COMP 251

Algorithms  Structures (Winter 2021)

Red-Black Trees

Announcements

- A1 has been relea
  - Please start early.
  - Try by yourself + Us ion board + OH.

Outline

- Introduction.
- Operations.

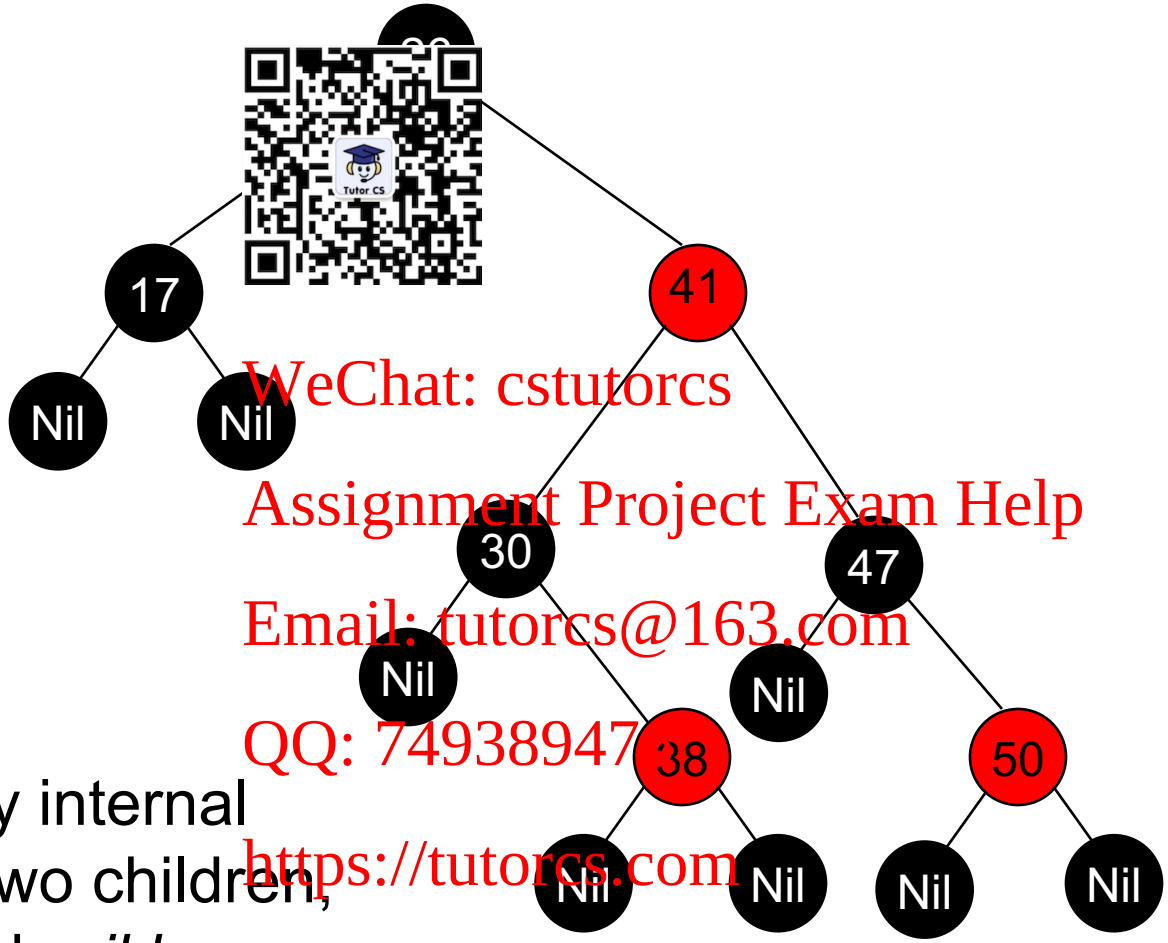
Introduction – Red-Black trees

- Red-black trees are a variation of binary search trees to ensure that the tree is 'approximately' **balanced**.
  - Height is $O(\lg n)$, where n is the number of nodes.
  - BST + 1 bit per node: the attribute color, which is either **red** or **black**.
  - All other attributes of BSTs are inherited (*key*, *left*, *right*, and *parent*).
  - no path (from the root to a leaf) is more than twice as long as any other path.
- Operations take $O(\lg n)$ time in the worst case.
- Invented by R. Bayer (1972).
- Modern definition by L.J. Guibas & R. Sedgewick (1978).

Red-Black trees - Properties

1. Every node is either red or black.
2. The root is black.
3. All leaves (*nil*) are black.
4. If a node is red, then its children are black (i.e. no 2 consecutive red nodes).
5. For each node, all paths from the node to descendant leaves contain the same number of black nodes (i.e. same black height).

Red-Black trees - example

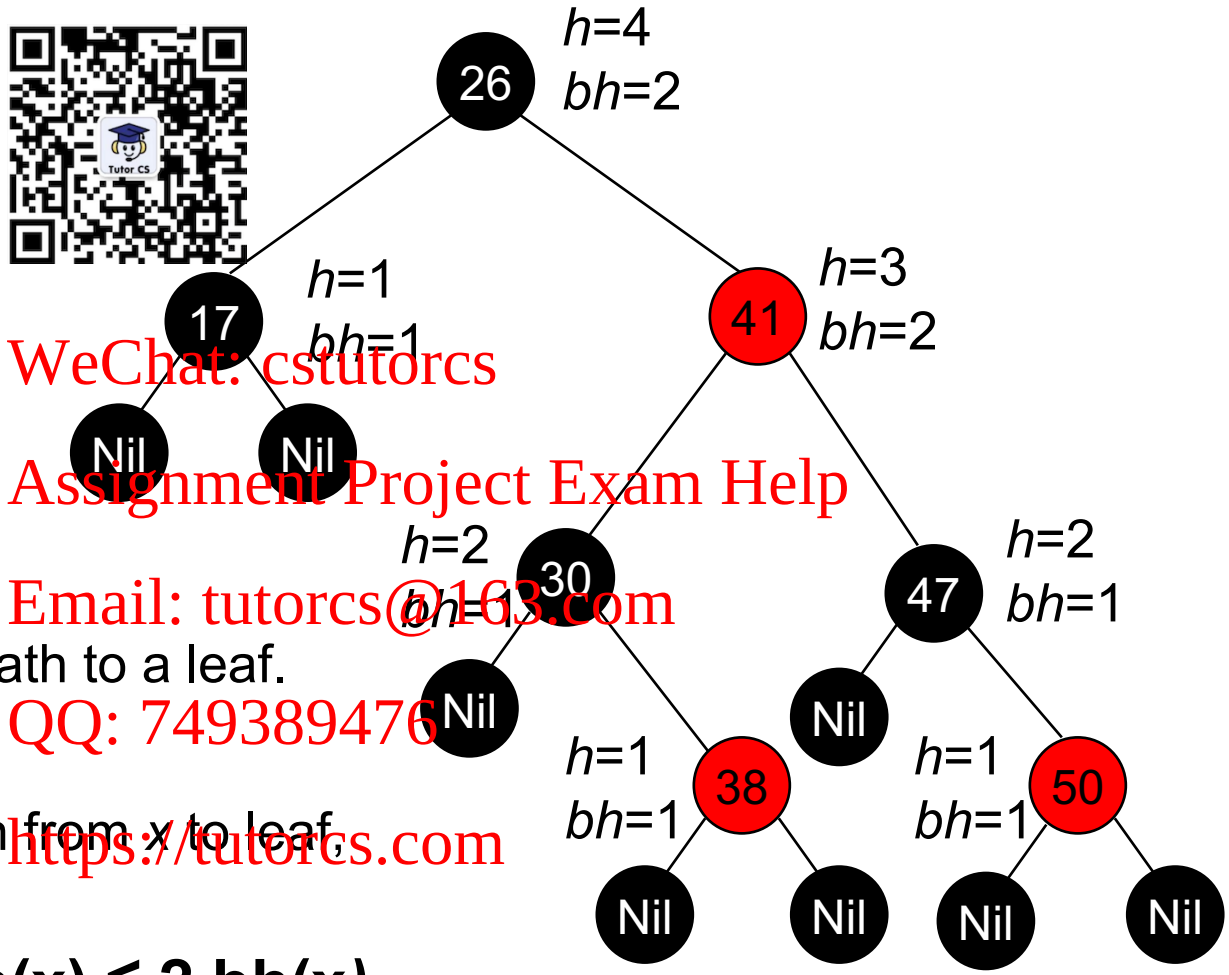


Note: every internal node has two children, even though *nil leaves are not usually shown*.

Red-Black trees - height

- Height of a node:
  - $h(x)$ = number of edges in a longest path to a leaf.
- Black-height of a node, $bh(x)$:
  - $bh(x)$ = number of black nodes (including $nil[T]$) on the path from x to leaf, not counting x.
- Black-height of a red-black tree is the black-height of its root.
  - By RB Property 5, black-height is well defined.

Red-Black trees - height



- Height h(x):

  #edges in a longest path to a leaf.

- Black-height bh(x):

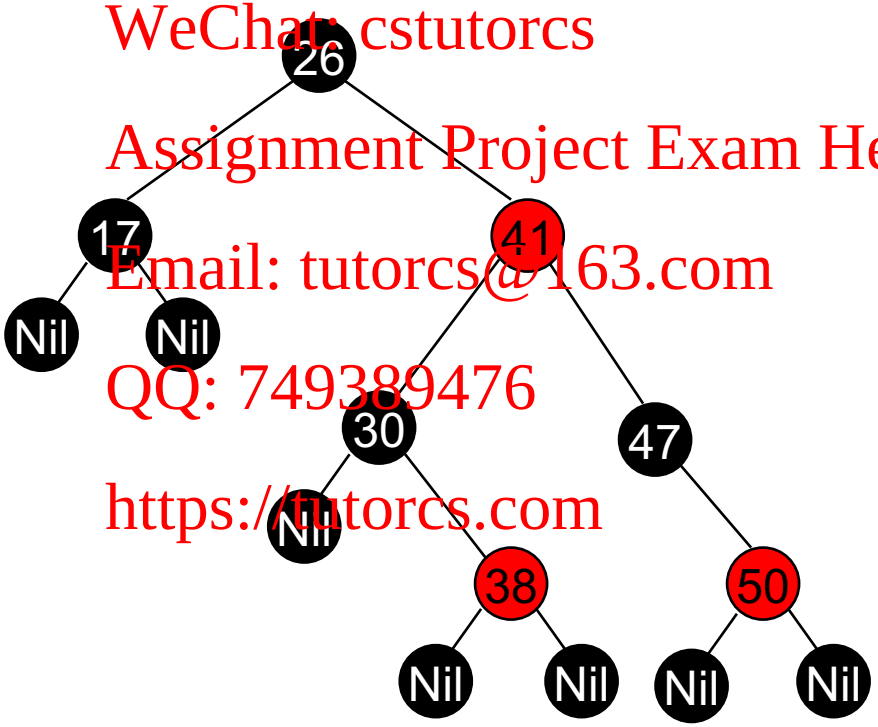
  # black nodes on path from *x* to leaf, *not counting x*.

- **Property: bh(x) ≤ h(x) ≤ 2 bh(x)**

Red-Black trees - height

Lemma 1: Any node x with height h(x) has a black-height bh(x) ≥ h(x)/2.

Proof: By RB prop h/2 nodes on the path from the node to a leaf are red. Hence ≥ h/2 are black. ■



Red-Black trees - height

Lemma 2: The subtree rooted at any node x contains $\geq 2^{bh(x)} - 1$ internal nodes.

Proof: By induction on height of x.

- **Base Case:** Height $h(x) = 0 \Rightarrow x$ is a leaf (Nil) $\Rightarrow$ bh(x) = 0. Subtree has $\geq 2^0 - 1 = 0$ nodes.
- **Induction Step:**
  - Each child of x has height $h(x) - 1$ and black-height either $bh(x)$ (child is red) or $bh(x) - 1$ (child is black).
  - By ind. hyp., each child has $\geq 2^{\text{bh}(x)-1} - 1$ internal nodes.
  - Subtree rooted at x has $\geq 2 \cdot (2^{\text{bh}(x)-1} - 1) + 1$ $= 2^{\text{bh}(x)} - 1$ internal nodes. ■

Red-Black trees - height

Lemma 1: Any node with height h(x) has a black-height bh(x) ≥ h(x)/2.

Lemma 2: The subtree rooted at any node x has $\geq 2^{bh(x)}-1$ internal nodes.

Lemma 3: A red-black tree with n internal nodes has height at most $2 \lg(n+1)$.

Proof:

- By lemma 2, $n \geq 2^{bh}-1$,
- By lemma 1, $bh \geq h/2$, thus $n \geq 2^{h/2}-1$.
- $\Rightarrow\ h \leq 2 \lg(n+1)$.

Outline

- Introduction.
- Operations.

RB trees - Insertion

- Insertion must preserve all red-black properties.
- Should an inserted node be colored Red? Black?
- Basic steps:
  - Use BST Tree-Insert to insert a node *x* into *T.*
    - Procedure **RB-Insert(x)**.
  - Color the node *x* red.
  - Fix the new tree by (1) re-coloring nodes, and (2) performing rotations to preserve RB tree property.
    - Procedure **RB-Insert-Fixup**.

RB trees - Insertion

RB-Insert(*T*, *z*)

```
1.   y ← nil[T]
2.   x ← root[T]
3.   while x ≠ nil[T]
4.       do y ← x
5.          if key[z] < key[x]
6.             then x ← left[x]
7.             else x ← right[x]
8.   p[z] ← y
9.   if y = nil[T]
10.      then root[T] ← z
11.  else if key[z] < key[y]
12.      then  left[y] ← z
13.  else right[y] ← z
```

RB-Insert(*T*, *z*) Contd.

```
14.  left[z] ← nil[T]
15.  right[z] ← nil[T]
16.  color[z] ← RED
17.  RB-Insert-Fixup (T, z)
```


Regular BST insert + color assignment + fixup.

RB trees – Insertion - Fixup

RB-Insert-Fixup

```
1.   while color[
2.       do if p[z] = left[p[p[z]]]
3.             then y ← right[p[p[z]]]
4.                  if color[y] = RED
5.                     then color[p[z]] ← BLACK  // Case 1
6.                          color[y] ← BLACK     // Case 1
7.                          color[p[p[z]]] ← RED // Case 1
8.                          z ← p[p[z]]          // Case 1
```

D

RB trees – Insertion - Fixup

RB-Insert-Fixup(ntd.)

9. **else** *ht*[*p*[*z*]] // color[*y*] ≠ RED

10. **then** *z* ← *p*[*z*] // Case 2

11. LEFT-ROTATE(*T*, *z*) // Case 2

12. *color*[*p*[*z*]] ← BLACK // Case 3

13. *color*[*p*[*p*[*z*]]] ← RED // Case 3

14. RIGHT-ROTATE(*T*, *p*[*p*[*z*]]) // Case 3

15. **else** (if *p*[*z*] = *right*[*p*[*p*[*z*]]])(same as **10-14**

16. with "right" and "left" exchanged)

17. *color*[*root*[*T*]] ← BLACK

RB trees – Insertion - Fixup

1. To determine what violations of the red-black properties are introduced in RB-INSERT when node *z* is inserted and colored red.
2. To understand each of the three cases within the *while* loop's body
3. To understand the overall goal of the while loop in lines 1-15.

RB trees – Insertion - Fixup

1. To determine what violations of the red-black properties are introduced in RB-Insert when node *z* is inserted and colored red.
   1. Every node is either red or black.
   2. The root is black.
      - Violated if z is the root.
   3. All leaves (*nil*) are black.
   4. If a node is red, then its children are black (i.e. no 2 consecutive red nodes).
      - Violated if z's parent is red
   5. For each node, all paths from the node to descendant leaves contain the same number of black nodes (i.e. same black height).
      - node z replaces the (black) sentinel, and node z is red with sentinel children.





RB trees – Insertion - Fixup

1. To determine what violations of the red-black properties are introduced in RB-INSERT when node *z* is inserted and colored red.
2. To understand each of the three cases within the *while* loop's body
3. To understand the overall goal of the while loop in lines 1-15.

RB trees – Insertion - Fixup

2. To understand each of the three cases within the *while* loop's body.

- We actually need to cor cases in the while loop, but three of them are symmetric to the oth
  - Line2: Three cases if th z is a left (right) child of the grandparent of z.
- The grand parent of z exists.
  - If the parent of z is the root (i.e., the grand parent will not exist), then the parent of z is black, but notice that we enter a loop iteration only if the parent of z is red, then we know that the parent of z cannot be the root. Hence, the grand parent of z exists.

RB-Insert-Fixup (*T*, *z*)

1. **while** *color*[*p*[*z*]] = RED
2. **do if** *p*[*z*] = *left*[*p*[*p*[*z*]]]
3. ……

RB trees – Insertion - Fixup

2. To understand each of the three cases within the *while* loop's body.

- In all three cases, the g[illegible]nt of *z* is black.
  - Since the parent of z is [illegible], and property 4 is violated only between z and its parent (p[z]).
- We distinguish case 1 from cases 2 and 3 by the color of z's parent's sibling, or "uncle".
  - Line3: Makes *y* point to *z's* uncle.
  - Line4: Test *y's* color. If *y* is red, then we execute case 1. Otherwise, control passes to cases 2 and 3.

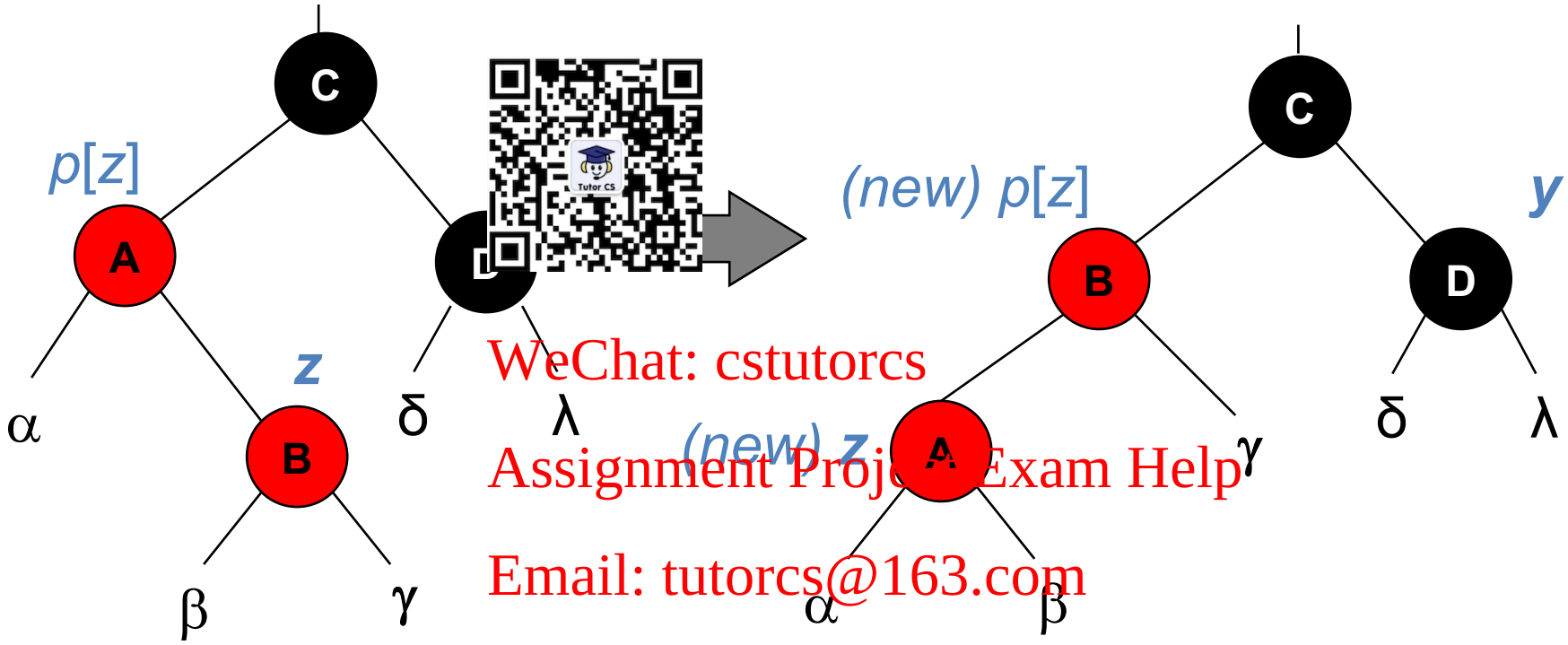
RB-Insert-Fixup (*T*, *z*)

```
1.    while color[p[z]] = RED
2.        do if p[z] = left[p[p[z]]]
3.              then y ← right[p[p[z]]]
4.                   if color[y] = RED
```

Insertion – Case1 – Uncle y is red

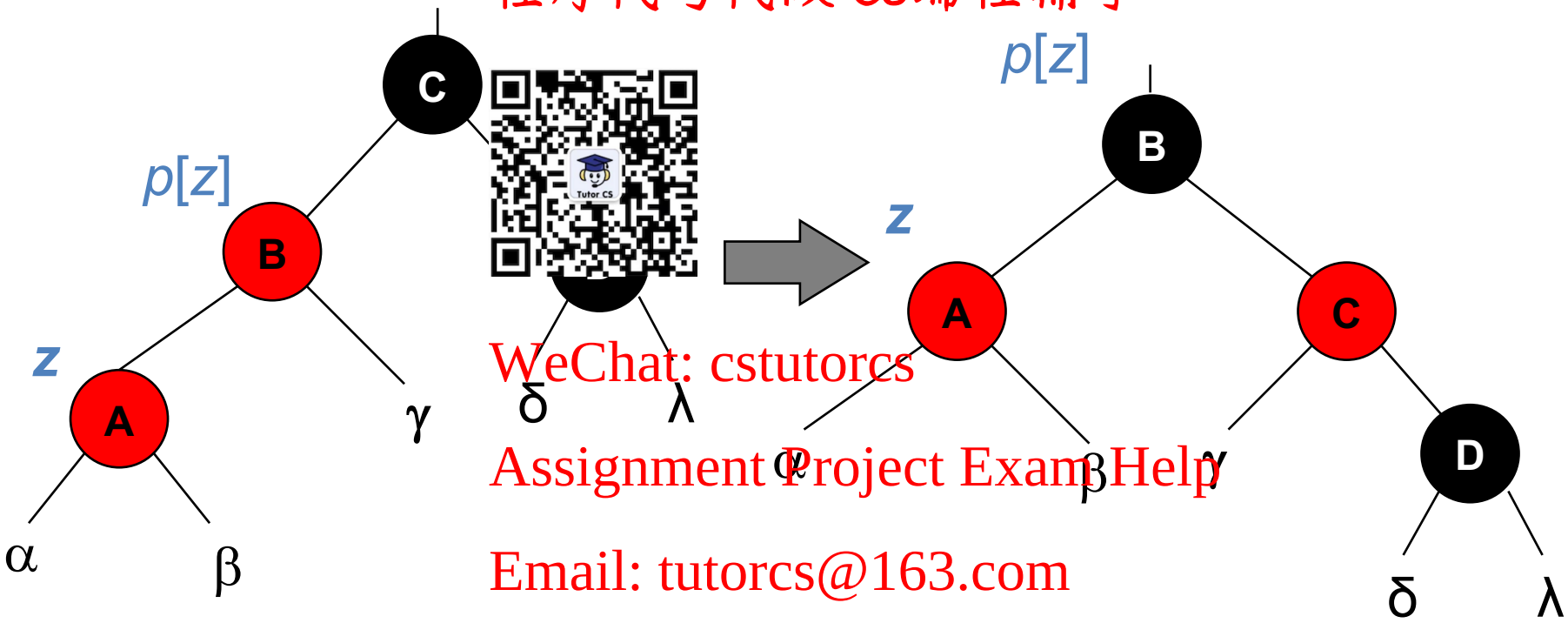
- $p[p[z]]$ (z's grandparent) must be black, since z and $p[z]$ are both red and there are no other violations of property 4.
- Make $p[z]$ and y black ⇒ now z and $p[z]$ are not both red. But property 5 might now be violated.
- Make $p[p[z]]$ red ⇒ restores property 5.
- The next iteration has $p[p[z]]$ as the new z (i.e., z moves up 2 levels).

Case2 – y is black, z is a right child



- Left rotate around $p[z]$, $p[z]$ and z switch roles $\Rightarrow$ now z is a left child, and both z and $p[z]$ are red.
- Takes us immediately to case 3.

Case3 – y is black, z is a left child



- Make *p*[*z*] black and *p*[*p*[*z*]] red.
- Then right rotate on *p*[*p*[*z*]] (in order to maintain property 4).
- No longer have 2 reds in a row.
- *p*[*z*] is now black ⇒ no more iterations.

RB trees – Insertion - Fixup

1. To determine what violations of the red-black properties are introduced in RB-INSERT when node *z* is inserted and colored red.
2. To understand each of the three cases within the *while* loop's body
3. To understand the overall goal of the while loop in lines 1-15.

RB trees – Insertion - Fixup

3. To understand the overall goal of the while loop in lines 1-15.
   - The loop maintains the [illegible]variant at the start of each iteration
   1. If the tree violates any [illegible]black properties, then:
      - The violation is either p[illegible]roperty 4.
      - It violates at most one of them.

Recall that we need to show that a loop invariant is true prior to the first iteration of the loop, that each iteration maintains the loop invariant, and that the loop invariant gives us a useful property at loop termination.

RB trees – Insertion - Fixup

3. To understand the overall goal of the while loop in lines 1-15.

- **Initialization:** Prior to [illegible]ation of the loop, we started with a red-black tree with no violations, and [illegible] a red node z.
  1. If the tree violates any [illegible]black properties, then the violation is at most one of property 2 or property 4:
     - If property 2 is violated: z must be the red root. Both children of z are sentinels, which are black. Then, the tree does not also violate property 4.
     - If property 4 is violated: z and its parent must be red. Given that both children of z are sentinels, which are black and that the tree had no other violations prior to z being added, the parent of z can not be the root. Then, the tree does not also violate property 2.

RB trees – Insertion - Fixup

3. To understand the overall goal of the while loop in lines 1-15.

- **Maintenance:** Case 1 [illegible] is red.

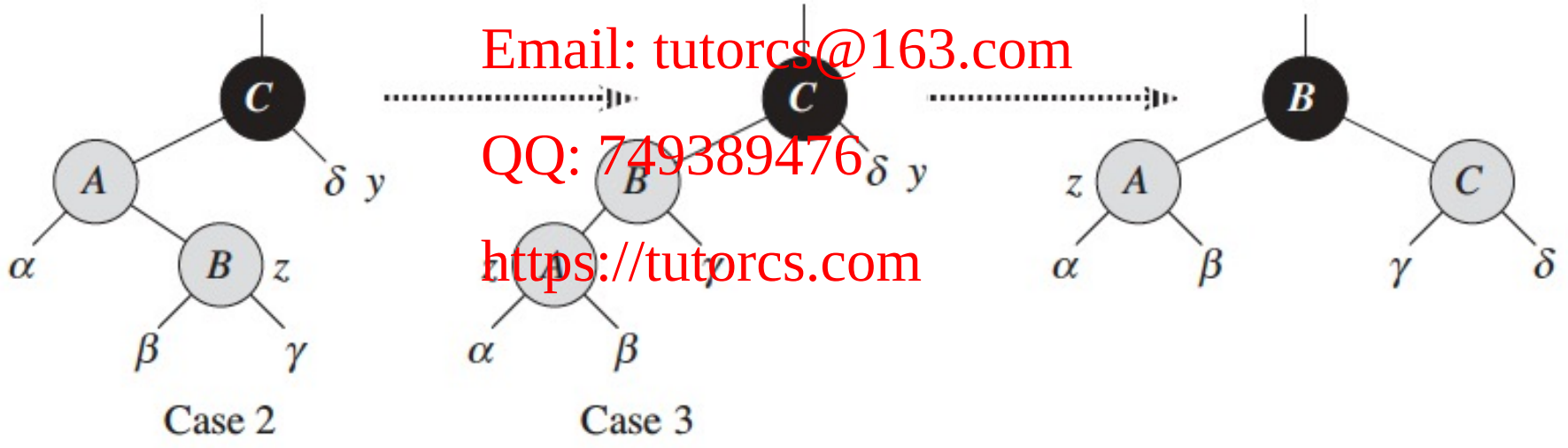
1. **If the tree violates a[illegible]d-black properties, then the violation is at most one of propert[illegible]erty 4:**
   - Case 1 maintains property 5, and it does not introduce a violation of properties 1 or 3.
   - If the new node z (i.e., the grand parent of the current z) is the root at the start of the next iteration, then case 1 corrected the lone violation of property 4. since the new node z is red and it is the root, property 2 becomes the only one that is violated.
   - If the new node z (i.e., the grand parent of the current z) is not the root at the start of the next iteration, then case 1 has not created a violation of property 2. The only possible violation in the next iteration will be to property 4.

RB trees – Insertion - Fixup

3. To understand the overall goal of the while loop in lines 1-15.

- **Maintenance:** Case 2 [illegible] uncle y is black.

1. **If the tree violates a[illegible]d-black properties, then the violation is at most one of propert[illegible]erty 4:**
   - Case 2 and 3 maintain properties 1, 3 and 5.
   - z is not the root in cases 2 and 3, then there is no violation of property 2.
   - Cases 2 and 3 correct the lone violation of property 4, and they do not introduce another violation.



RB trees – Insertion - Fixup

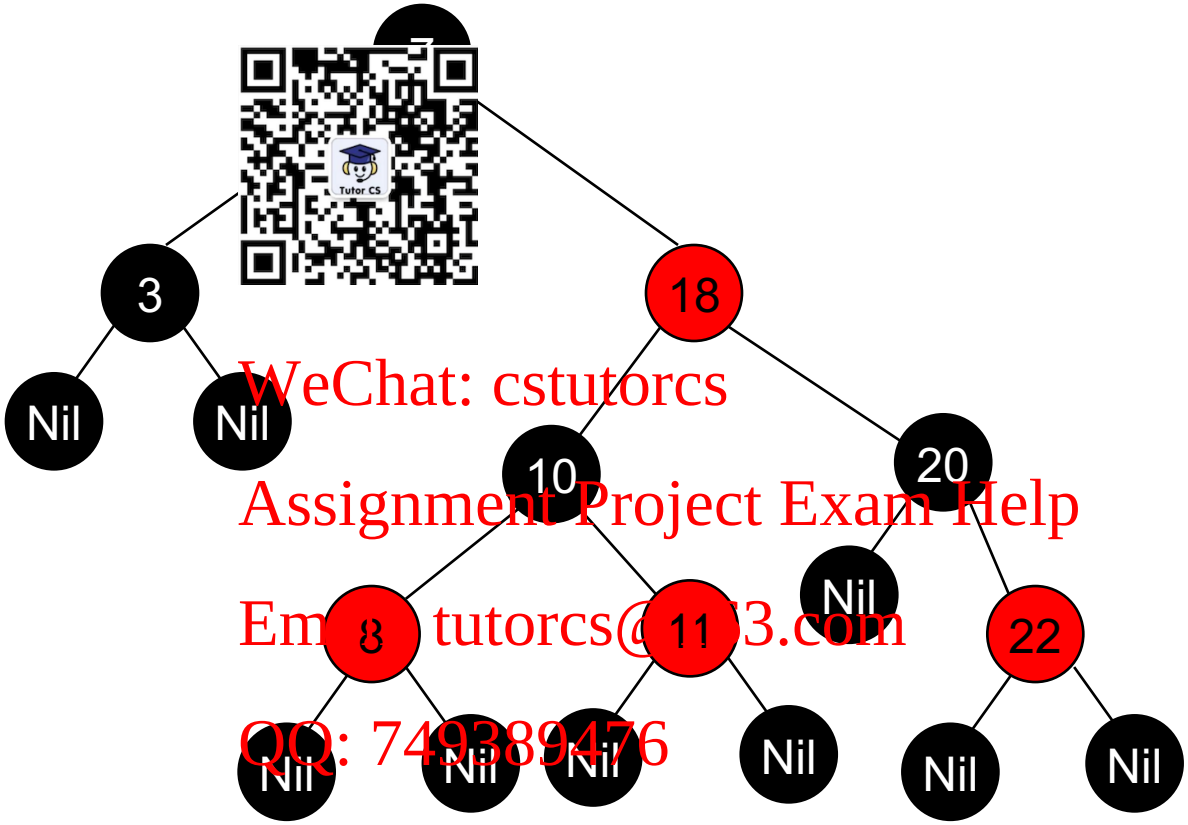
3. To understand the overall goal of the while loop in lines 1-15.

- **Termination:** When th nates, it does so because p[z] is black. Thus, the tree does not violate p loop termination. By the loop invariant, the only property that might fa property 2. Line 17 restores this property.

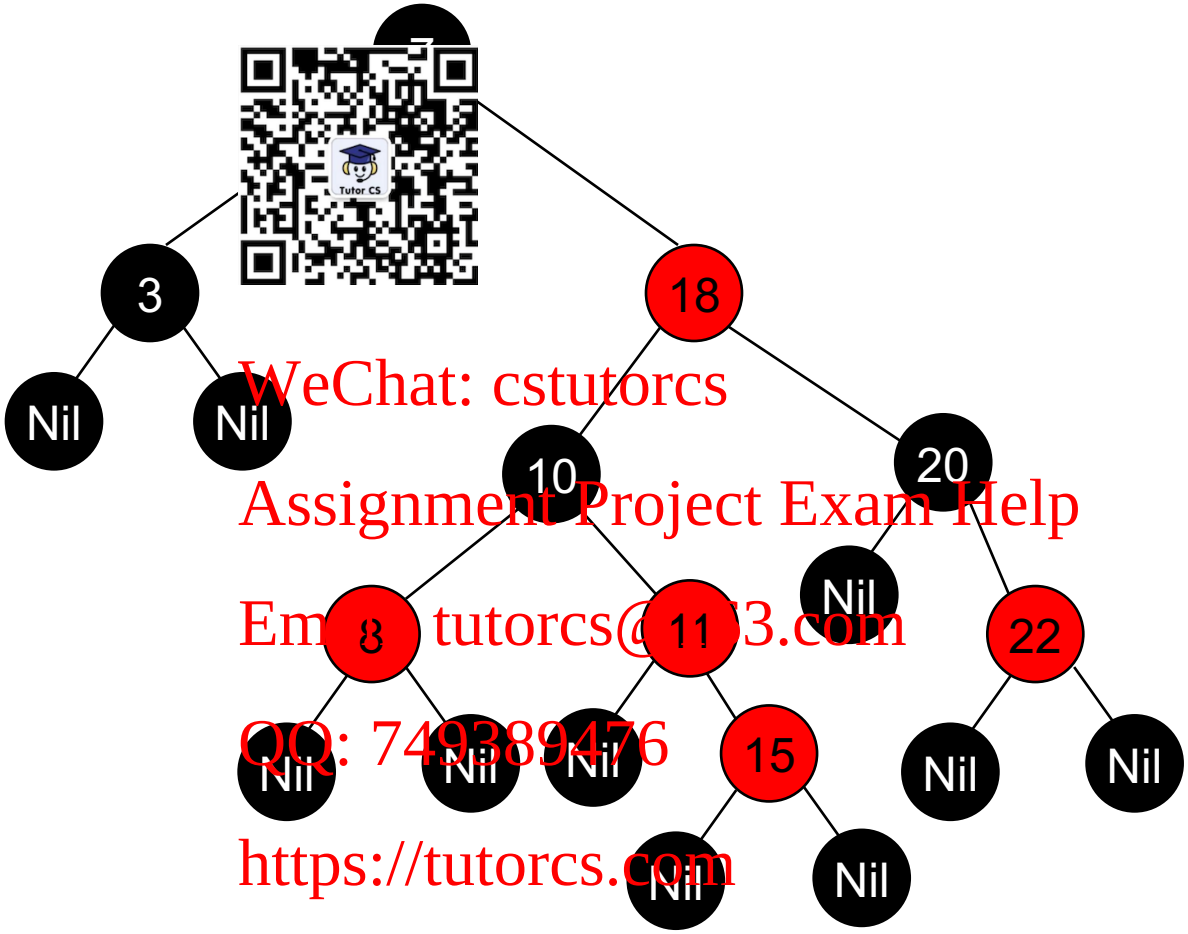
RB-Insert-Fixup(*T*, *z*) (cont'd.)

```
1.    while color[p[z]] = RED
2.        do if p[z] = left[p[p[z]]]
3.    ……
15.       else (if p[z] = right[p[p[z]]])(same as 10-14
16.                  with "right" and "left" exchanged)
17. color[root[T ]] ← BLACK
```

Insert RB tree - Example

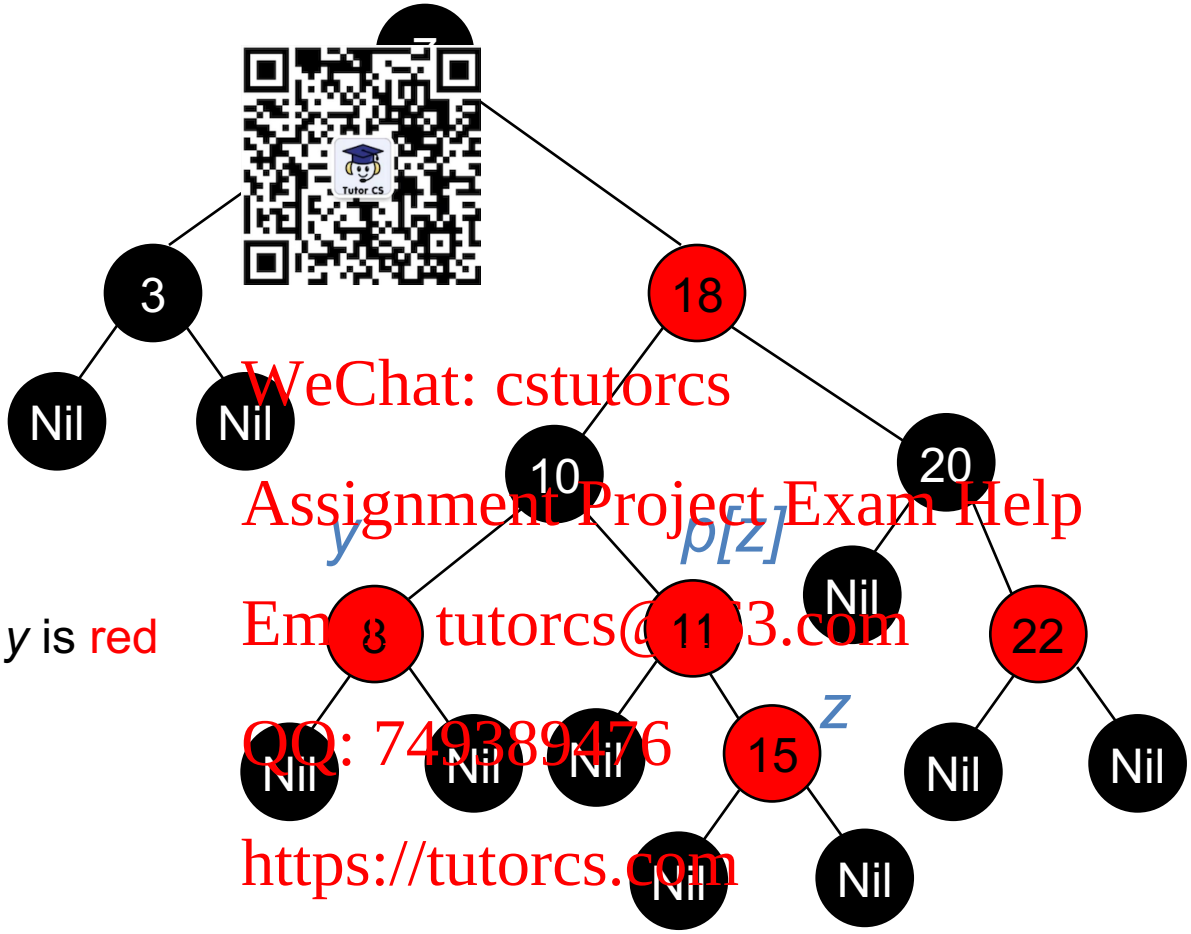


Insert RB tree - Example



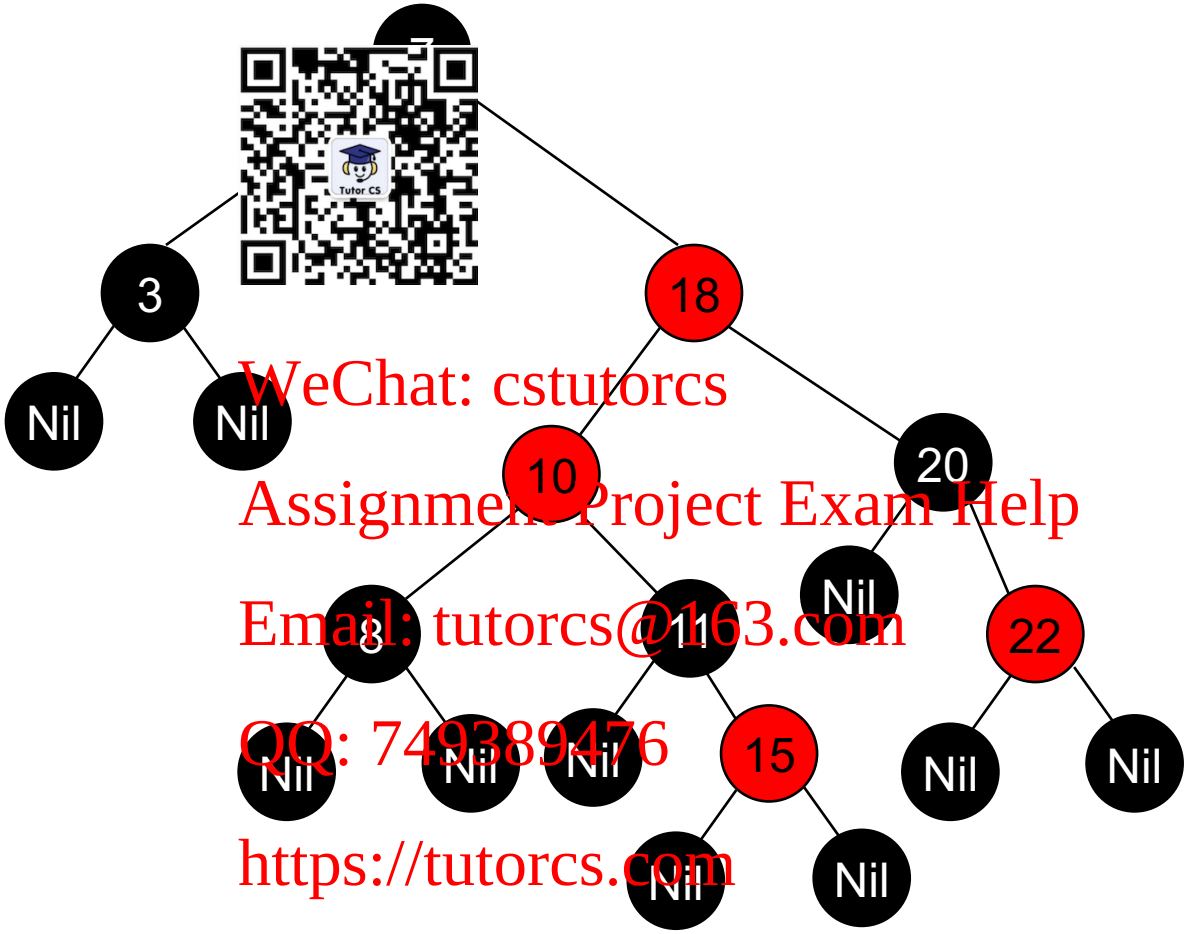
Insert(T,15)

Insert RB tree - Example



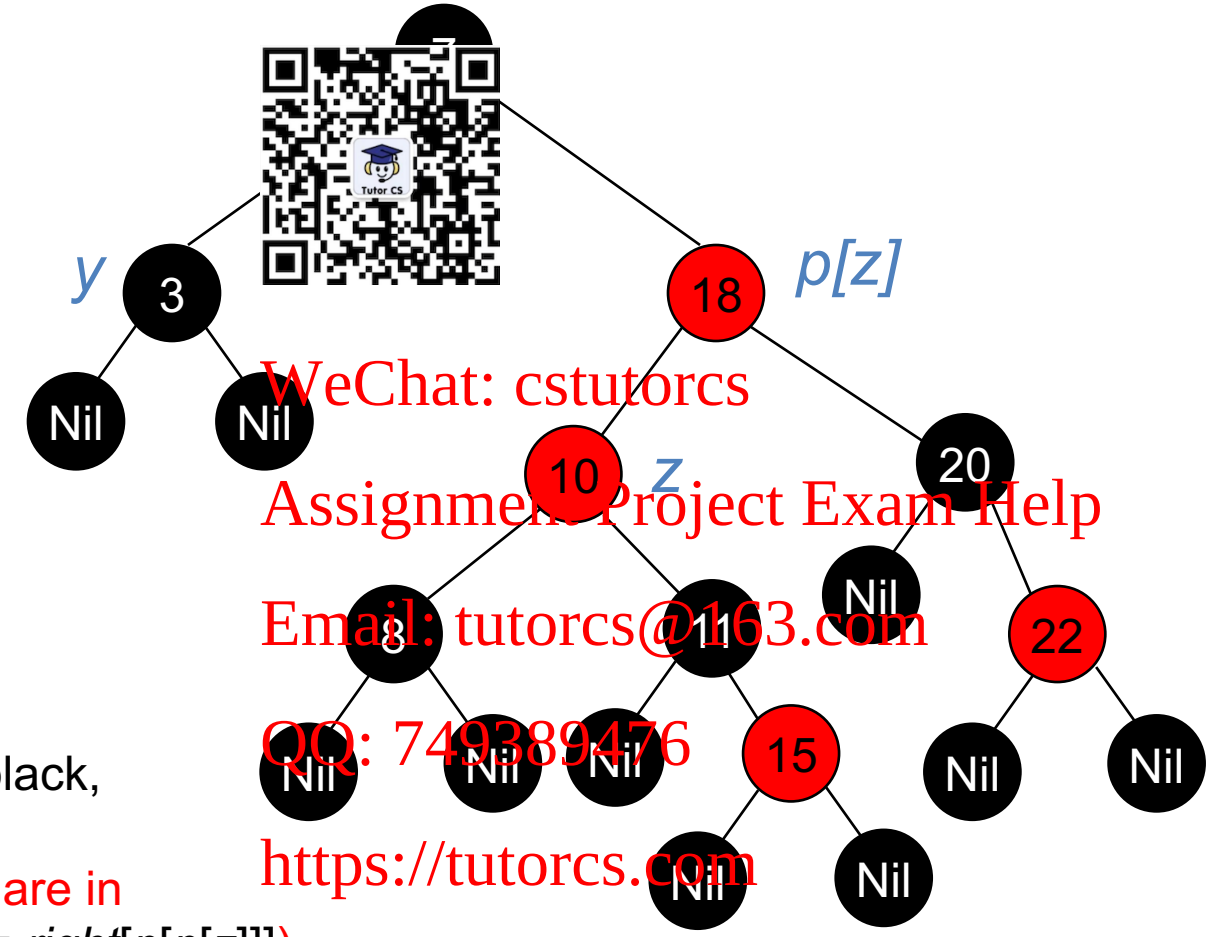
Case 1 – uncle *y* is red

Insert RB tree - Example



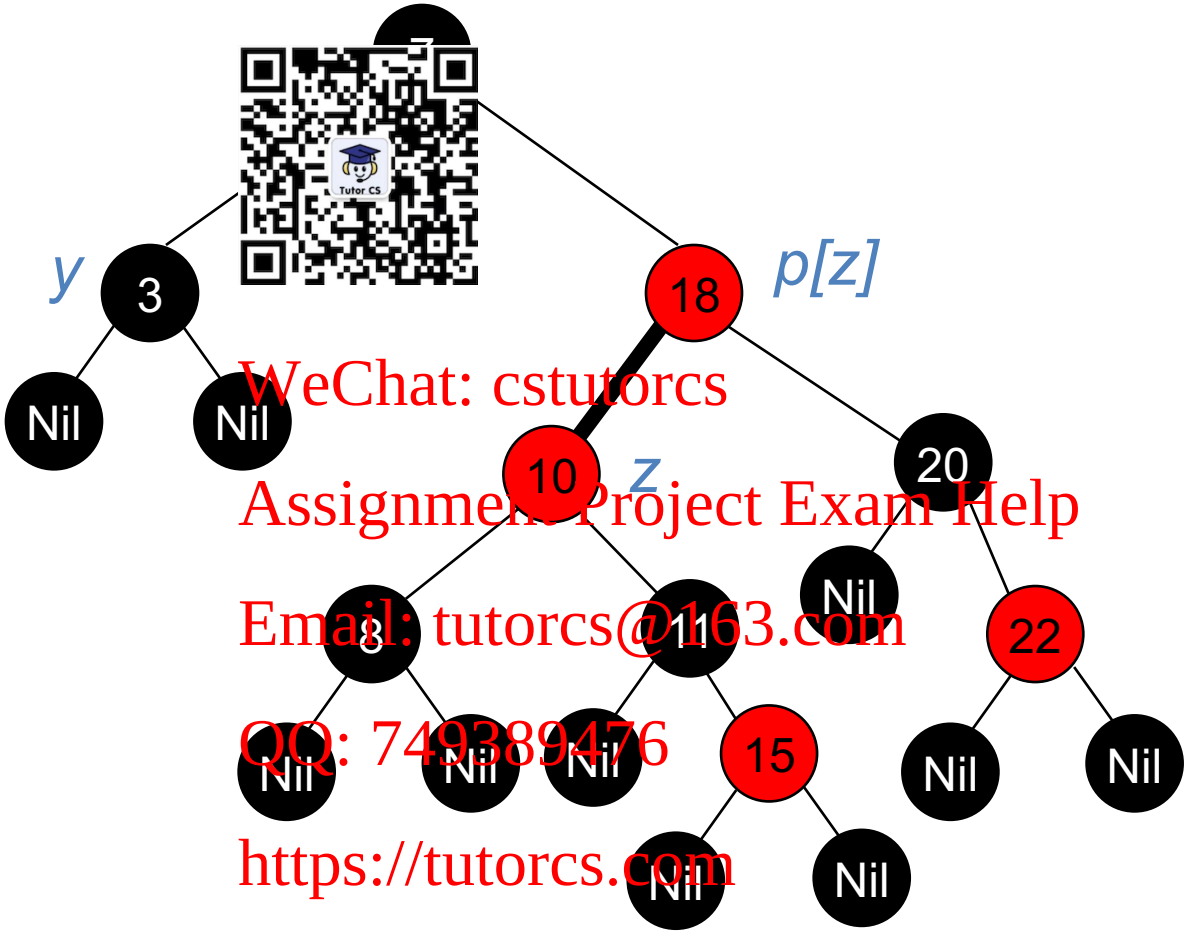
Recolor 10, 8 &11

Insert RB tree - Example



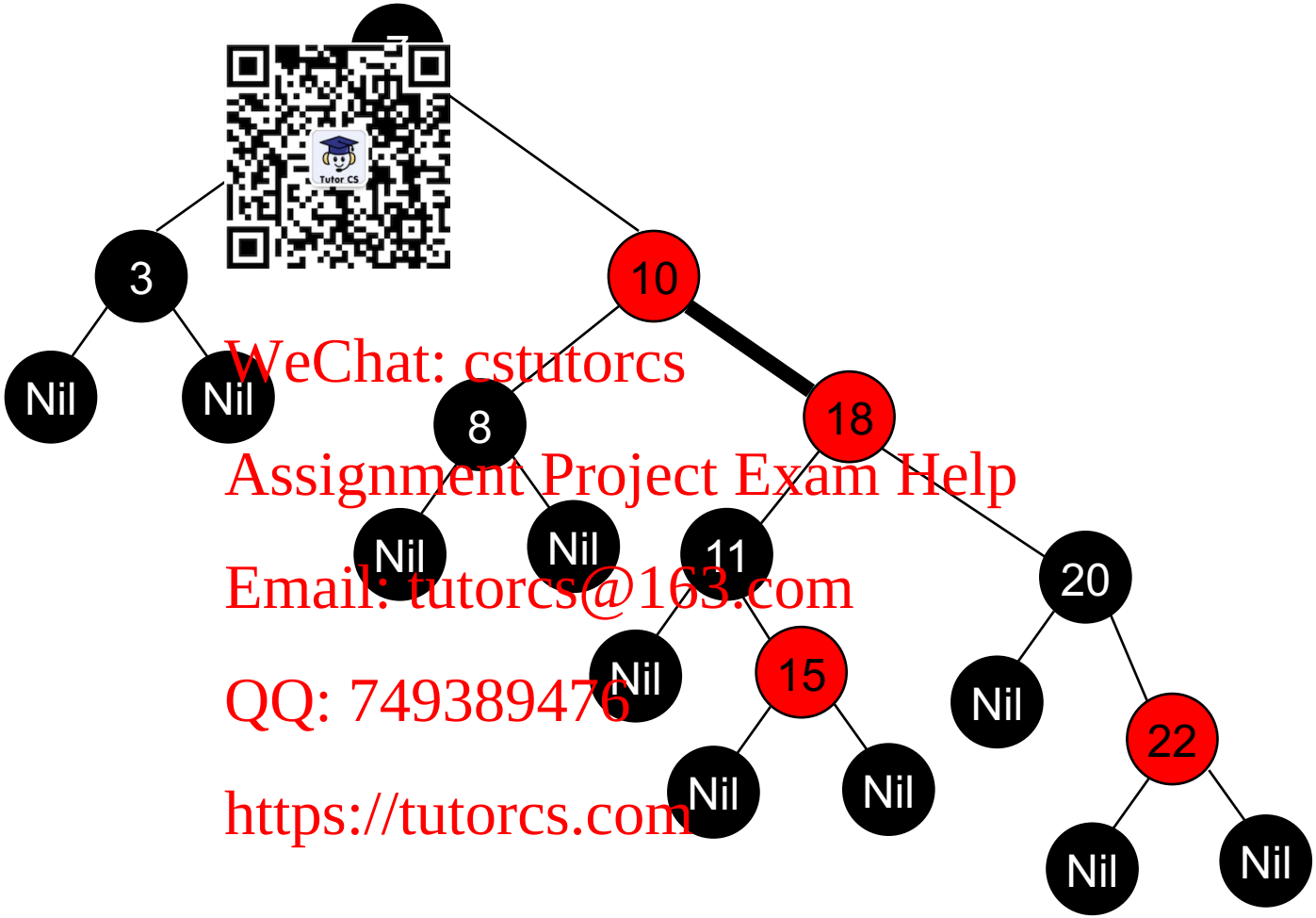
Case 2 – *y* is black,
z is a left child.
Notice that we are in line 15 (if *p*[*z*] = *right*[*p*[*p*[*z*]]]).

Insert RB tree - Example



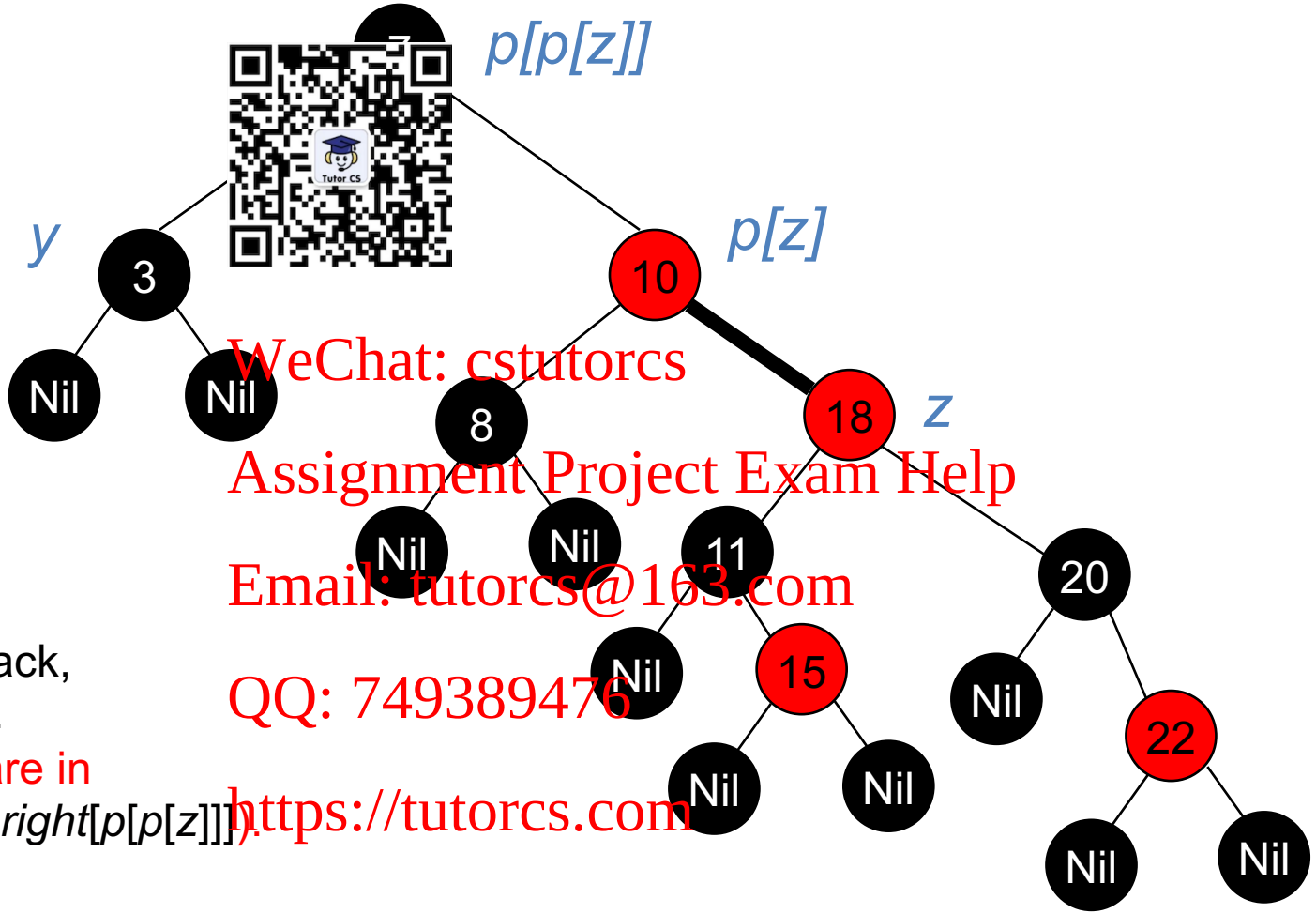
Right rotate at 18 p[z]

Insert RB tree - Example



Parent & child with conflict are now aligned with the root.

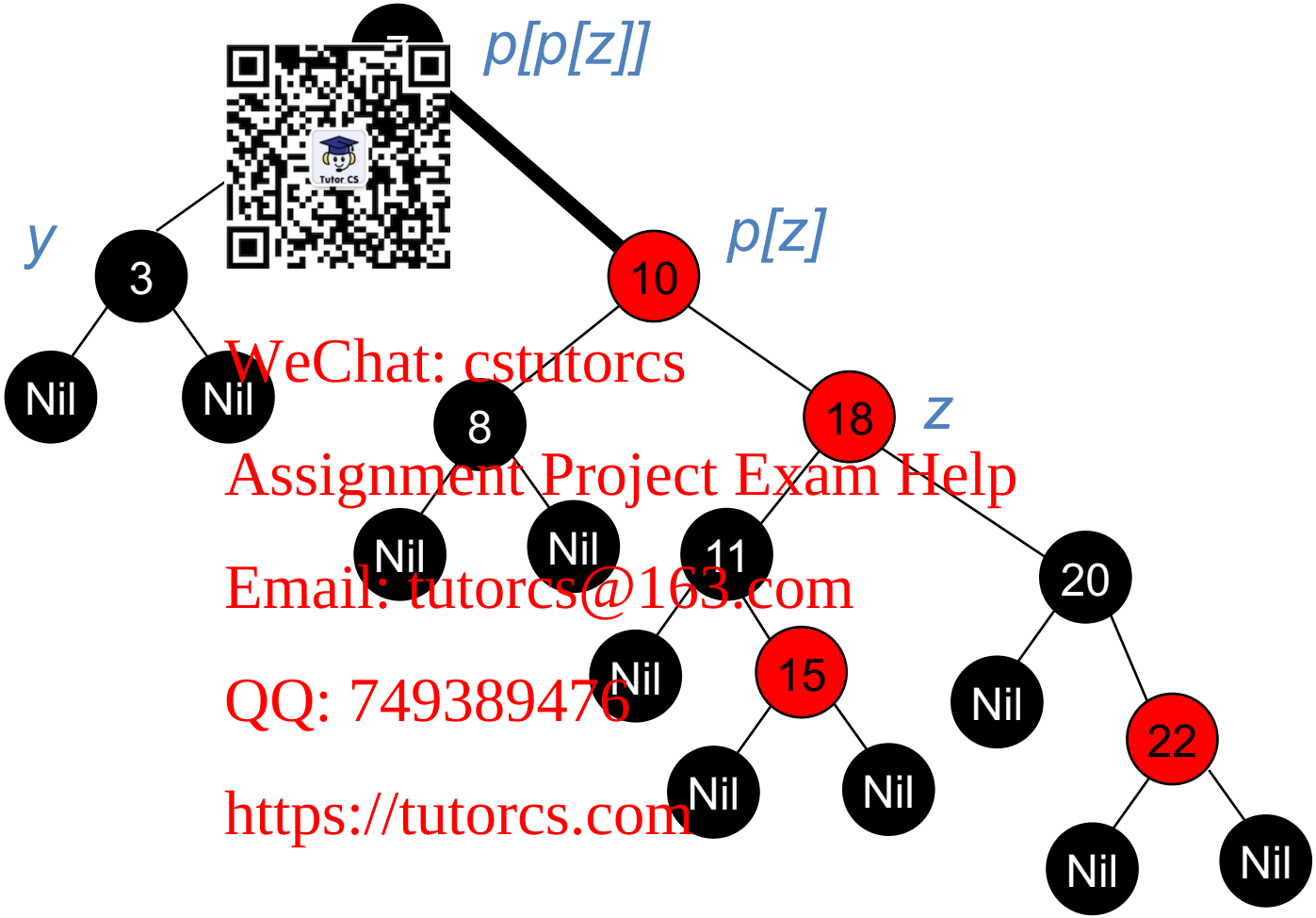
Insert RB tree - Example



Case 3 – *y* is black,
z is a right child.
Notice that we are in line 15 (if *p*[*z*] = *right*[*p*[*p*[*z*]]]).

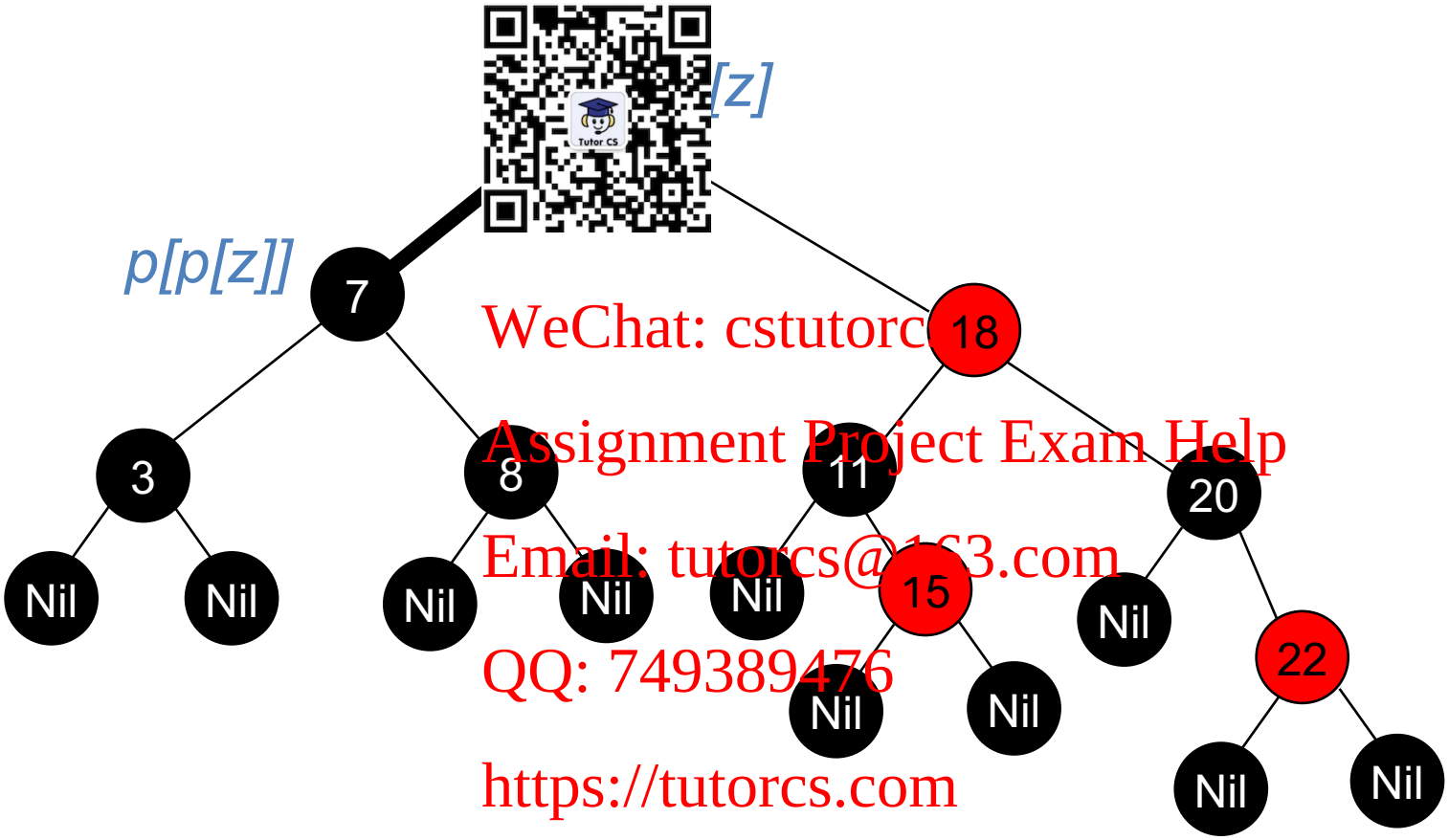
Parent & child with conflict are now aligned with the root.

Insert RB tree - Example

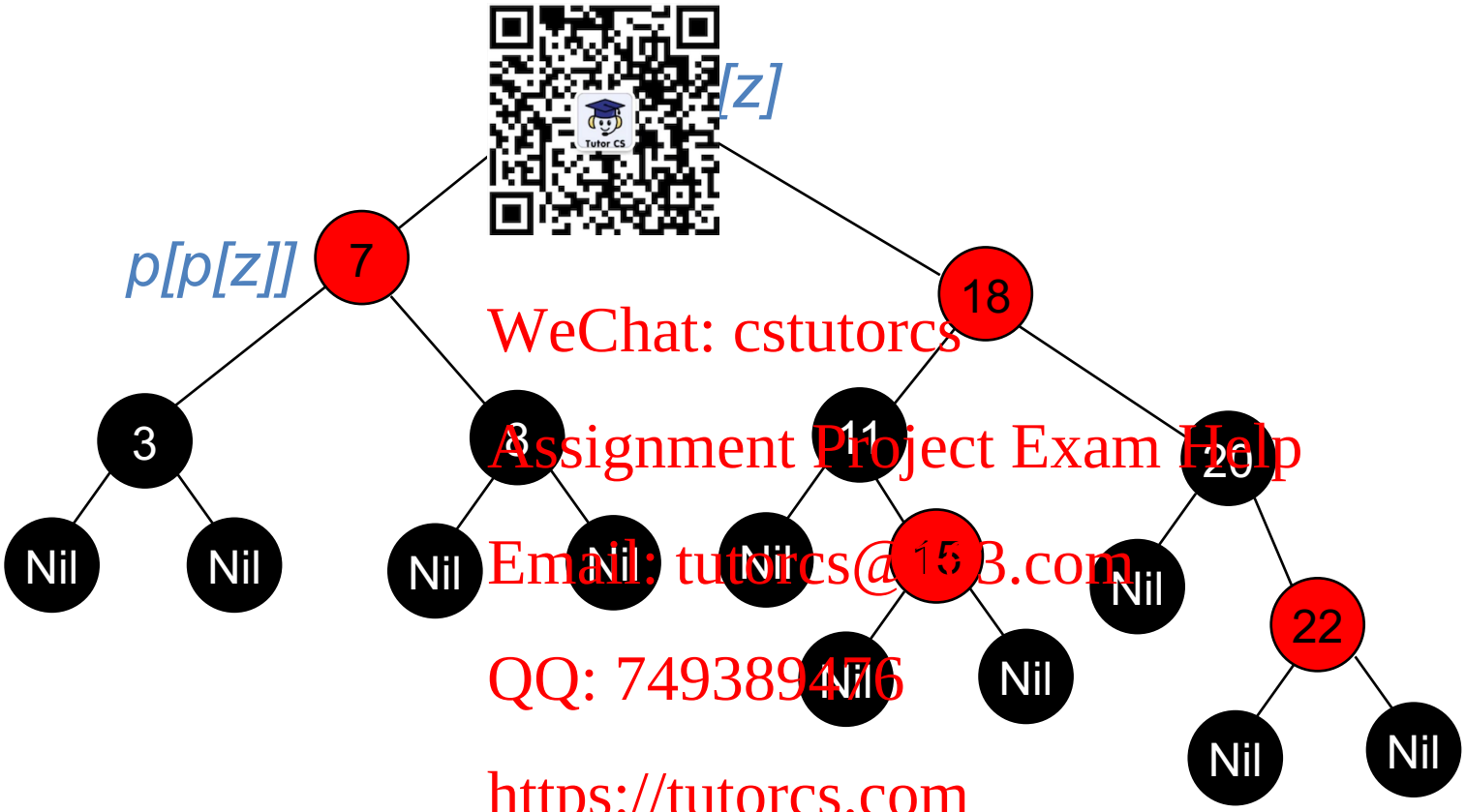


Left rotate at 7 p[p[z]]

Insert RB tree - Example



Insert RB tree - Example



Recolor 10 p[z] & 7 p[p[z]] (root must be black!)

Insert RB tree - Complexity

- $O(\lg n)$ time to get through RB-Insert up to the call of RB-Insert-Fixup.
- Within RB-Insert-Fixup:
  - Each iteration takes $O(1)$ time.
  - The while loop repeats only if case 1 occurs.
    - Each iteration but the last moves z up 2 levels.
    - $O(\lg n)$ levels $\Rightarrow$ $O(\lg n)$ time.
    - Thus, insertion in a red-black tree takes $O(\lg n)$ time.
  - Note: there are at most 2 rotations overall.
    - Since the while loop terminates if case 2 or case 3 is executed.

AVL vs Red-Black Trees

- AVL trees are more strictly balanced ⇒ faster search

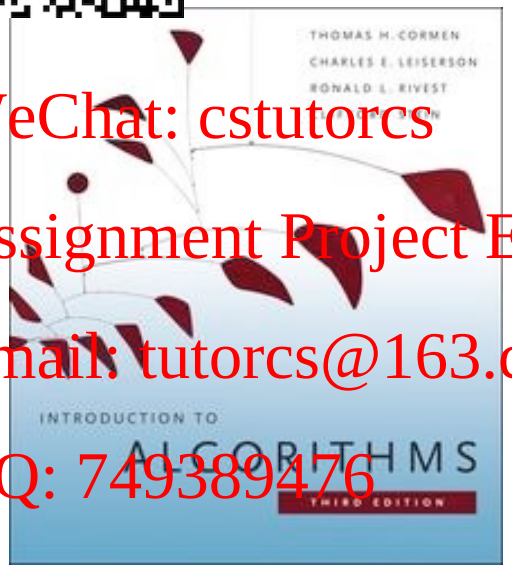
- Red Black Trees h[illegible] constraints and insert/remove operations require[illegible]ations ⇒ faster insertion and removal

- AVL trees store balance factors or heights with each node

- Red Black Tree requires only 1 bit of information per node

Further readings

[CLRS2009] Co[illegible]eiserson, Rivest, & Stein, *Introduction to A[illegible]s*. (available as E-book)

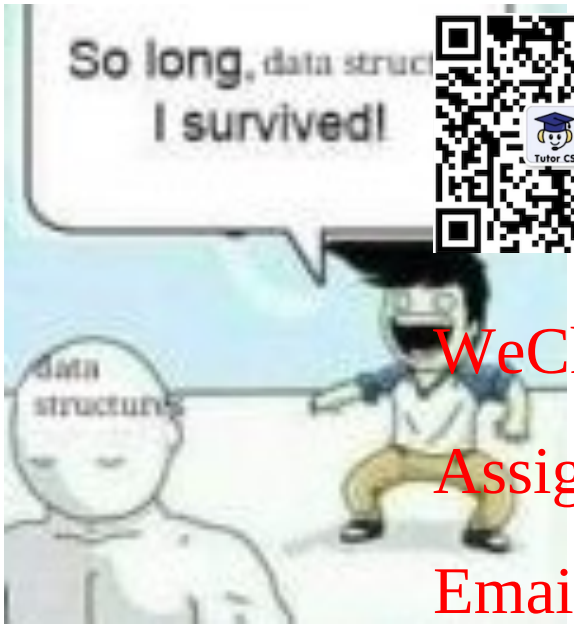


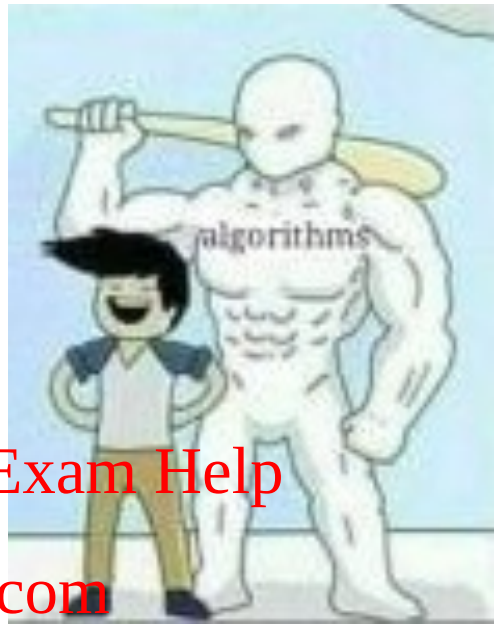
See Chapter 13 for the complete proofs & deletion

Outline

- Introduction.
- Operations.

Outline – Course so far





First 7 lectures

Next 8 lectures

Modified image initially taken from Programmer Humor

Algorithmic Paradigms

- General approaches to the construction of *efficient* solutions to problems.
- Such methods are of interest because:
  - They provide templates suited to solving a broad range of diverse problems.
  - They can be translated into common control and data structures provided by most high-level languages.
  - The temporal and spatial requirements of the algorithms which result can be precisely analyzed.
- Although more than one technique may be applicable to a specific problem, it is often the case that an algorithm constructed by one approach is clearly superior to equivalent solutions built using alternative techniques.

Algorithmic Paradigms

- Complete Search.
- Divide and Conque
- Dynamic Programm
- Greedy
- Conclusions