

i	1	1	1	2	2												
$P[i]$		b		a													
$P[j]$		a		a													
j	0	$\xrightarrow{b}$	0	$\xrightarrow{a}$	1												
$F[i]$			0		1												

KMP failure function – fast computation

$P =$ `ababaca`

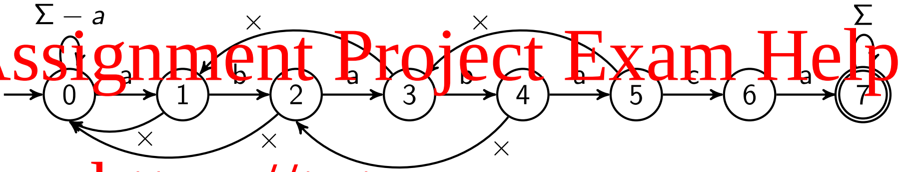


Parse $P[1..m-1] =$ babaca while adding failure arcs:

i	1	1	1	2	2	3	3										
$P[i]$		b		a		b											
$P[j]$		a		a		b											
j	0	$\xrightarrow{b}$	0	$\xrightarrow{a}$	1	$\xrightarrow{b}$	2										
$F[i]$			0		1		2										

KMP failure function – fast computation

$P =$ `ababaca`

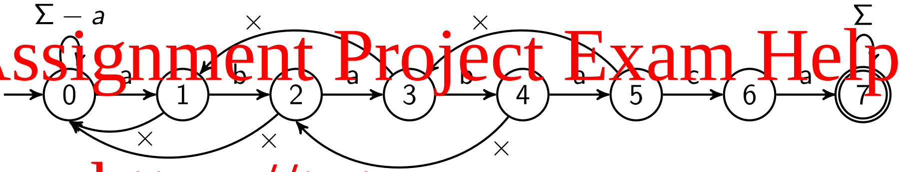


Parse $P[1..m-1] =$ babaca while adding failure arcs:

i	1	1	1	2	2	3	3	4	4								
$P[i]$		b		a		b		a									
$P[j]$		a		a		b		a									
j	0	$\xrightarrow{b}$	0	$\xrightarrow{a}$	1	$\xrightarrow{b}$	2	$\xrightarrow{a}$	3								
$F[i]$			0		1		2		3								

KMP failure function – fast computation

$P =$ `ababaca`

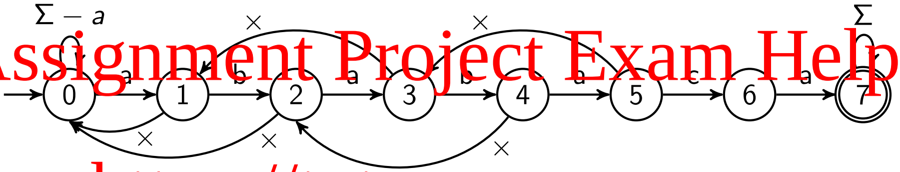


Parse $P[1..m-1] =$ babaca while adding failure arcs:

i	1	1	1	2	2	3	3	4	4	5							
$P[i]$		b		a		b		a		c							
$P[j]$		a		a		b		a		b							
j	0	$\xrightarrow{b}$	0	$\xrightarrow{a}$	1	$\xrightarrow{b}$	2	$\xrightarrow{a}$	3	$\xrightarrow{\times}$							
$F[i]$			0		1		2		3								

KMP failure function – fast computation

$P =$ `ababaca`

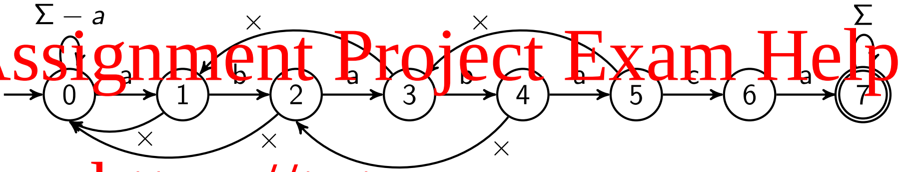


Parse $P[1..m-1] =$ babaca while adding failure arcs:

i	1	1	1	2	2	3	3	4	4	5	5						
$P[i]$		b		a		b		a		c							
$P[j]$		a		a		b		a		b							
j	0	$\xrightarrow{b}$	0	$\xrightarrow{a}$	1	$\xrightarrow{b}$	2	$\xrightarrow{a}$	3	$\xrightarrow{\times}$	1						
$F[i]$			0		1		2		3								

KMP failure function – fast computation

$P = $ `ababaca`

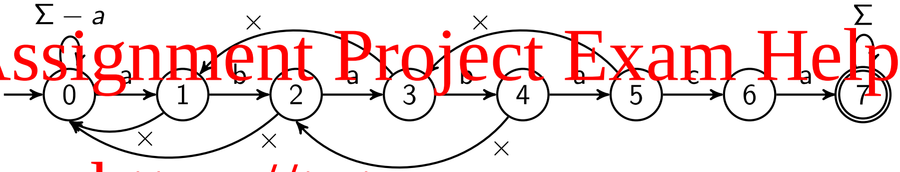


Parse $P[1..m-1] =$ babaca while adding failure arcs:

i	1	1	1	2	2	3	3	4	4	5	5	5				
$P[i]$		b		a		b		a		c		c				
$P[j]$		a		a		b		a		b		b				
j	0	$\xrightarrow{b}$	0	$\xrightarrow{a}$	1	$\xrightarrow{b}$	2	$\xrightarrow{a}$	3	$\xrightarrow{\times}$	1	$\xrightarrow{\times}$				
$F[i]$			0		1		2		3							

KMP failure function – fast computation

$P =$ `ababaca`

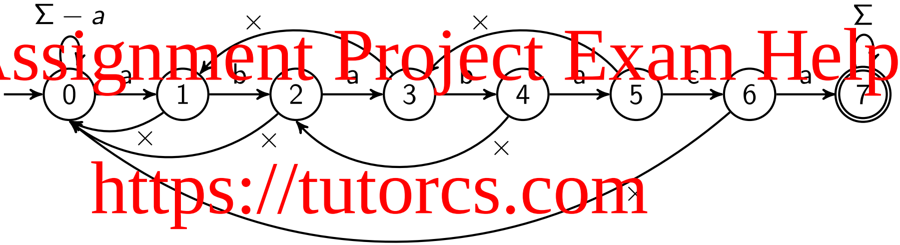


Parse $P[1..m-1]$ = `babaca` while adding failure arcs:

i	1	1	1	2	2	3	3	4	4	5	5	5	5				
$P[i]$		b		a		b		a		c		c					
$P[j]$		a		a		b		a		b		b					
j	0	$\xrightarrow{b}$	0	$\xrightarrow{a}$	1	$\xrightarrow{b}$	2	$\xrightarrow{a}$	3	$\xrightarrow{\times}$	1	$\xrightarrow{\times}$	0				
$F[i]$			0		1		2		3								

KMP failure function – fast computation

$P =$ `ababaca`

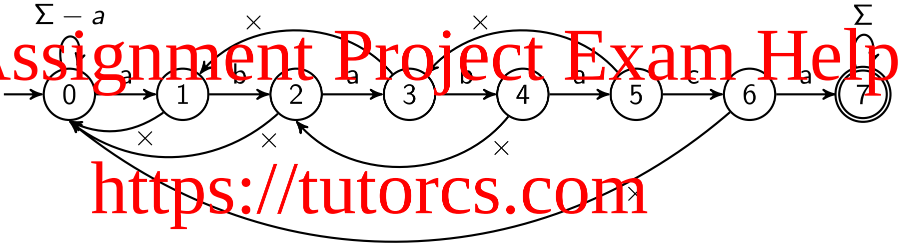


Parse $P[1..m-1] =$ babaca while adding failure arcs:

i	1	1	1	2	2	3	3	4	4	5	5	5	5	5	5		
$P[i]$		b		a		b		a		c		c		c			
$P[j]$		a		a		b		a		b		b		a			
j	0	$\xrightarrow{b}$	0	$\xrightarrow{a}$	1	$\xrightarrow{b}$	2	$\xrightarrow{a}$	3	$\xrightarrow{\times}$	1	$\xrightarrow{\times}$	0	$\xrightarrow{c}$	0		
$F[i]$			0		1		2		3						0		

KMP failure function – fast computation

$P =$ `ababaca`



Parse $P[1..m-1] =$ babaca while adding failure arcs:

i	1	1	1	2	2	3	3	4	4	5	5	5	5	5	5	6	6
$P[i]$		b		a		b		a		c		c		c		a	
$P[j]$		a		a		b		a		b		b		a		a	
j	0	$\xrightarrow{b}$	0	$\xrightarrow{a}$	1	$\xrightarrow{b}$	2	$\xrightarrow{a}$	3	$\xrightarrow{\times}$	1	$\xrightarrow{\times}$	0	$\xrightarrow{c}$	0	$\xrightarrow{a}$	1
$F[i]$			0		1		2		3						0		1

KMP – Runtime

failureArray

- Consider how $2i - j$ changes in each iteration of the while loop
  - i and j both increase by $1 \Rightarrow 2i - j$ increases –OR–
  - j decreases ($F[j-1] < j$) $\Rightarrow 2i - j$ increases –OR–
  - i increases $\Rightarrow 2i - j$ increases
- Initially $2i - j \geq 0$, at the end $2i - j \leq 2m$
- So no more than $2m$ iterations of the while loop.
- Running time: $\Theta(m)$

KMP – Runtime

failureArray

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- So no more than $2m$ iterations of the while loop.
- Running time: $\Theta(m)$

KMP main function

- failureArray can be computed in $\Theta(m)$ time
- Same analysis gives at most $2n$ iterations of the while loop since $2i - j \leq 2n$.
- Running time KMP altogether: $\Theta(n + m)$

Outline

1. String Matching
   - Introduction
   - Karp-Rabin Algorithm
   - String Matching with Finite Automata
   - Knuth-Morris-Pratt algorithm
   - Boyer-Moore Algorithm
   - Suffix Trees
   - Conclusion

Boyer-Moore Algorithm

Brute-force search with three changes:

- Reverse-order searching: Compare P with a guess moving backwards
- Bad character jumps: When a mismatch occurs, then eliminate guesses where P does not agree with this char of T
- Good suffix jumps: When a mismatch occurs, then use recently seen suffix of P to eliminate guesses.
- This gives two possible shifts (locations of next guess to try). Use the one that moves forward more.
- In practice large parts of T will not be looked at.

Boyer-Moore Algorithm

```
boyer-moore(T,P)
    L ← last occurrence array computed from P
    S ← good suffix array computed from P
    i ← m − 1,       j ← m − 1
    while i < n and j ≥ 0 do
        if T[i] = P[j]
            i ← i − 1
            j ← j − 1
        else
            i ← i + m − 1 − min(L[T[i]], S[j])
            j ← m − 1
    if j = −1 return i + 1
    else return FAIL
```

L and S will be explained below.

Bad character heuristic

P:	p	a	n	i	n	i									
T:	O	c	e	a	n	i	s	t	o	o	d	e	e	p	!

Bad character heuristic

P : p a n i n i

T : O c e a n i s t o o d e e p !

Bad character heuristic

P : p a n i n i

T : O c e a n i s t o o d e e p !

					i									

Bad character heuristic

P:	p	a	n	i	n	i									
T:	O	c	e	a	n	i	s	t	o	o	d	e	e	p	!
					n	i									

Bad character heuristic

P:	p	a	n	i	n	i									
T:	O	c	e	a	n	i	s	t	o	o	d	e	e	p	!
				i	n	i									

Bad character heuristic

P: p a n i n i

T: O c e a n i s t o o d e e p !

O	c	e	a	n	i	s	t	o	o	d	e	e	p	!	
			i	n	i										Shift to where 'a' fits

Bad character heuristic

P : p a n i n i

T : O c e a n i s t o o d e e p !

			i	n	i									
			[a]											

Shift to where 'a' fits

Bad character heuristic

P : p a n i n i

T :	O	c	e	a	n	i	s	t	o	o	d	e	e	p	!	
				i	n	i										Shift to where 'a' fits
				[a]				i								

Bad character heuristic

P:	p	a	n	i	n	i										
T:	O	c	e	a	n	i	s	t	o	o	d	e	e	p	!	
				i	n	i										Shift to where 'a' fits
				[a]				i								't' $\notin P \Rightarrow$ shift past 't'

Bad character heuristic

P : p a n i n i

T : O c e a n i s t o o d e e p !

O	c	e	a	n	i	s	t	o	o	d	e	e	p	!	
			i	n	i										Shift to where 'a' fits
			[a]				i								't' $\notin P \Rightarrow$ shift past 't'

Bad character heuristic

P : p a n i n i

T:	O	c	e	a	n	i	s	t	o	o	d	e	e	p	!	
				i	n	i										Shift to where 'a' fits
				[a]				i								't' $\notin P \Rightarrow$ shift past 't'
														i		

Bad character heuristic

P: p a n i n i

T: O	c	e	a	n	i	s	t	o	o	d	e	e	p	!	
			i	n	i										Shift to where ‘a’ fits
			[a]				i								‘t’ $\notin P \Rightarrow$ shift past ‘t’
													i		Shift to where ‘p’ fits

Bad character heuristic

P: p a n i n i

T:	O	c	e	a	n	i	s	t	o	o	d	e	e	p	!	
				i	n	i										Shift to where 'a' fits
				[a]				i								't' $\notin P \Rightarrow$ shift past 't'
														i		Shift to where 'p' fits
														[p]		

Bad character heuristic

P : p a n i n i

T : O c e a n i s t o o d e e p !

O	c	e	a	n	i	s	t	o	o	d	e	e	p	!	
			i	n	i										Shift to where 'a' fits
			[a]				i								't' $\notin P \Rightarrow$ shift past 't'
													i		Shift to where 'p' fits
													[p]		$i > n$, so P not in T

Bad character heuristic

P : p a n i n i

T: O	c	e	a	n	i	s	t	o	o	d	e	e	p	!	
			i	n	i										Shift to where 'a' fits
			[a]				i								't' $\notin P \Rightarrow$ shift past 't'
													i		Shift to where 'p' fits
													[p]		$i > n$, so P not in T

- Build the last-occurrence array L mapping Σ to integers
- $L(c)$ is the largest index i such that $P[i] = c$

  (or -1 if no such index exists)

c	p	a	n	i	all others
$L(c)$	0	1	4	5	-1

- Can build this in time $O(m + |\Sigma|)$ with simple for-loop
- Guesses are updated by aligning $T[i]$ with $P[L(T[i])]$

Good suffix heuristic

$P =$ onobobo

o	n	o	o	o	b	o	o	o	i	b	b	o	u	n	d	a	r	y

Good suffix heuristic

$P =$ onobobo

o	n	o	o	o	b	o	o	o	i	b	b	o	u	n	d	a	r	y
			b	o	b	o												
Do smallest shift so that obo fits in the new guess.																		

Good suffix heuristic

$P =$ onobobo

o	n	o	o	o	b	o	o	o	i	b	b	o	u	n	d	a	r	y
			b	o	b	o												
Do smallest shift so that obo fits in the new guess.																		
				(o)	(b)	(o)												

Good suffix heuristic

$P =$ onobobo

<table>
<tr><td>o</td><td>n</td><td>o</td><td>o</td><td>o</td><td>b</td><td>o</td><td>o</td><td>o</td><td>i</td><td>b</td><td>b</td><td>o</td><td>u</td><td>n</td><td>d</td><td>a</td><td>r</td><td>y</td></tr>
<tr><td></td><td></td><td></td><td>b</td><td>o</td><td>b</td><td>o</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="19">Do smallest shift so that obo fits in the new guess.</td></tr>
<tr><td></td><td></td><td></td><td></td><td>(o)</td><td>(b)</td><td>(o)</td><td>b</td><td>o</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="19">Do smallest shift so that o fits in the new guess.</td></tr>
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<tr><td colspan="19"></td></tr>
</table>

Good suffix heuristic

$P =$ onobobo

o	n	o	o	o	b	o	o	o	i	b	b	o	u	n	d	a	r	y
			b	o	b	o												
Do smallest shift so that obo fits in the new guess.																		
				(o)	(b)	(o)	b	o										
Do smallest shift so that o fits in the new guess.																		
								(o)										

Good suffix heuristic

$P =$ onobobo

o	n	o	o	o	b	o	o	o	i	b	b	o	u	n	d	a	r	y
			b	o	b	o												
Do smallest shift so that obo fits in the new guess.																		
				(o)	(b)	(o)	b	o										
Do smallest shift so that o fits in the new guess.																		
								(o)										
But this *has* to fail at b, so could shift farther right away																		
							(not b)	(o)										

Good suffix heuristic

$P =$ onobobo

o	n	o	o	o	b	o	o	o	i	b	b	o	u	n	d	a	r	y
			b	o	b	o												
Do smallest shift so that obo fits in the new guess.																		
				(o)	(b)	(o)	b	o										
Do smallest shift so that o fits in the new guess.																		
								(o)										
But this *has* to fail at b, so could shift farther right away																		
							(not b)	(o)		o	b	o						
Again: the shift that matches bo would fail at o, so shift farther.																		

Good suffix heuristic

P = onobobo

o	n	o	o	o	b	o	o	o	i	b	b	o	u	n	d	a	r	y
			b	o	b	o												
Do smallest shift so that obo fits in the new guess.																		
				(o)	(b)	(o)	b	o										
Do smallest shift so that o fits in the new guess.																		
								(o)										
But this *has* to fail at b, so could shift farther right away																		
							(not b)	(o)		o	b	o						
Again: the shift that matches bo would fail at o, so shift farther.																		
										(o)	(b)	(o)						
												(o)						

Good suffix heuristic

P = onobobo

o	n	o	o	o	b	o	o	o	i	b	b	o	u	m	d	a	n	y
			b	o	b	o												
Do smallest shift so that obo fits in the new guess.																		
				(o)	(b)	(o)	b	o										
Do smallest shift so that o fits in the new guess.																		
								(o)										
But this *has* to fail at b, so could shift farther right away																		
							(not b)	(o)		o	b	o						
Again: the shift that matches bo would fail at o, so shift farther.																		
										(o)	(b)	(o)						
												(o)						
Note that we could not match all of bo but match as much as we can.																		

Suffix skip array

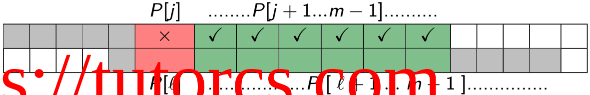
- For $0 \le j < m$, if search failed at $T[i] \neq P[j]$
  - Had $T[i{+}1..k{+}m-1] = P[j{+}1..m{-}1]$ and $T[i] \neq P[j]$

Suffix skip array

- For $0 \leq j < m$, if search failed at $T[i] \neq P[j]$
  - Had $T[i{+}1..k{+}m-1] = P[j{+}1..m{-}1]$ and $T[i] \neq P[j]$
  - Can precompute *suffix skip array* of where to shift

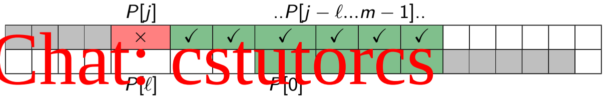
$S[j]$ is the maximum ℓ such that

- ★ $P[j{+}1 \ldots m{-}1]$ is a prefix of $P[\ell{+}1 \ldots m{-}1]$ and $P[j] \neq P[\ell]$



–OR–

- ★ $P[j-\ell \ldots m-1]$ is a prefix of P and $\ell < 0$.



–OR–

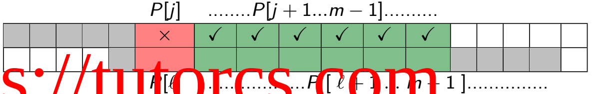
- ★ $\ell = -j$ if neither of the above is possible

Suffix skip array

- For $0 \le j < m$, if search failed at $T[i] \neq P[j]$
  - Had $T[i{+}1..k{+}m-1] = P[j{+}1..m{-}1]$ and $T[i] \neq P[j]$
  - Can precompute *suffix skip array* of where to shift

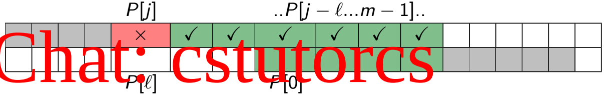
$S[j]$ is the maximum ℓ such that

★ $P[j{+}1\ldots m{-}1]$ is a prefix of $P[\ell{+}1\ldots m{-}1]$ and $P[j] \neq P[\ell]$



–OR–

★ $P[j-\ell\ldots m-1]$ is a prefix of P and $\ell < 0$.



–OR–

★ $\ell = -j$ if neither of the above is possible

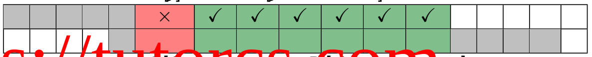
- Then can update guess by aligning $T[i]$ with $P[S[j]]$

Suffix skip array

- For $0 \leq j < m$, if search failed at $T[i] \neq P[j]$
  - Had $T[i{+}1..k{+}m-1] = P[j{+}1..m{-}1]$ and $T[i] \neq P[j]$
  - Can precompute *suffix skip array* of where to shift

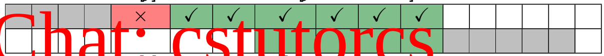
$S[j]$ is the maximum ℓ such that

- ★ $P[j{+}1\ldots m{-}1]$ is a prefix of $P[\ell{+}1\ldots m{-}1]$ and $P[j] \neq P[\ell]$



–OR–

- ★ $P[j-\ell\ldots m-1]$ is a prefix of P and $\ell < 0$.



–OR–

- ★ $\ell = -j$ if neither of the above is possible

  - Then can update guess by aligning $T[i]$ with $P[S[j]]$

- $S[\cdot]$ computable (similar to KMP failure function) in $\Theta(m)$ time.

Suffix skip array example

Example: bonobobo

i	0	1	2	3	4	5	6	7
$P[i]$	b	o	n	o	b	o	b	o
$S[i]$								

Suffix skip array example

Example: bonobobo

i	0	1	2	3	4	5	6	7
$P[i]$	b	o	n	o	b	o	b	o
$S[i]$								6

Suffix skip array example

Example: bonobobo

i	0	1	2	3	4	5	6	7
$P[i]$	b	o	n	o	b	o	b	o
$S[i]$							2	6

Suffix skip array example

Example: bonobobo

i	0	1	2	3	4	5	6	7
$P[i]$	b	o	n	o	b	o	b	o
$S[i]$						-1	2	6

Suffix skip array example

Example: bonobobo

i	0	1	2	3	4	5	6	7
$P[i]$	b	o	n	o	b	o	b	o
$S[i]$					2	-1	2	6

Suffix skip array example

Example: bonobobo

i	0	1	2	3	4	5	6	7
$P[i]$	b	o	n	o	b	o	b	o
$S[i]$				-3	2	-1	2	6

Suffix skip array example

Example: bonobobo

i	0	1	2	3	4	5	6	7
$P[i]$	b	o	n	o	b	o	b	o
$S[i]$	-6	-5	-4	-3	2	-1	2	6

Outline

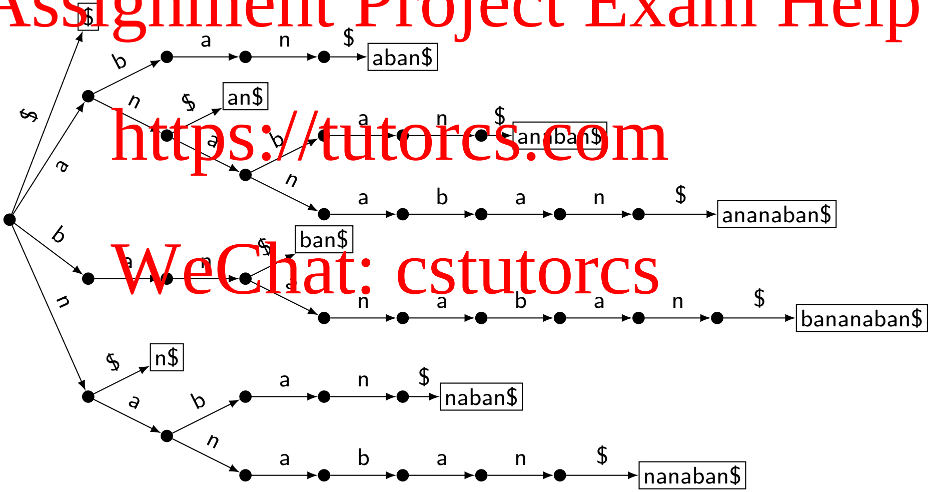
Tries of Suffixes and Suffix Trees

- What if we want to search for many patterns P within the same fixed text T?
- Idea: Preprocess the text T rather than the pattern P
- Observation: P is a substring of T if and only if P is a prefix of some suffix of T.
- So want to store all suffixes of T in a trie.
- To save space:
  - Use a compressed trie.
  - Store suffixes implicitly via indices into T.
- This is called a **suffix tree**.

Trie of suffixes: Example

$T =$ bananaban has suffixes

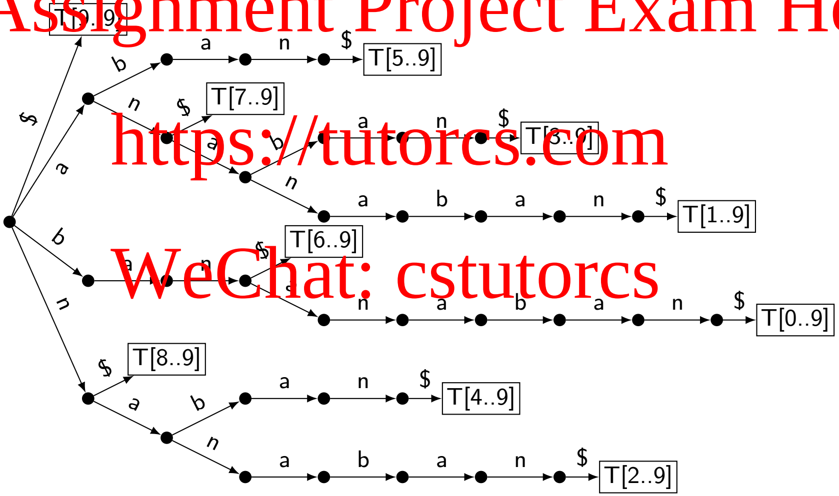
{bananaban, ananaban, nanaban, anaban, naban, aban, ban, an, n, Λ}



Tries of suffixes

Store suffixes via indices: $T =$

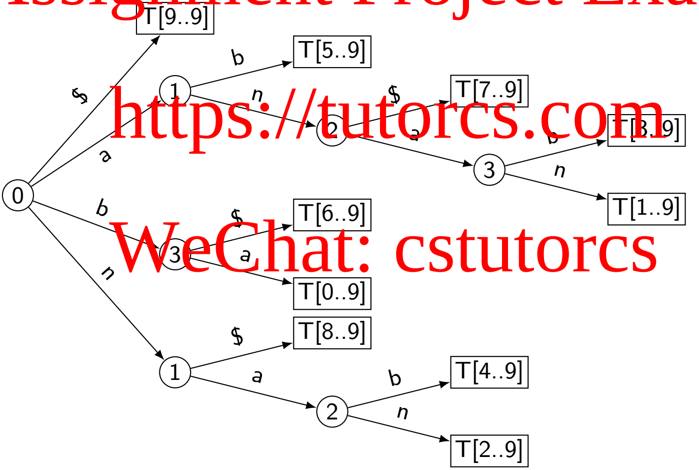
0	1	2	3	4	5	6	7	8	9
b	a	n	a	n	a	b	a	n	$



Suffix tree

Suffix tree: Compressed trie of suffixes

	0	1	2	3	4	5	6	7	8	9
$T =$	b	a	n	a	n	a	b	a	n	\$



Building Suffix Trees

- Text T has n characters and $n+1$ suffixes
- We can build the suffix tree by inserting each suffix of T into a compressed trie.
  This takes time $\Theta(n^2)$.
- There *is* a way to build a suffix tree of T in $\Theta(n)$ time.
  This is quite complicated and beyond the scope of the course.
- For pattern matching, suffix trees additionally need:
  - Every interior node v stores a reference $v.leaf$ to the leaf in its subtree with the longest suffix.
  - This can be found in $O(n)$ time by traversing the suffix tree.

Suffix Trees: String Matching

- In the *uncompressed* trie, searching for P would be easy.
- In the *compressed* suffix tree, search as in a compressed trie.
  Stop the search once P has run out of characters.

SuffixTreePM$(T[0..n-1], P[0..m-1], \mathcal{T})$
T: text, P: pattern, $\mathcal{T}$: Suffix tree of T

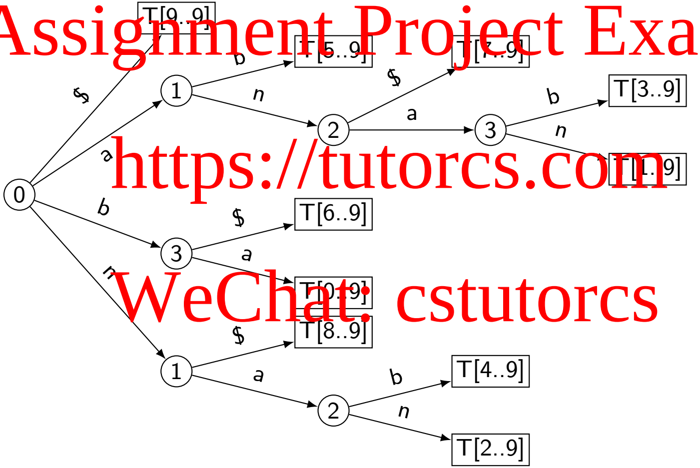

```
v ← T.root
repeat
    w ← child of v corresponding to P[v.index]
    if there is no such child return FAIL
    if w is leaf or w.index ≥ m // have gone beyond pattern P
        ℓ ← w.leaf
        i ← ℓ.start
        if (i+m ≤ n and strcmp(T[i..i+m−1], P) = 0)
            return "found at guess i"
        else return FAIL
    v ← w
```

Pattern Matching in Suffix Tree: Example 1

$T =$

0	1	2	3	4	5	6	7	8	9
b	a	n	a	n	a	b	a	n	$

$P =$ ann

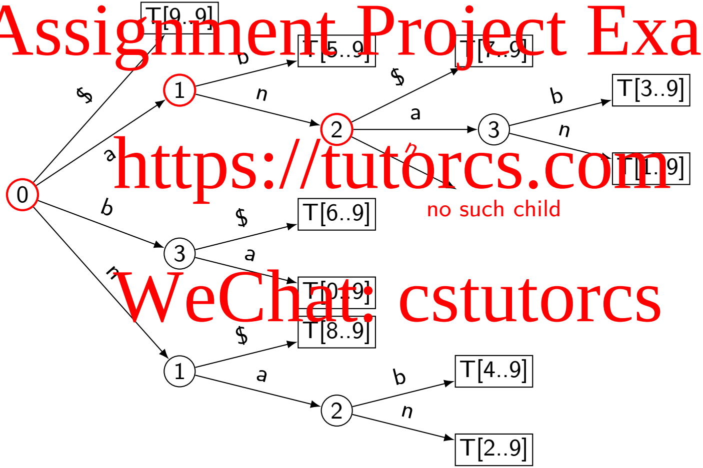


Pattern Matching in Suffix Tree: Example 1

	0	1	2	3	4	5	6	7	8	9
$T =$	b	a	n	a	n	a	b	a	n	\$

$P =$ ann

FAIL

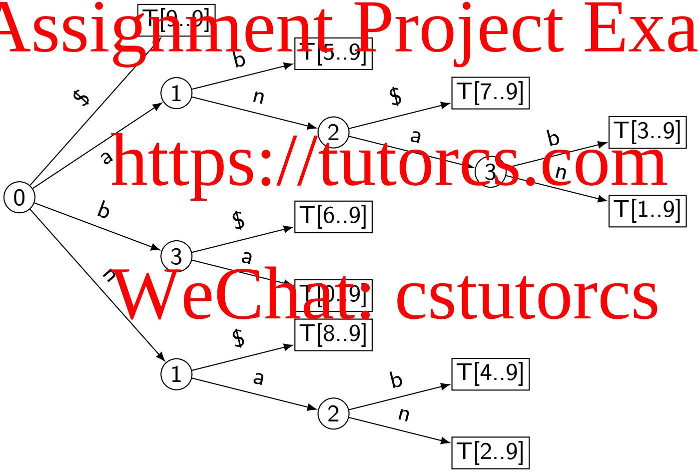


Pattern Matching in Suffix Tree: Example 2

$T =$

0	1	2	3	4	5	6	7	8	9
b	a	n	a	n	a	b	a	n	$

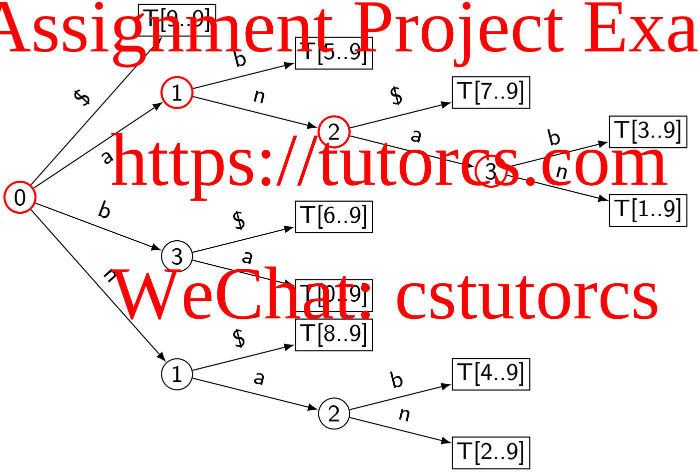
$P =$ ana



Pattern Matching in Suffix Tree: Example 2

	0	1	2	3	4	5	6	7	8	9
$T =$	b	a	n	a	n	a	b	a	n	\$

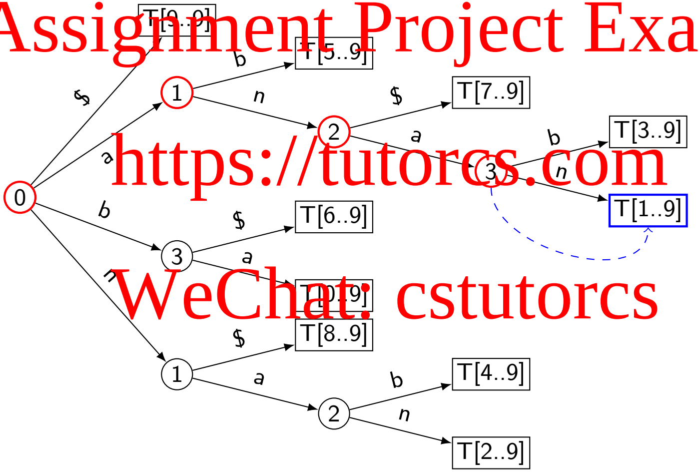
$P =$ ana



Pattern Matching in Suffix Tree: Example 2

	0	1	2	3	4	5	6	7	8	9
$T =$	b	a	n	a	n	a	b	a	n	$

$P =$ ana

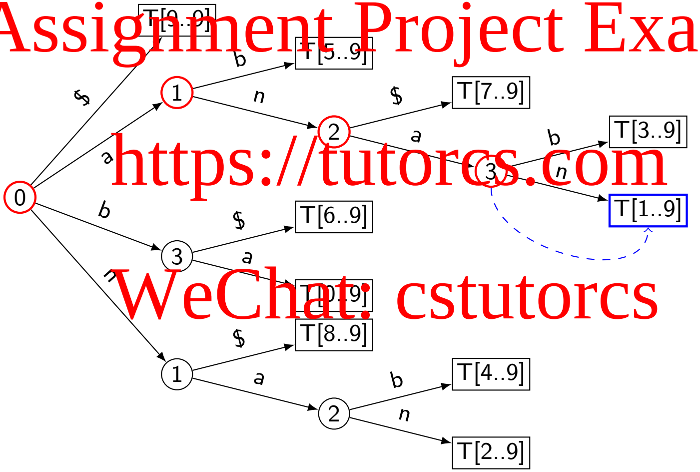


Pattern Matching in Suffix Tree: Example 2

	0	1	2	3	4	5	6	7	8	9
$T =$	b	a	n	a	n	a	b	a	n	$

$P =$ ana

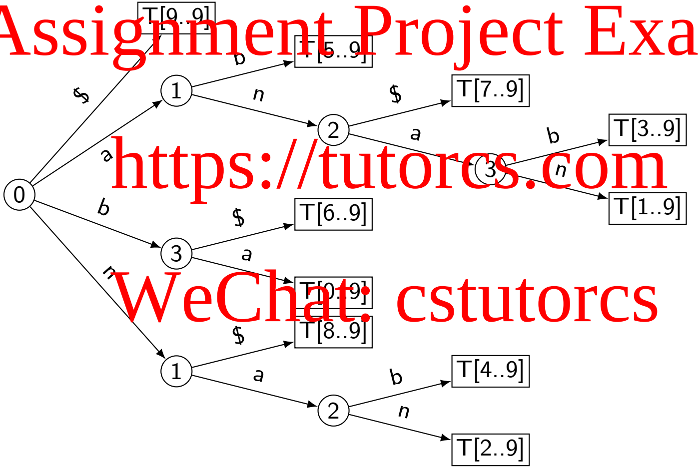
"found at guess 1"



Pattern Matching in Suffix Tree: Example 3

	0	1	2	3	4	5	6	7	8	9
$T =$	b	a	n	a	n	a	b	a	n	\$

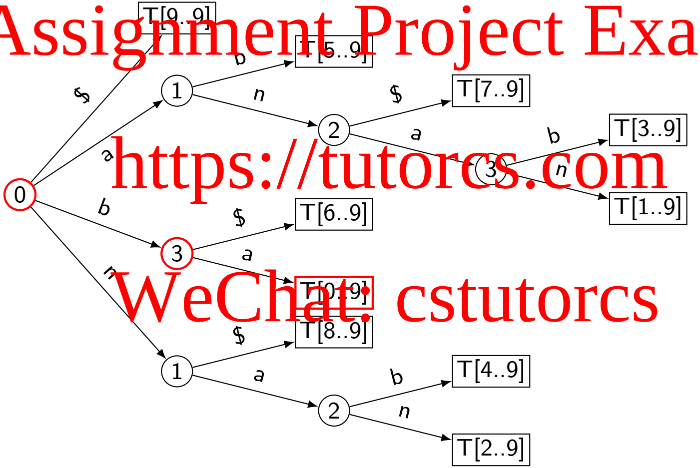
$P =$ `briar`



Pattern Matching in Suffix Tree: Example 3

	0	1	2	3	4	5	6	7	8	9
$T =$	b	a	n	a	n	a	b	a	n	$

$P =$ `briar`

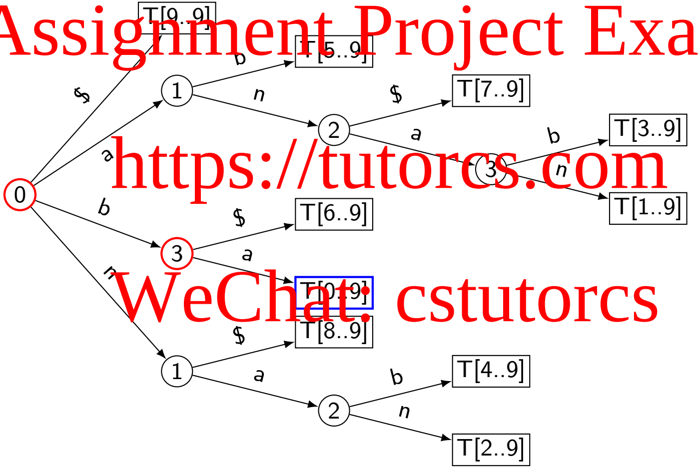


Pattern Matching in Suffix Tree: Example 3

$T =$

0	1	2	3	4	5	6	7	8	9
b	a	n	a	n	a	b	a	n	$

$P =$ `briar`

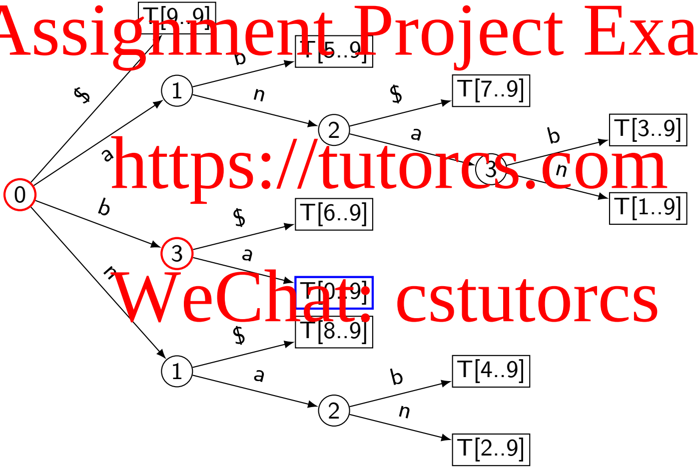


Pattern Matching in Suffix Tree: Example 3

	0	1	2	3	4	5	6	7	8	9
$T =$	b	a	n	a	n	a	b	a	n	\$

$P =$ `briar`

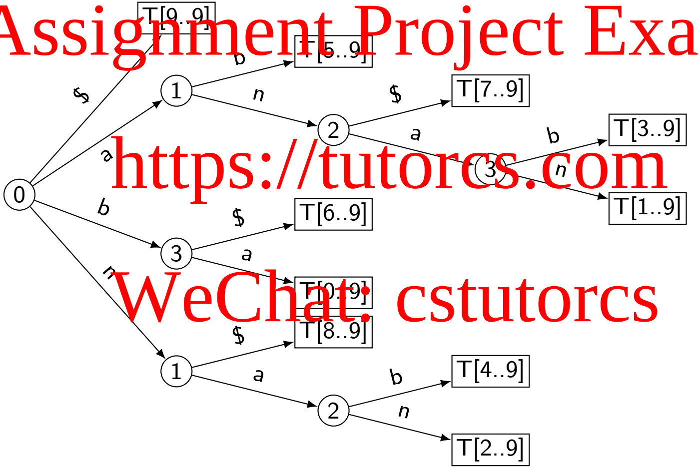
FAIL



Pattern Matching in Suffix Tree: Example 4

	0	1	2	3	4	5	6	7	8	9
$T =$	b	a	n	a	n	a	b	a	n	$

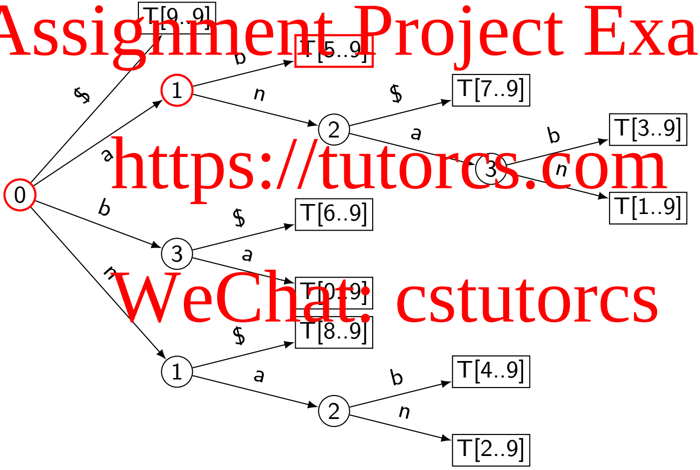
$P =$ abando



Pattern Matching in Suffix Tree: Example 4

	0	1	2	3	4	5	6	7	8	9
$T =$	b	a	n	a	n	a	b	a	n	\$

$P =$ abando

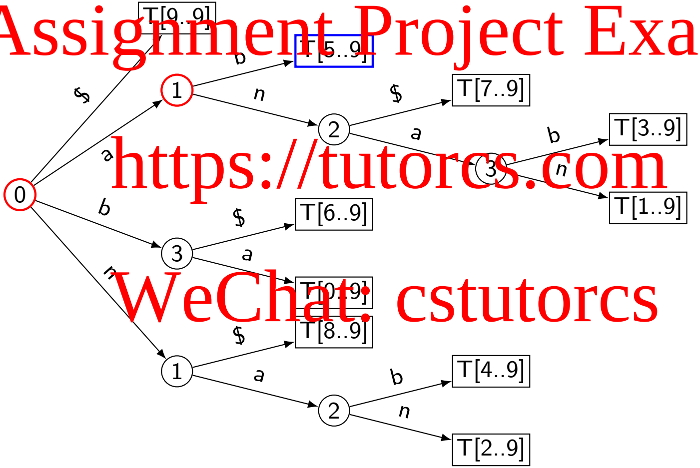


Pattern Matching in Suffix Tree: Example 4

	0	1	2	3	4	5	6	7	8	9
$T =$	b	a	n	a	n	a	b	a	n	\$

$P =$ `abando`

FAIL



Outline

1. String Matching
   - Introduction
   - Karp-Rabin Algorithm
   - String Matching with Finite Automata
   - Knuth-Morris-Pratt algorithm
   - Boyer-Moore Algorithm
   - Suffix Trees
   - Conclusion

String Matching Conclusion

	Brute-Force	Karp-Rabin	DFA	Knuth-Morris-Pratt	Boyer-Moore	Suffix Tree	Suffix Array[1]
Preproc.	—	$O(m)$	$O(m\|\Sigma\|)$	$O(m)$	$O(m+\|\Sigma\|)$	$O(n^2)$ $[O(n)]$	$O(n \log n)$ $[O(n)]$
Search time	$O(nm)$	$O(n+m)$ expected	$O(n)$	$O(n)$	$O(n)$ or better	$O(m)$	$O(m \log n)$
Extra space	—	$O(1)$	$O(m\|\Sigma\|)$	$O(m)$	$O(m+\|\Sigma\|)$	$O(n)$	$O(n)$

- Our algorithms stopped once they have found one occurrence.
- Most of them can be adapted to find *all* occurrences within the same worst-case run-time.

[1] studied only in the enriched section