COMP 251

Algorithms  Structures (Winter 2022)

Algorithm Paradigms – Divide and Conquer

Announcements

Outline

- Complete Search
- Divide and Conque
  - Introduction.
  - Examples.
- Dynamic Programming.
- Greedy.

Algorithmic Paradigms – Divide and Conquer

- It is a problem solving paradigm where we try to make a problem [illegible] by 'dividing' it into smaller parts and 'conqu[illegible]hem.
- Recursive in structure
  - ***Divide*** the problem into sub-problems that are similar to the original but smaller in size
    - Usually by half or nearly half.
  - ***Conquer*** the sub-problems by solving them recursively. If they are small enough, just solve them in a straightforward manner.
  - ***Combine*** the solutions to create a solution to the original problem

Decrease and Conquer

- Sometimes we're not actually dividing the problem into many subproblems, but only one smaller subproblem
- Usually called decrease and conquer
- The most common example of this is binary search O(log n).

  - Given a sorted array of elements.
    1. Base case: the array is empty, return false
    2. Compare x to the element in the middle of the array
    3. If it's equal, then we found x and we return true
    4. If it's less, then x must be in the left half of the array
       4.1 Binary search the element (recursively) in the left half
    5. If it's greater, then x must be in the right half of the array
       5.1 Binary search the element (recursively) in the right half

Decrease and Conquer

Example: Does the following **sorted** array A contains the number 6?

5	6	7	9	9

Search(A, 0, 7, 6)

1	1	3	5	6	7	9	9	Search [0:7]

5 < 6 ⇒ look into right half of the array

1	1	3	5	6	7	9	9	Search [4:7]

7 > 6 ⇒ look into left half of the array

1	1	3	5	6	7	9	9	Search [4:4]

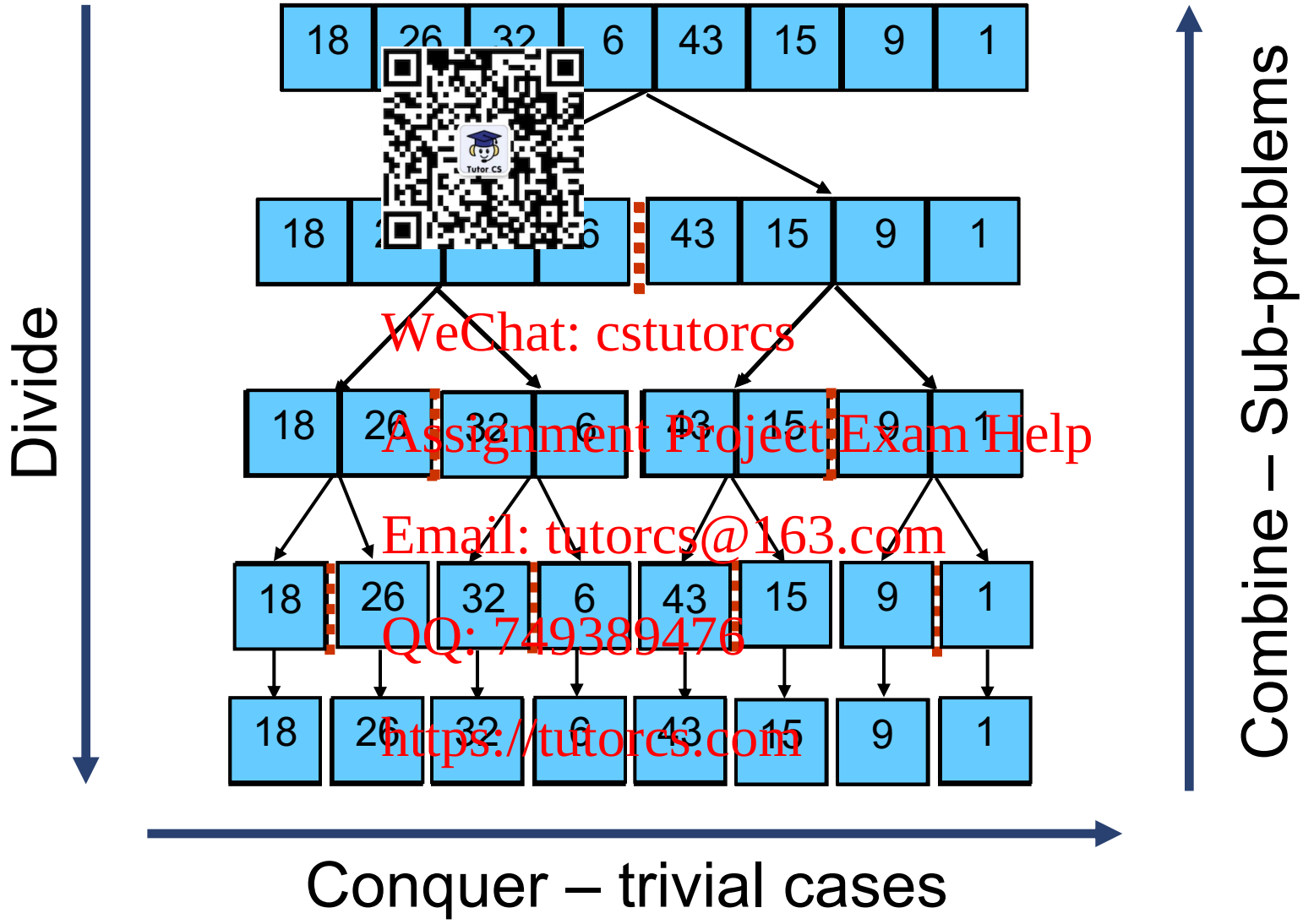
6 is found. Return 4 (index)

Divide and Conquer – Merge Sort

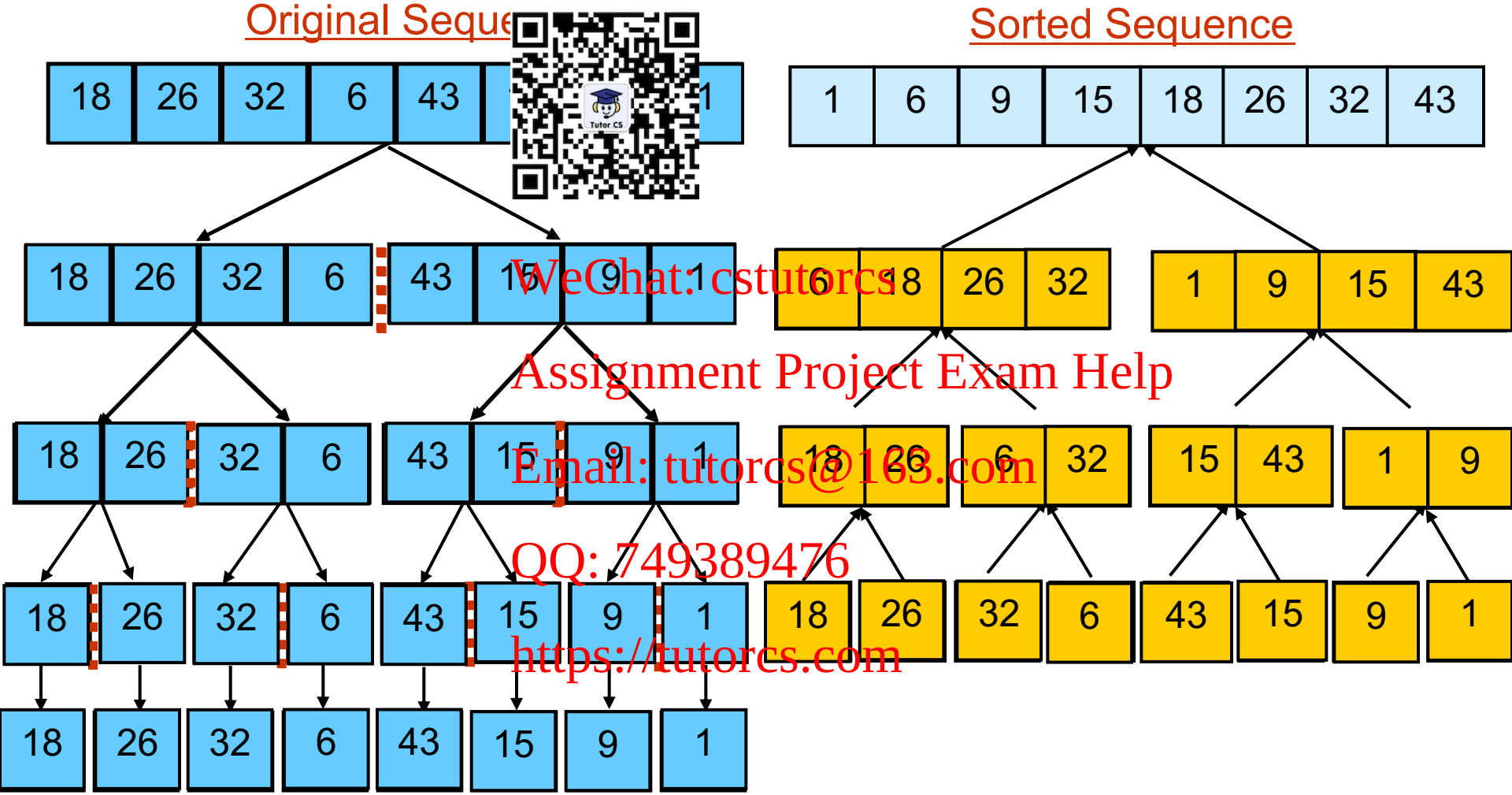
Sorting Problem: Sort a sequence of *n* elements into non-decreasing order.

- ***Divide***: Divide the *n*-element sequence to be sorted into two subsequences of *n*/2 elements each.
- ***Conquer:*** Sort the two subsequences recursively using merge sort.
- ***Combine***: Merge the two sorted subsequences to produce the sorted answer.

Divide and Conquer – Merge Sort



Divide and Conquer – Merge Sort



MergeSort(A, p, r)

INPUT: a sequen numbers stored in array A

OUTPUT: an ord quence of n numbers

```
MergeSort (A, p, r)  // sort A[p..r] by divide & conquer
1   if p < r
2       then q ← ⌊(p+r)/2⌋
3            MergeSort (A, p, q)
4            MergeSort (A, q+1, r)
5            Merge (A, p, q, r)  // merges A[p..q] with A[q+1..r]
```


Initial Call: MergeSort(A, 1, n)

Merge(A, p, q, r)

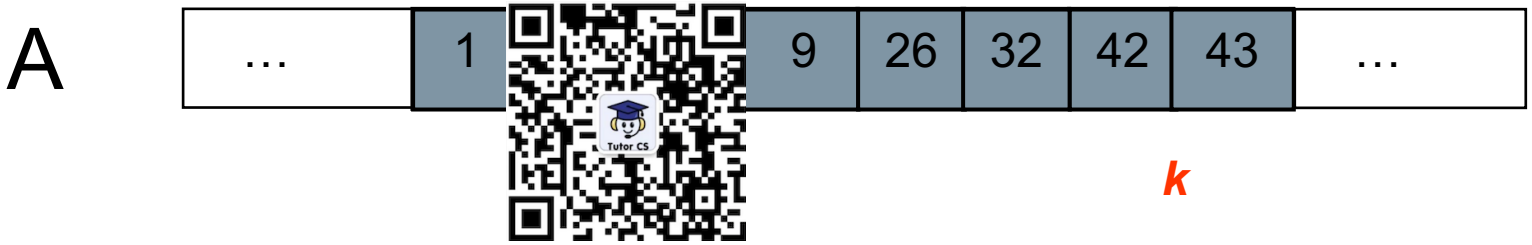

```
Merge(A, p, q, r)
1.    n1 ← q – p + 1
2.    n2 ← r – q
3.    for i ← 1 to n1
4.        do L[i] ← A[p + i – 1]
5.    for j ← 1 to n2
6.        do R[j] ← A[q + j]
7.    L[n1+1] ← ∞
8.    R[n2+1] ← ∞
9.    i ← 1
10.   j ← 1
11.   for k ← p to r
12.       do if L[i] ≤ R[j]
13.          then A[k] ← L[i]
14.               i ← i + 1
15.          else A[k] ← R[j]
16.               j ← j + 1
```

Input: Array containing sorted subarrays $A[p..q]$ and $A[q+1..r]$.

Output: Merged sorted subarray in $A[p..r]$.

Sentinels, to avoid having to check if either subarray is fully copied at each step.

Merge - Example



Merge - Correctness


```
Merge(A, p, q, r)
1.   n1 ← q – p + 1
2.   n2 ← r – q
3.   for i ← 1 to n1
4.       do L[i] ← A[p + i – 1]
5.   for j ← 1 to n2
6.       do R[j] ← A[q + j]
7.   L[n1+1] ← ∞
8.   R[n2+1] ← ∞
9.   i ← 1
10.  j ← 1
11.  for k ← p to r
12.      do if L[i] ≤ R[j]
13.           then A[k] ← L[i]
14.                i ← i + 1
15.           else A[k] ← R[j]
16.                j ← j + 1
```

Loop Invariant property (main *for* loop)

- At the start of each iteration of the for loop, subarray $A[p..k-1]$ contains the $k-p$ smallest elements of L and R in sorted order.
- $L[i]$ and $R[j]$ are the smallest elements of L and R that have not been copied back into A.

Initialization:

Before the first iteration:

- $A[p..k-1]$ is empty, k = p => k – p = 0.
- $i = j = 1$.
- $L[1]$ and $R[1]$ are the smallest elements of L and R not copied to A.

Merge - Correctness


```
Merge(A, p, q, r)
1.    n1 ← q – p + 1
2.    n2 ← r – q
3.    for i ← 1 to n1
4.        do L[i] ← A[p + i]
5.    for j ← 1 to n2
6.        do R[j] ← A[q + j]
7.    L[n1+1] ← ∞
8.    R[n2+1] ← ∞
9.    i ← 1
10.   j ← 1
11.   for k ←p to r
12.       do if L[i] ≤ R[j]
13.          then A[k] ← L[i]
14.               i ← i + 1
15.          else A[k] ← R[j]
16.               j ← j + 1
```

Maintenance:

Case 1: $L[i] \le R[j]$

- By LI, A contains $k - p$ smallest elements of L and R in sorted order.
- By LI, $L[i]$ and $R[j]$ are the smallest elements of L and R not yet copied into A.
- Line 13 results in A containing $k - p + 1$ smallest elements (again in sorted order). Incrementing i and k reestablishes the LI for the next iteration.

Case 2: Similar arguments with $L[i] > R[j]$

Termination:

- On termination, $k = r + 1$. By LI, *A[p..k-1], which is A[p..r],* contains $k\text{-}p = r - p + 1$ smallest elements of L and R in sorted order.
- L and R together *contain* $n_1 + n_2 + 2 = r - p + 3$ elements including the two sentinels. All but the two largest (i.e., sentinels) have been copied in A.

MergeSort - Analysis

- Running time of Merge Sort:
  - falls out from the th of the basic paradigm
- Divide: computing the middle takes $O(1)$
- Conquer: solving 2 subproblems takes $2T(n/2)$
- Combine: merging n elements takes $O(n)$
- Total:

$$T(n) = O(1) \quad \text{if } n = 1$$
$$T(n) = 2T(n/2) + O(n) + O(1) \quad \text{if } n > 1$$

$\Rightarrow T(n) = O(n \lg n)$

In general – Analysis - Recurrence

Trivial case of size nt

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \le c\ , \\ aT(n/b) + D(n) + C(n) & \text{otherwise}\ . \end{cases}$$

# of subproblems

size of subproblems

divide time

combine time

Solving recurrences

- ***Substitution meth***uess a bound and then use mathematical induc rove that our guess is correct.
- ***Recursion-tree method:*** converts the recurrence into a tree whose nodes represent the costs incurred at various levels of the recursion. We use techniques for bounding summations to solve the recurrence.
- ***Master method***: provides bounds for recurrences of the form.

$$T(n) = aT(n/b) + f(n)$$

where $a \geq 1$, $b > 1$, and $f(n)$ is a given function

MergeSort – Substitution method

Proposition. If $T(n)$ satisfies the following recurrence, then $T(n) = n \log_2 n$.

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2\,T(n/2) + n & \text{otherwise} \end{cases}$$

assuming n is a power of 2

Pf 2. [by induction on n]

- Base case: when $n = 1$, $T(1) = 0$.
- Inductive hypothesis: assume $T(n) = n \log_2 n$.
- Goal: show that $T(2n) = 2n \log_2 (2n)$.

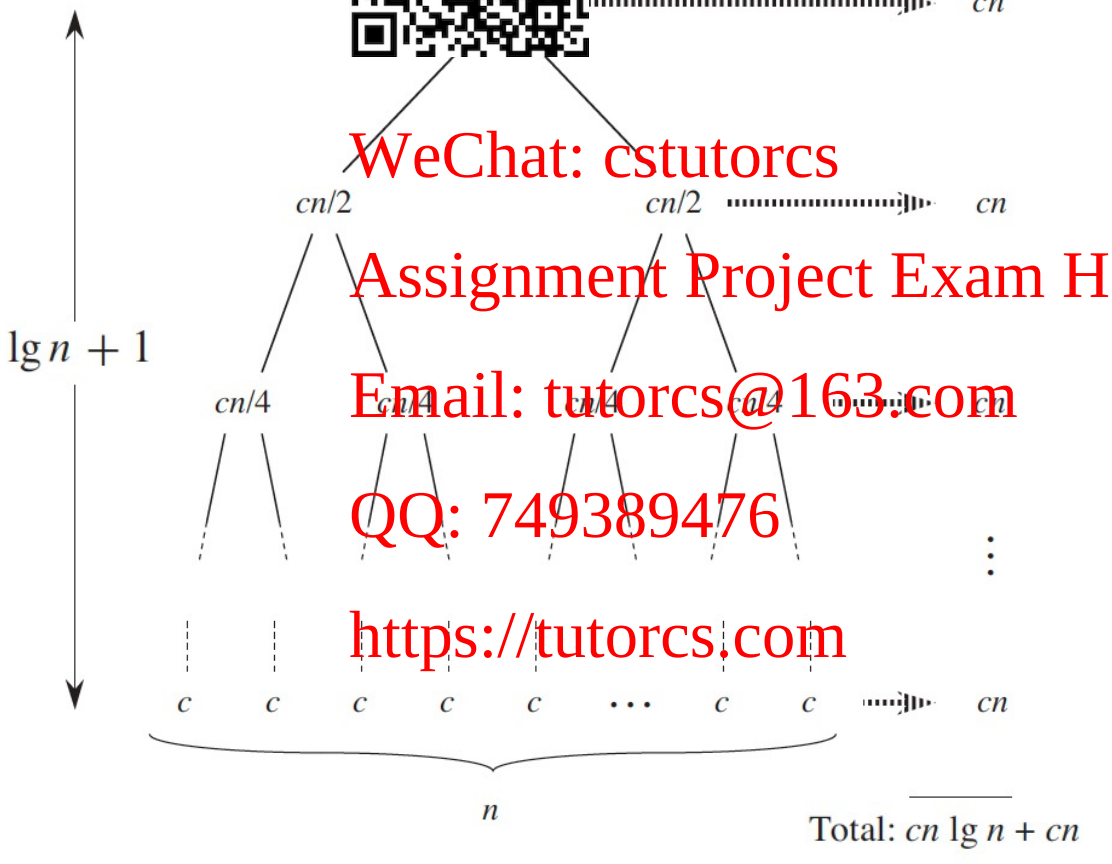
$\log_2 2n = \log_2 2 + \log_2 n$
$\log_2 2n - 1 = \log_2 n$

$$\begin{aligned} T(2n) &= 2\,T(n) + 2n \\ &= 2n \log_2 n + 2n \\ &= 2n\,(\log_2 (2n) - 1) + 2n \\ &= 2n \log_2 (2n). \quad \blacksquare \end{aligned}$$

MergeSort – Recursion Tree

$$T(n) = \begin{cases} c & \text{if } n = 1, \\ 2T(n/2) + cn & \text{if } n > 1, \end{cases}$$

assuming n is a power of 2



Recursion Tree

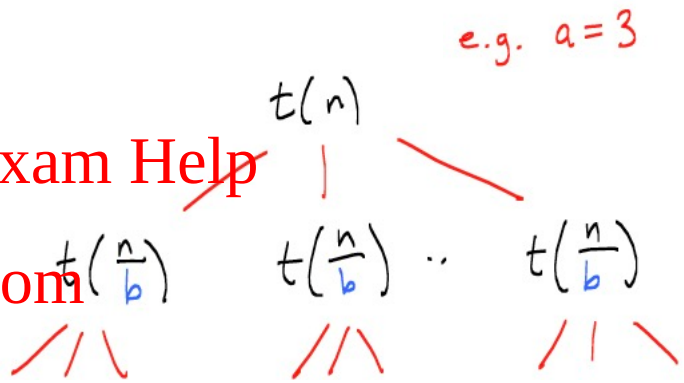
Suppose we have a divide and conquer algorithm that gives recurrence:

$$t(n) = a\, t\left(\frac{n}{b}\right) + c\, n^d$$

Notice that a, b and d are independent of n

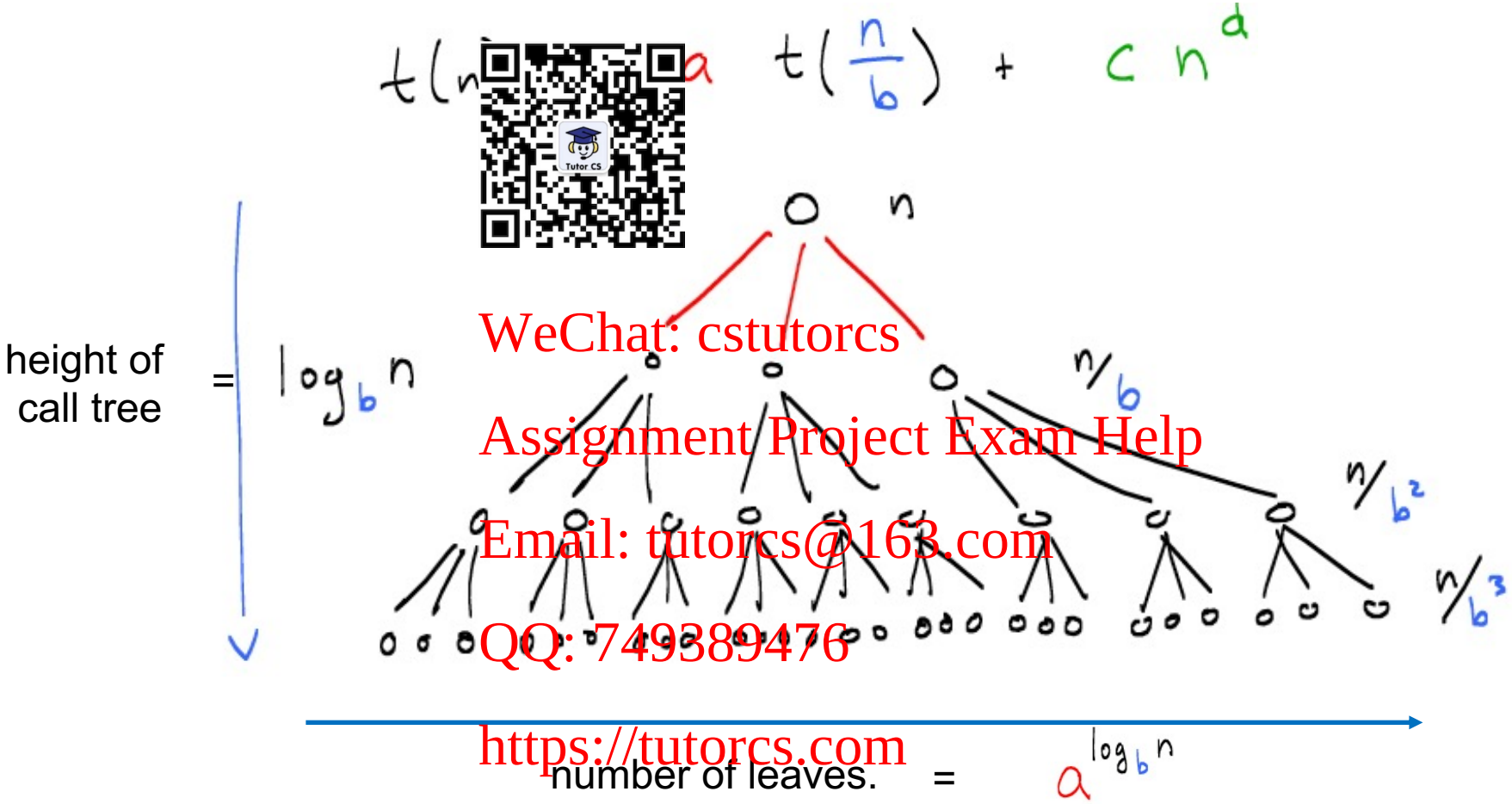
- a is the number of subproblems
- $\frac{n}{b}$ is the size of each subproblem
- n^d is the overhead for problem of size n (to partition and combine solutions)

I'll set $c = 1$.



Note that n^d represents the work done outside the recursive call

Recursion Tree



Recursion stops at the base case, typically when problem size is a small number

Recursion Tree – Good VS Evil

$t(n) = t\left(\frac{n}{b}\right) + c\,n^{d}$

assume $c = 1$

Tim Roughgarden:

"the forces of good and evil"

good - the size of the problem shrinks with each recursive call ($b > 1$)

evil - the number of subproblems increases at each level of the call tree. ($a > 1$)

Let's the battle begin

Taken from pinterest

Let's the battle begin (supplemental)

- But first lets define (recall) the allowed 'super powers'.
- Geometric series p[illegible]nvergence power).
  - Sum of a number of t[illegible] have a constant ratio between successive terms.

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\cdots$$

  - In general:

$$S_n = \sum_{k=0}^{n} r^k = 1 + r + r^2 + \cdots + r^n$$

  - Multiplying both sides by r gives.

$$rS_n = r + r^2 + r^3 + \cdots + r^{n+1}$$

Let's the battle begin (supplemental)

- Geometric series (convergence power).
  - Substracting the two equations.

$$(1-r)S_n = (1 + r + r^n) - (r + r^2 + r^3 + \cdots + r^{n+1})$$

$$(1-r)S_n = 1 - r^{n+1}$$

$$S_n = \sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

- For -1 < r < 1, the sum converges as $r^n \to \infty$, in which case

$$S_\infty = \sum_{k=0}^{\infty} r^k = \frac{1}{1 - r}$$

Let's the battle begin (supplemental)

- Exponents and logs (manipulation power).

$$x^{(y^z)} \neq (x^y)^z$$

e.g. $x^{2 \cdot 3} = (x^2)^3 \neq x^{(2^3)}$

$= (x^2)(x^2)(x^2)$

$= x^8$

e.g. $(b^d)^{\log_b n} = b^{d \log_b n}$

$= (b^{\log_b n})^d$

$= n^d$

Let's the battle begin (supplemental)

- Exponents and logs (manipulation power).

for $a, b, x > 0$

$$\log_b x = \log_b a \cdot \log_a x$$

Why?

take $\log_b$ of both sides

$$x = a^{\log_a x}$$

$$\log_b x = \log_b \left(a^{\log_a x} \right)$$

$$= \log_a x \cdot \log_b a$$

Let's the battle begin (supplemental)

- Exponents and logs (manipulation power).

Claim: $b^{\log_b n} = n^{\log_b a}$

Proof:

$$a^{\log_b n} = a^{\log_b a \log_a n}$$

$$= \left(a^{\log_a n}\right)^{\log_b a}$$

$$= n^{\log_b a}$$

Good VS Bad – battle

Assume $n = b^k$ for simplicity

$$t(n) = a\,t\left(\frac{n}{b}\right) + n^d$$

$$= a\left[a\,t\left(\frac{n}{b^2}\right) + \left(\frac{n}{b}\right)^d\right] + n^d$$

level 2

$$= a^2 t\left(\frac{n}{b^2}\right) + a\left(\frac{n}{b}\right)^d + n^d$$

level 3

$$= a^3 t\left(\frac{n}{b^3}\right) + a^2\left(\frac{n}{b^2}\right)^d + a\left(\frac{n}{b}\right)^d + n^d$$

Good VS Bad – battle

Good VS Bad – battle

$$t(n) = a^{\log_b n} + \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^d$$

$$= n^d \sum_{i=0}^{\log_b n} \left(\frac{a}{b^d}\right)^i$$

Note that the battle is this ratio.

Assume $t(1) = 1$, and note $\frac{n}{b^{\log_b n}} = 1$.

Good VS Bad – battle possible results

$$t(n) = \sum_{i=0}^{\log_b n} \left(\frac{a}{b^d}\right)^i$$

- If $a < b^d$ (good wins)
  - The amount of work is decreasing with the recursion level i.
  - Worst case is in the root (i.e., $i = 0$)
    - Might expect $O(n^d)$ => The work n^d of the root dominates
- If $a = b^d$ (there is a tie)
  - The amount of work is the same at every recursion level i.
  - All levels have the same 'worst' case.
    - Might expect $O(n^d \log n)$ => n^d for all the *log* levels
- If $a > b^d$ (evil wins)
  - The amount of work is increasing with the recursion level i.
  - Worst case is in the leaves (i.e., $i = log_b n$)
    - Might expect => $O(n^{\log(a)})$ => O(#leaves) because leaves dominates

Good VS Bad – battle possible results

$$t(n) = n^d \left(\frac{a}{b^d}\right)^i$$

geometric series

$$\sum_{i=0}^{k} r^i$$

three cases

1.) $r = 1$

2.) $r < 1$

3.) $r > 1$

If $a = b^d$ (there is a tie)
If $a < b^d$ (good wins)
If $a > b^d$ (evil wins)

Case 1 (r = 1) : $a = b^d$

Here we have the amount of work at each

$$1 + r + r^2 + r + \dots r^k$$

$$= 1 + 1 + 1 + 1 + \dots 1$$

$$= k + 1$$

$$= \log_b n$$

$$\Rightarrow t(n) = O(n^d \log_b n)$$

$$t(n) = n^d \sum_{i=0}^{\log_b n} \left(\frac{a}{b^d}\right)^i$$

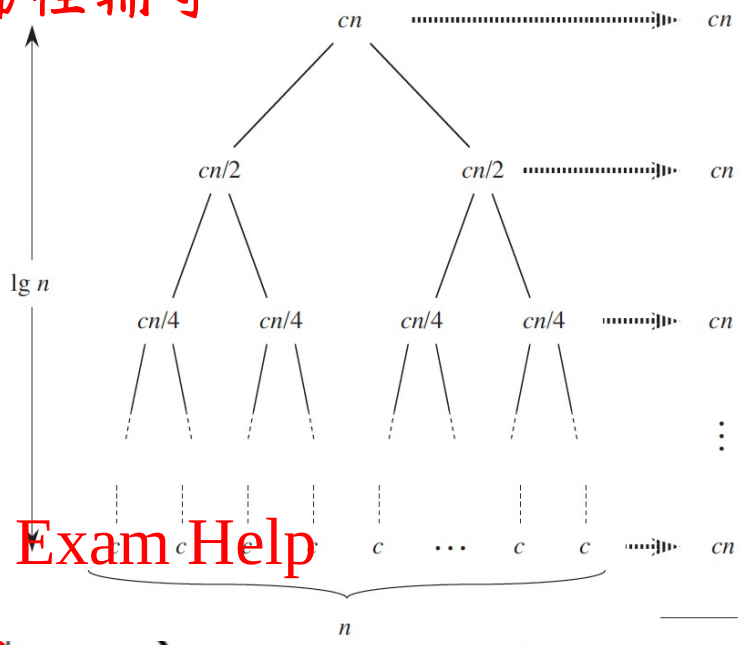
$$\sum_{i=0}^{k} r^i$$

Case 1 (r = 1) : $a = b^d$

eg. mergesort

$t(n) = a\,t(\quad) + cn^d$

$a = 2, \quad b = 2, \quad d = 1$



$t(n) = O(n^d \log_b n) = O(n \log_2 n)$

The same amount of work done at each level i, namely $O(n)$.

Case 2 (r < 1) : $a < b^d$

Here we have a decreasing amount of work at each level.

$$1 + r + r^2 + \cdots + r^k$$

$$= \frac{1 - r^{k+1}}{1 - r}$$

$$< \frac{1}{1 - r}, \quad \text{if } r < 1$$

$$= \text{constant} \quad (\text{independent of } n)$$

$$\Rightarrow t(n) = O(n^d)$$

$$t(n) = n^d \sum_{i=0}^{\log_b n} \left(\frac{a}{b^d}\right)^i$$

$$\sum_{i=0}^{k} r^i$$

Case 3 (r > 1) : $a > b^d$

Here we have increasing amount of work to do each level.

The leaves dominate.

$$1 + r + r^2 + r^3 + \ldots r^k$$

$$= \frac{r^{k+1} - 1}{r - 1}$$

$$< c r^k, \quad \text{for some } c \text{ which depends on } r$$

$$t(n) = n^d \sum_{i=0}^{\log_b n} \left(\frac{a}{b^d}\right)^i$$

$$\sum_{i=0}^{k} r^i$$

Case 3 (r > 1) : $a > b^d$

$$r^k = \left(\frac{a}{b^d}\right)^k$$

let $k = \log_b n$

$$= \left(\frac{a}{b^d}\right)^{\log_b n}$$

$$= a^{\log_b n} / \left(b^d\right)^{\log_b n}$$

$$= n^{\log_b a} / n^d$$

$$t(n) = n^d \sum_{i=0}^{\log_b n} \left(\frac{a}{b^d}\right)^i$$

$$\sum_{i=0}^{k} r^i$$

See Supplemental Slide

Case 3 (r > 1) : $a > b^d$

$$t(n) = n^d \sum_{i=0}^{\log_b n} \left(\frac{a}{b^d}\right)^i$$

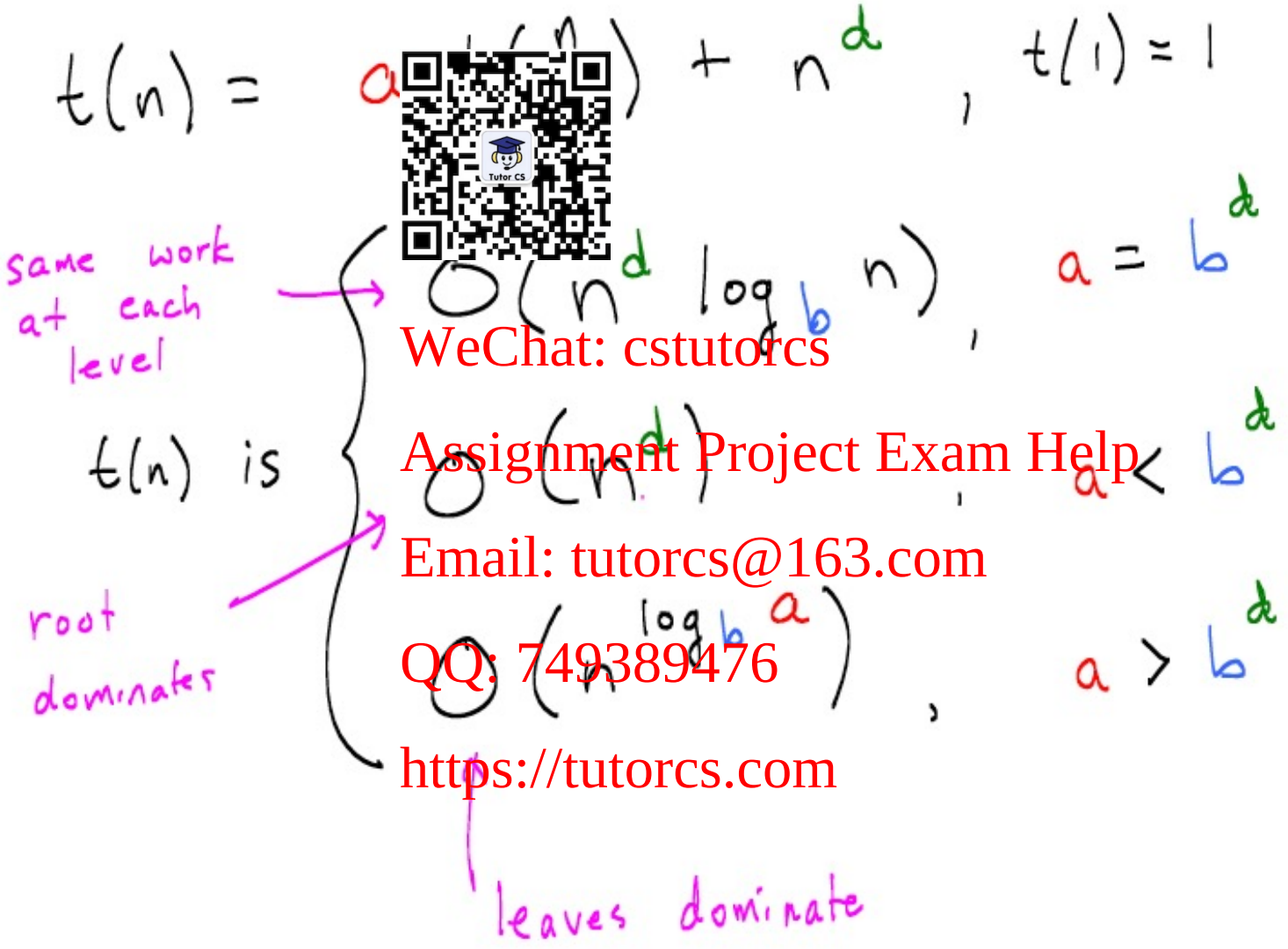
$$\sum_{i=0}^{k} r^i$$

$$t(n) < n^d \, c \left(\frac{a}{b^d}\right)^{\log_b n}$$

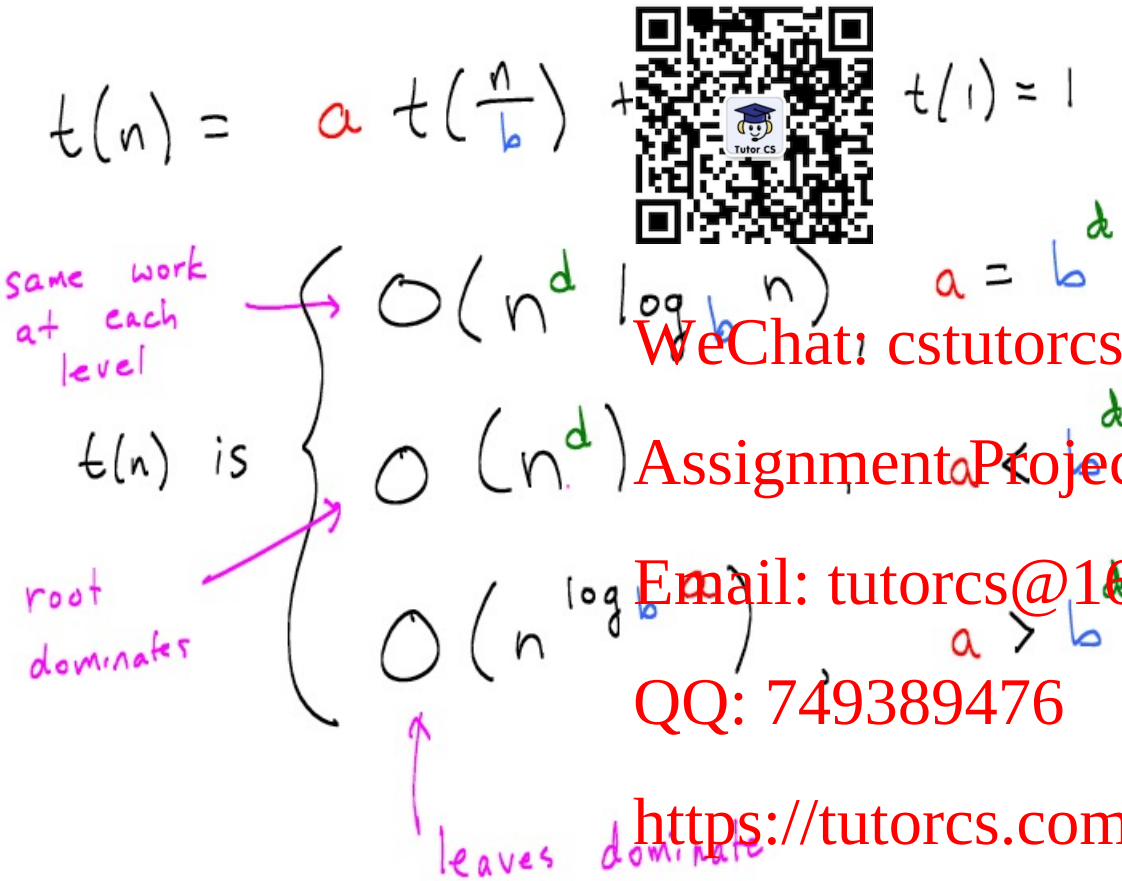
$$< n^d \, c \frac{n^{\log_b a}}{n^d}$$

$$= c \, n^{\log_b a}$$

Master Method (summary)



Master Method (summary)



- Limitations:
  - Defined for worst case.
  - Sub-problems need to have the same size.
  - Applicable to recursions of divide-and-conquer solutions
  - Etc
  - Etc

Master theorem

- Goal: Recipe for solving common recurrences.

$$T(n) = aT(n/b) + f(n)$$

a = # subproblems

b = factor by which the subproblem size decreases

f(n) = work to divide/merge subproblems

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$. bad wins
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$. tie
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \le cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$. ■

good wins

Note that the three cases do not cover all the possibilities for $f(n)$.

Master theorem – Case 1

Master theorem. Suppose that $T(n)$ is a function on the nonnegative integers that satisfies the r

$$T(r\left(\frac{n}{b}\right) + f(n)$$

where n/b means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Let $k = \log_b a$. Then,

Case 1. If $f(n) = O(n^{k-\varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^k)$.

Ex. $T(n) = 3\,T(n/2) + n$.

- $a = 3,\ b = 2,\ f(n) = n,\ k = \log_2 3$.
- $T(n) = \Theta(n^{\lg 3})$.

The formula works with $\varepsilon = \log_2 3 - 1 > 0$

$$f(n) = n = O\left(n^{\log_2 3 - (\log_2 3 - 1)}\right)$$

Master theorem – Case 2

Master theorem. Suppose that $T(n)$ is a function on the nonnegative integers that satisfies the r

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

where n/b means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Let $k = \log_b a$. Then,

Case 2. If $f(n) = \Theta(n^k \log^p n)$, then $T(n) = \Theta(n^k \log^{p+1} n)$.

Ex. $T(n) = 2T(n/2) + \Theta(n \log n)$.

- $a = 2,\ b = 2,\ f(n) = n \log n,\ k = \log_2 2 = 1,\ p = 1$.
- $T(n) = \Theta(n \log^2 n)$.

$$f(n) = \Theta(n \log n) = \Theta(n^{\log_2 2} \log n)$$

Master theorem – Case 3

Master theorem. Suppose that $T(n)$ is a function on the nonnegative integers that satisfies the re

$$T(n) \quad + \quad f(n)$$

where n/b means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Let $k = \log_b a$. Then,

regularity condition holds if $f(n) = \Theta(n^{k+\varepsilon})$

Case 3. If $f(n) = \Omega(n^{k+\varepsilon})$ for some constant $\varepsilon > 0$ and if $a\,f(n/b) \le c\,f(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.

Ex. $T(n) = 3\,T(n/4) + n^5$.

- $a = 3,\ b = 4,\ f(n) = n^5,\ k = \log_4 3$.
- $T(n) = \Theta(n^5)$.

1^{st} *property satisfied with* $\varepsilon = 4 - \log_4 3$

$$f(n) = n^5 = \Omega(n^{\log_4 3 + (4 - \log_4 3)})$$

2^{nd} *property satisfied with* $c = \frac{3}{4}$

$$3 \cdot \left(\frac{n}{4}\right)^5 \le c \cdot n^5$$

Master theorem – Applications

$k = \log_2 1 = 0; f(n) = 2^n$
$2^n = \Omega\left(n^{0+\log 2}\right)$
$1 \cdot 2^{\frac{n}{2}} \leq \frac{1}{2} \cdot 2^n$

T(n/2) + n²
Θ(n²) (case 3)

(n/2) + 2ⁿ
⇒ T(n) = Θ(2ⁿ) (case 3)

T(n) = 16 * T(n/4) + n
⇒ T(n) = Θ(n²) (case 1)

T(n) = 2 * T(n/2) + n log n

⇒ T(n) = n log² n (case 2)

T(n) = 2ⁿ * T(n/2) + nⁿ

⇒ Does not apply!!

$k = \log_2 3; f(n) = n^2$
$n^2 = \Omega\left(n^{\log_2 3+(2-\log_2 3)}\right)$
$3 \cdot \left(\frac{n}{2}\right)^2 \leq \frac{3}{4} \cdot n^2$

$k = \log_4 16 = 2; f(n) = n$
$n = O(n^{2-1})$

$k = \log_2 2 = 1; f(n) = n \log n$
$n \log n = \Theta(n^1 log^1 n)$

Master theorem – Other variants – Akra-Bazzi

Desiderata. Generalizes master theorem to divide-and-conquer algorithms where subproblems have su different sizes.

Theorem. [Akra-Bazzi] Give s $a_i > 0$ and $0 < b_i \le 1$, functions $h_i(n) = O(n / \log^2 n)$ and $g(n) = O(n^c)$, if the function $T(n)$ satisfies the recurrence:

$$T(n) = \sum_{i=1}^{k} a_i T(b_i n + h_i(n)) + g(n)$$

a_i subproblems of size b_i n

small perturbation to handle floors and ceilings

Then $T(n) = \Theta\left(n^p \left(1 + \int_1^n \frac{g(u)}{u^{p+1}}\, du\right)\right)$ where p satisfies $\sum_{i=1}^{k} a_i b_i^p = 1$.

Ex. $T(n) = 7/4\ T(\lfloor n/2 \rfloor) + T(\lceil 3/4\, n \rceil) + n^2$.

- $a_1 = 7/4,\ b_1 = 1/2,\ a_2 = 1,\ b_2 = 3/4 \Rightarrow p = 2.$
- $h_1(n) = \lfloor 1/2\, n \rfloor - 1/2\, n,\ h_2(n) = \lceil 3/4\, n \rceil - 3/4\, n.$
- $g(n) = n^2 \Rightarrow T(n) = \Theta(n^2 \log n).$

Outline

- Complete Search
- Divide and Conque
  - Introduction.
  - Examples.
- Dynamic Programming.
- Greedy.