COMP 251

Algorithms Structures (Winter 2022)

Disjoint Sets

Outline

- Introduction.
- Operations.

Introduction - Motivation

- You have a set of nodes (numbered 1-9) on a network. You are given a sequence of pairwise connections between them:

[illegible]7

8-2

1-6

5-7

4-8

3-5

Q: Are nodes 2 and 4 (indirectly) connected?

Q: Are nodes 3 and 8 connected?

Q: Are any of the paired connections redundant?

Q: How many sub-networks do we have?

Introduction - Motivation

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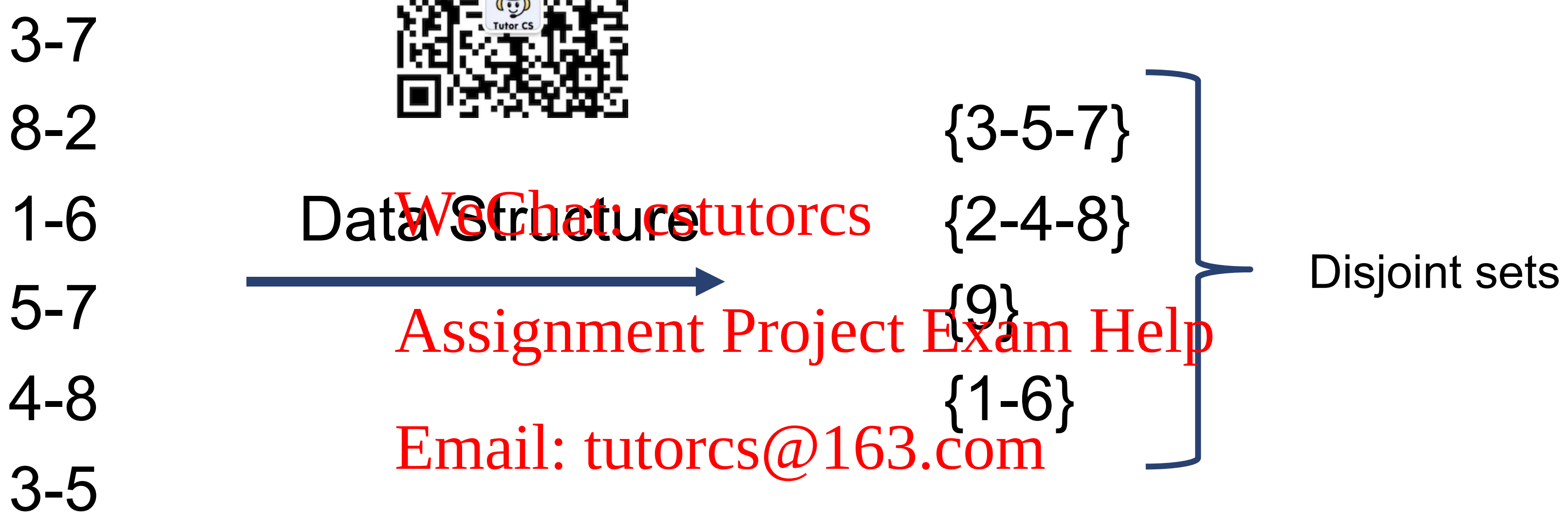

3-7
8-2
1-6
5-7
4-8
3-5

Data Structure

{3-5-7}
{2-4-8}
{9}
{1-6}

Disjoint sets

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Introduction - Motivation

Q: Are nodes 2 and 4 (indirectly) connected?

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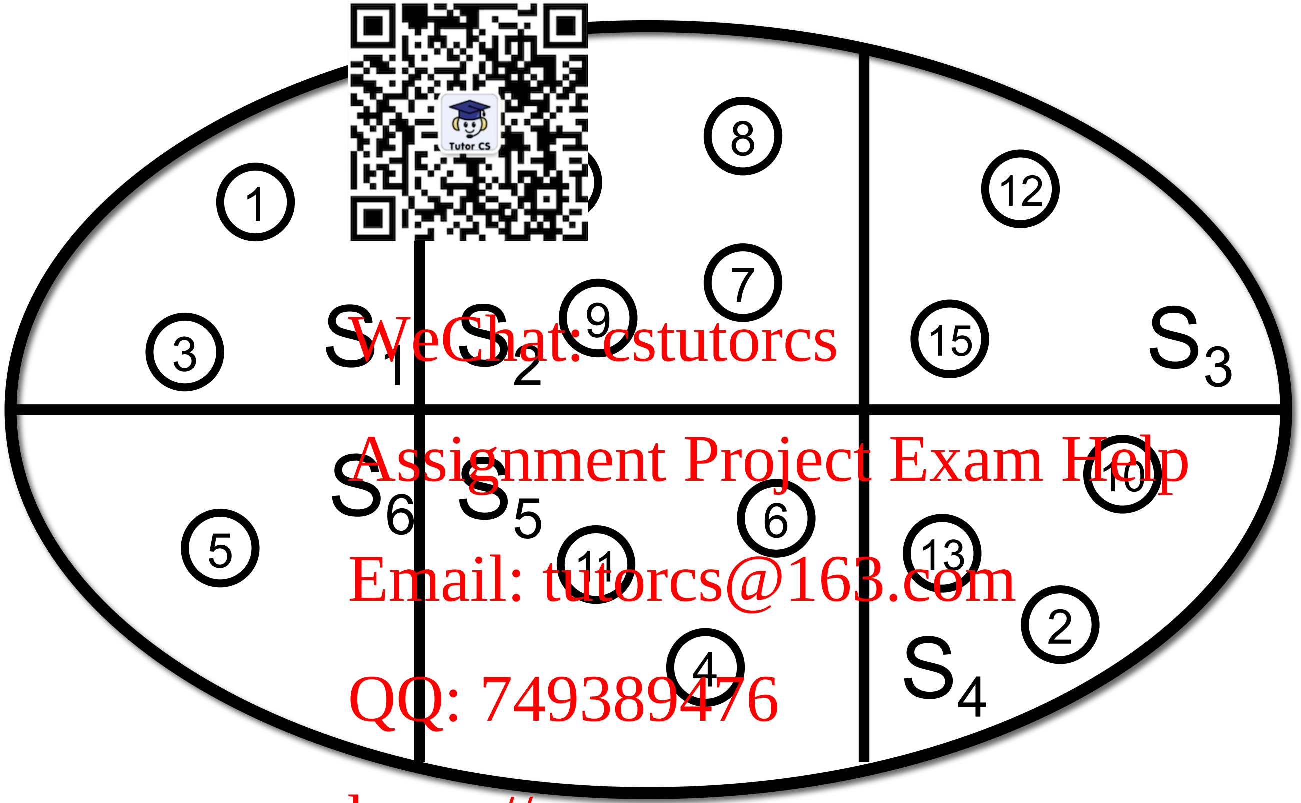
Q: Are any of the pai[illegible]ections redundant?

Q: How many sub-networks do we have?

- These kind of questions arises in a huge number of areas.
  - Networks
  - Transistor interconnects
  - Compilers
  - Image segmentation
  - Graph problems (upcoming topic)
  - Etc

Introduction - Partition

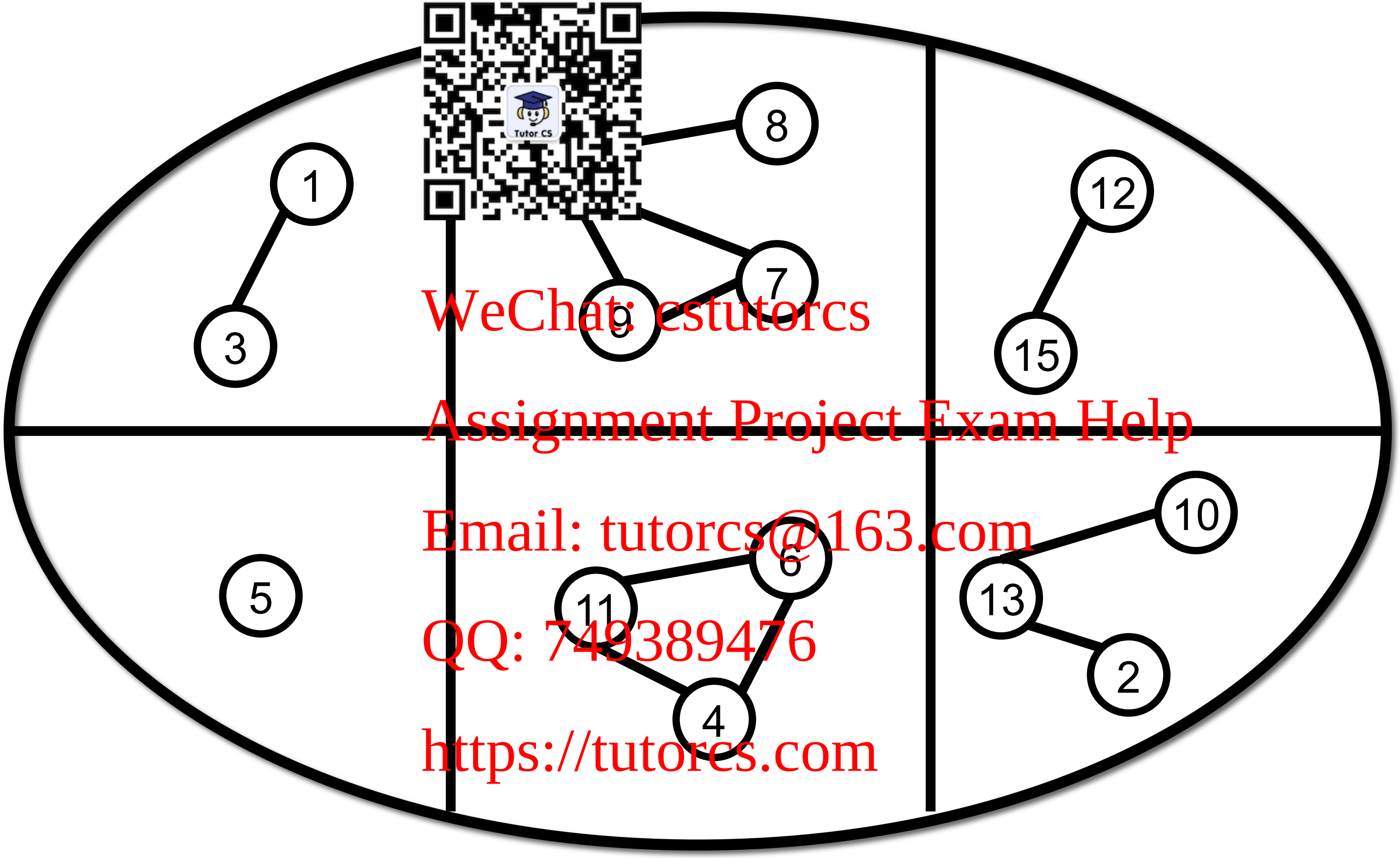
Generalization: Set of object partitioned into disjoint subsets.



$$S = S_1 \cup S_2 \cup \ldots \cup S_n \begin{cases} S_i \neq \emptyset \ \forall i \in \{1,\ldots,n\} \\ S_i \cap S_j = \emptyset \ iff \ i \neq j \end{cases}$$

Introduction – Graph example

Connected component: Set of nodes connected by a path.



Question: Given 2 nodes A & B, are they in the same component?

Introduction – Equivalence relation

A relation that is:

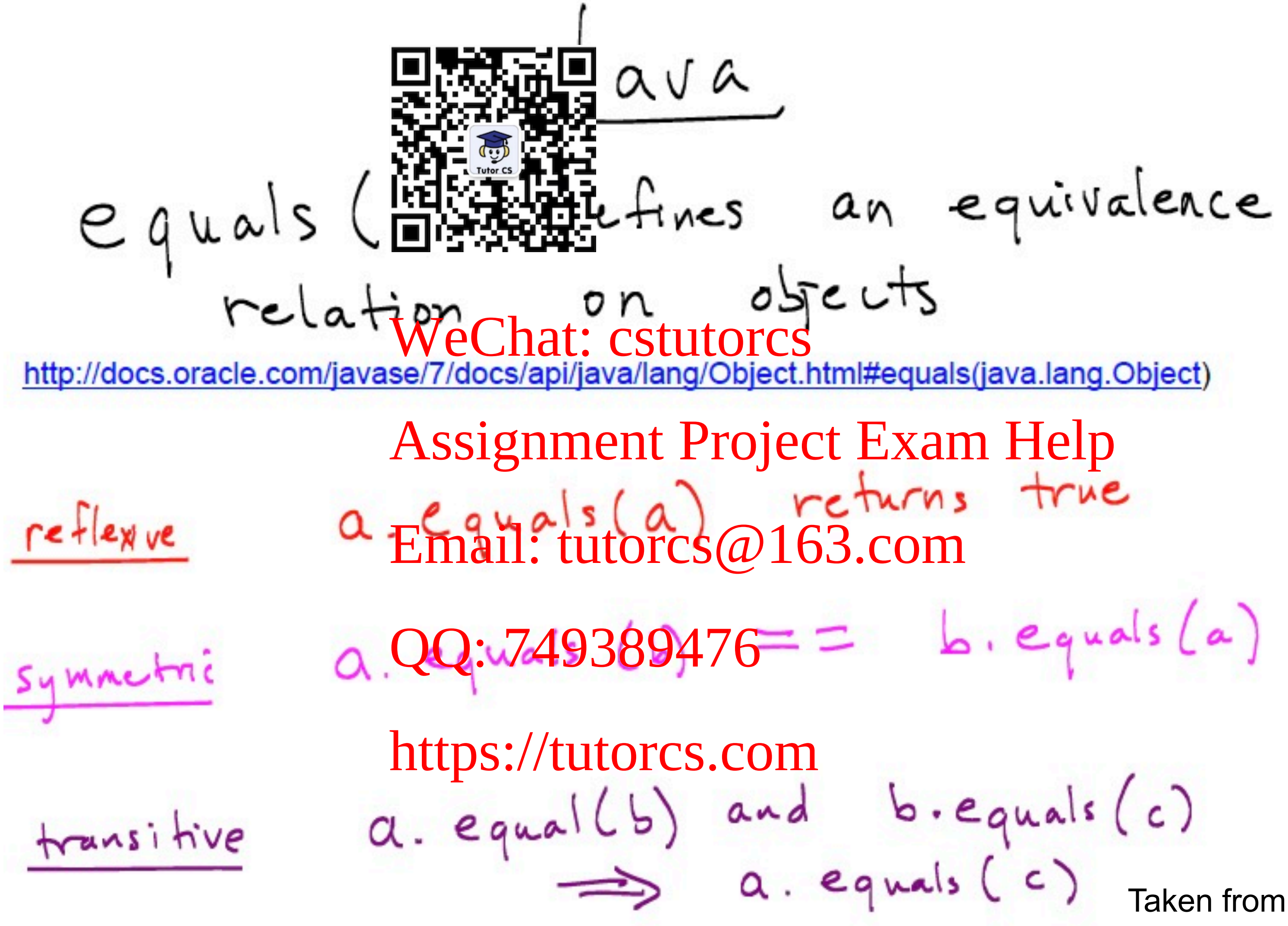
- Reflexive $\forall a \in$ [illegible] $\in R$
- Symmetric $\forall a, b$ [illegible] $) \in R \Rightarrow (b, a) \in R$
- Transitive $\forall a, b, c \in S, (a, b) \in R \text{ and } (b, c) \in R \Rightarrow (a, c) \in R$

Example:

For any undirected graph, the connections define an equivalence relation on vertices.

- For all u ∈ V, there is a path of length 0 from u to u.
- For all u,v ∈ V, There is a path from u to v, iff there is a path from v to u.
- For all u,v,w ∈ V, if there is a path from u to v and a path from v to w, then there is a path from u to w.

Introduction – Equivalence relation



Java

equals([illegible] efines an equivalence relation on objects

http://docs.oracle.com/javase/7/docs/api/java/lang/Object.html#equals(java.lang.Object)

reflexive   a.equals(a) returns true

symmetric   a.equals(b) == b.equals(a)

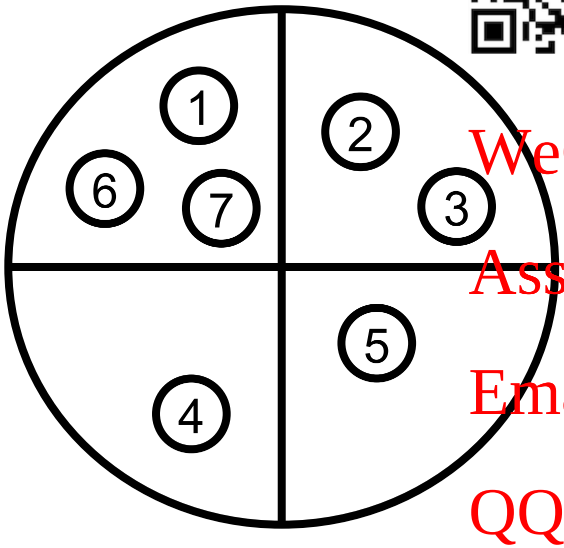
transitive   a.equal(b) and b.equals(c) ⇒ a.equals(c)

Taken from Langer2014

Introduction – Equivalence relation

An equivalence relation is the same as a partition:

Each equivalence relation p[illegible]artition of the underlying set into disjoint equivalent classes ([illegible]

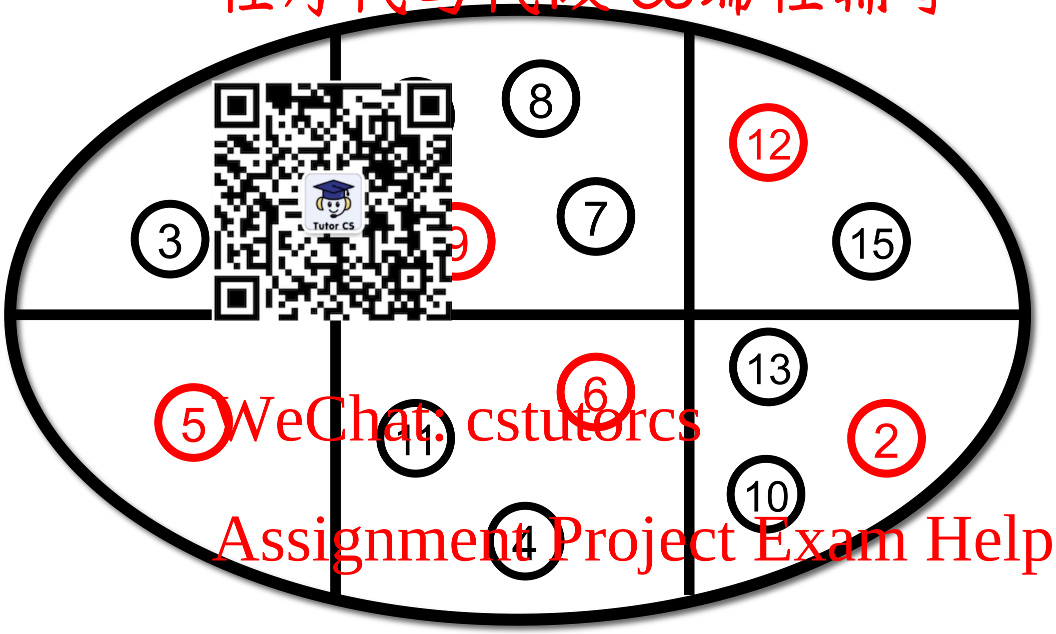


$$
\begin{array}{c|ccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
6 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
7 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
\hline
\end{array}
$$

i is equivalent to j if they belong to the same set.

(more constrained that general relation)

Introduction – Disjoint set ADT



i is equivalent to j if they belong to the same set.

Each set in the partition has a unique name (a representative member for convenience).

- find(i) returns the representative of the set that contains i.

- sameset(i,j) returns the boolean value find(i)==find(j)
- union(i,j) merges the sets containing i and j.

Introduction - Operations

- Maintain disjoint sets:
  - {3, 5, 7}, {4, 2, 8}, {9}
- Each set has a representative:
  - {3, 5, 7}, {4, 2, 8}, {9}, {1, 6}
- Find(x) returns the representative of the set containing x
  - Find(1) = 5
  - Find(4) = 8
- Union(x,y) takes the union of the two sets that contains x and y
  - Union(5,1) = {3, 5, 7, 1, 6}, {4, 2, 8}, {9}
  - Union(9,6) = {3, 5, 7}, {4, 2, 8}, {9, 1, 6}

Introduction - Operations

- Maintain disjoint sets:
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- Each set has a repr ve:
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- Q: Are nodes 2 and 4 (indirectly) connected?
  - Find(2) == Find(4)
- Q: Are nodes 3 and 8 connected?
  - Find(3) == Find(8)
- Q: How many sub-networks do we have?
  - Number of representatives.
- Q: Are any of the paired connections redundant?
  - Connections – (|S1-1| + |S2-1| + |S3-1| + |S4-1)| = 6 – (2+2+0+1) = 1

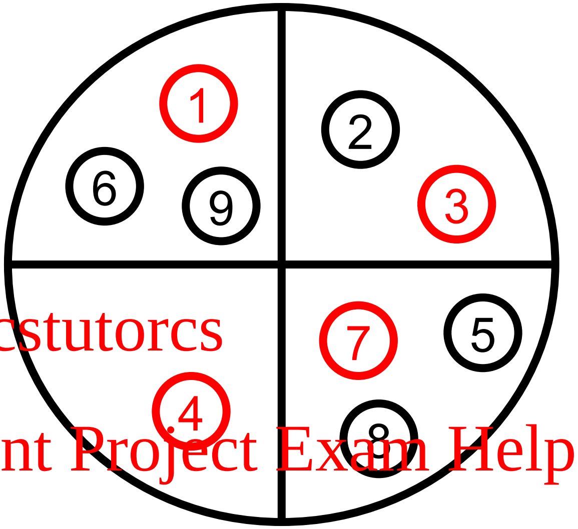
Outline

- Introduction.
- Operations.

Operations – find

Rep[]

1	1
2	3
3	3
4	4
5	7
6	1
7	7
8	7
9	1



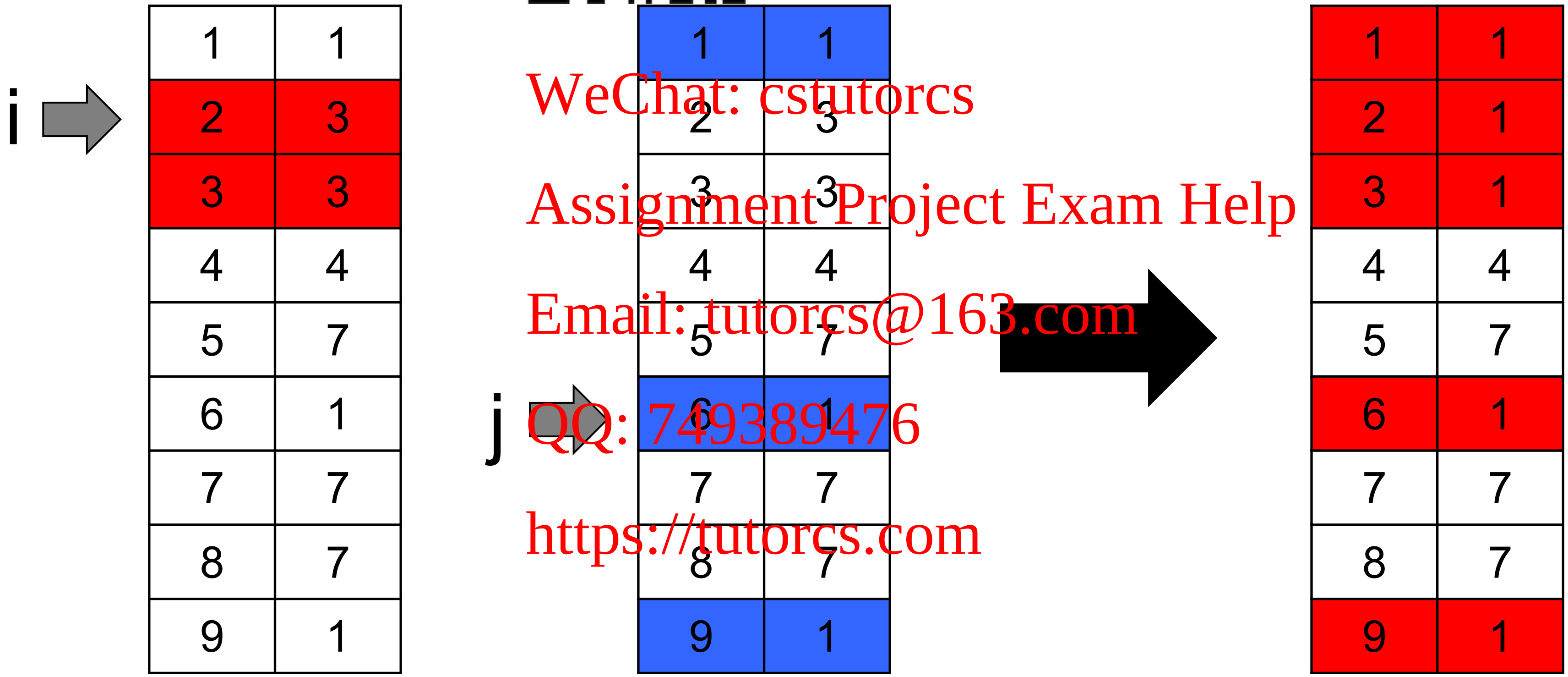
Let Rep[i] ∈ { 1, 2, … , n } be the representative of the set containing i.

find(i) { return rep[i]; }

Operations – union

- union(i,j) merges the sets containing i and j.

Example: unio [illegible]



1	1
2	3
3	3
4	4
5	7
6	1
7	7
8	7
9	1

1	1
2	3
3	3
4	4
5	7
6	1
7	7
8	7
9	1

1	1
2	1
3	1
4	4
5	7
6	1
7	7
8	7
9	1

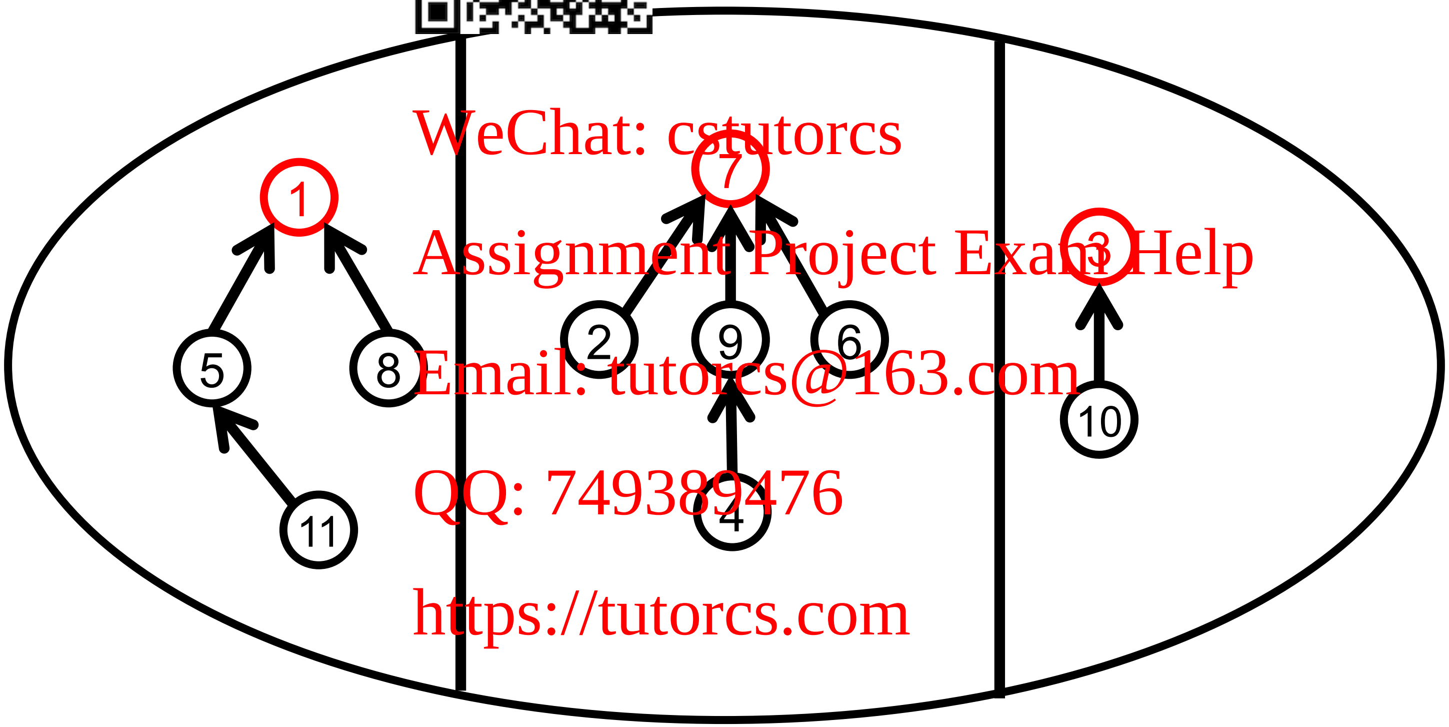
Operations – union

```
union(i,j) {
    if rep[i] != rep[j] {
        prevrepi = rep[i];
        for (k=1; k<=n; k++) {
            if rep[k] == prevrepi {
                rep[k] = rep[j];
            }
        }
    }
}
```

- store value of rep[i] because it may change during the execution of the algorithm.
- O(n) running time… slow.

Union – forest & tree representation

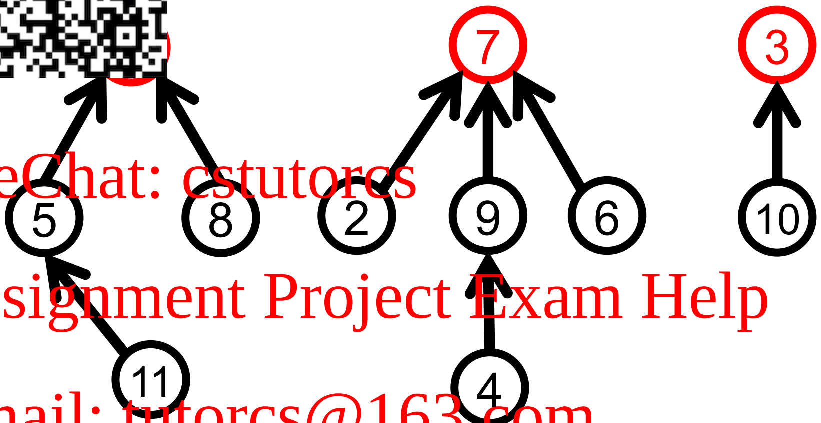
- Represent the disjoint sets by a forest of rooted trees.
  - Roots are the repr[illegible]e (i.e. find(i) == findroot(i)).
  - Each node points [illegible]nt.



Note:
The tree structure does not necessarily represent the relationship between the stored objects.

Union – table representation

	p[]
1	1
2	7
3	3
4	9
5	1
6	7
7	7
8	1
9	7
10	3
11	5

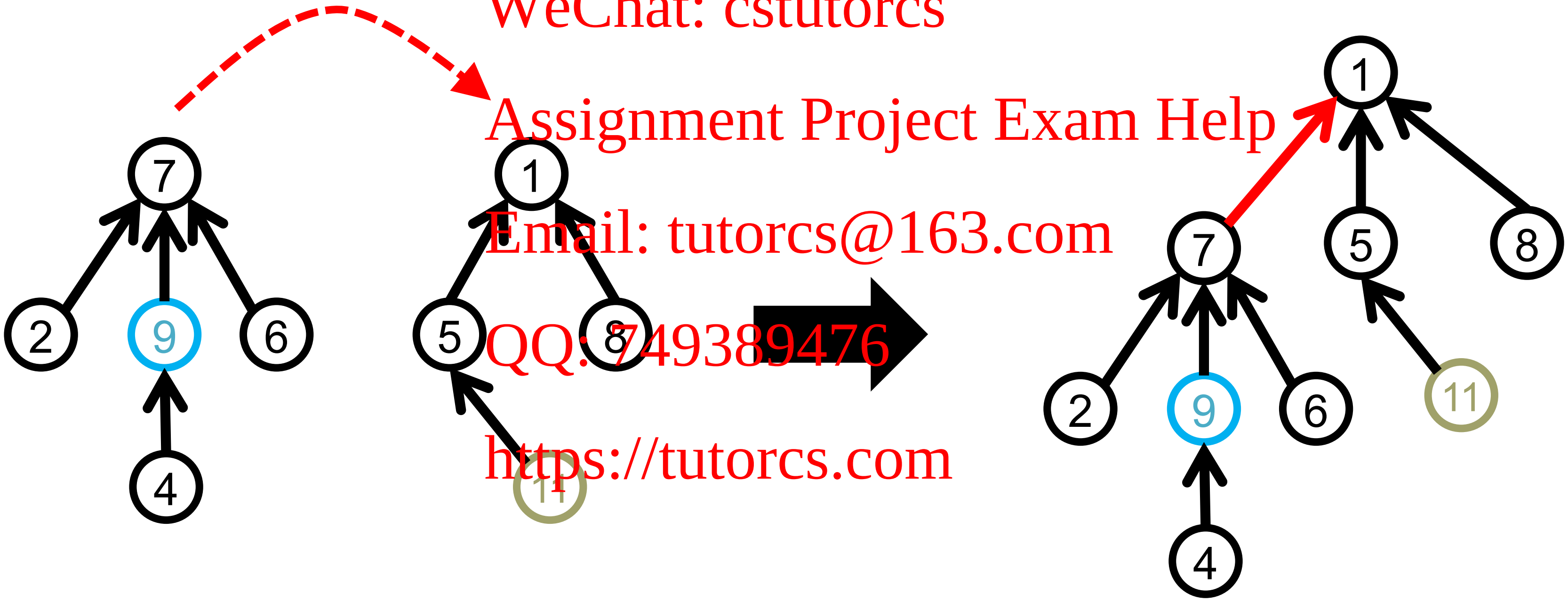


- Non-root nodes hold index of their parent.
- Root nodes store their own value.

Operations – Union

Union(9,11)

Root of the tree of 11 becomes the parent of the root of the tree of 9.



Operations – Find & Union

```
find(i) {
      if p[i] =
            retu
      } else {
            return find(p[i]);
      }
}

union(i,j) {
      if find(i) != find(j) {
            p[find(i)] = find(j);
      }
}
```

Remark: Arbitrarily merge the set on i into the set of j.

Operations – Union - > Worst case

union(1,2)

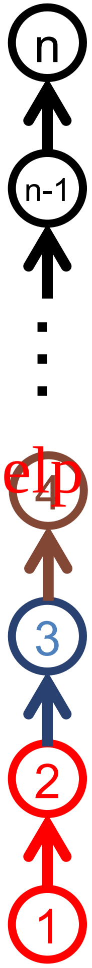
union(1,3)

union(1,4)

…

union(1,n)

Then, find(1) is O(n)…



Union – Heuristic – Definitions

- The **depth** of a node is the number of edges from the node to the tree's root node. A root node will [illegible] of 0.
- The **height** of a node is [illegible] of edges on the *longest path* from the node to a leaf. A leaf node [illegible] a height of 0.

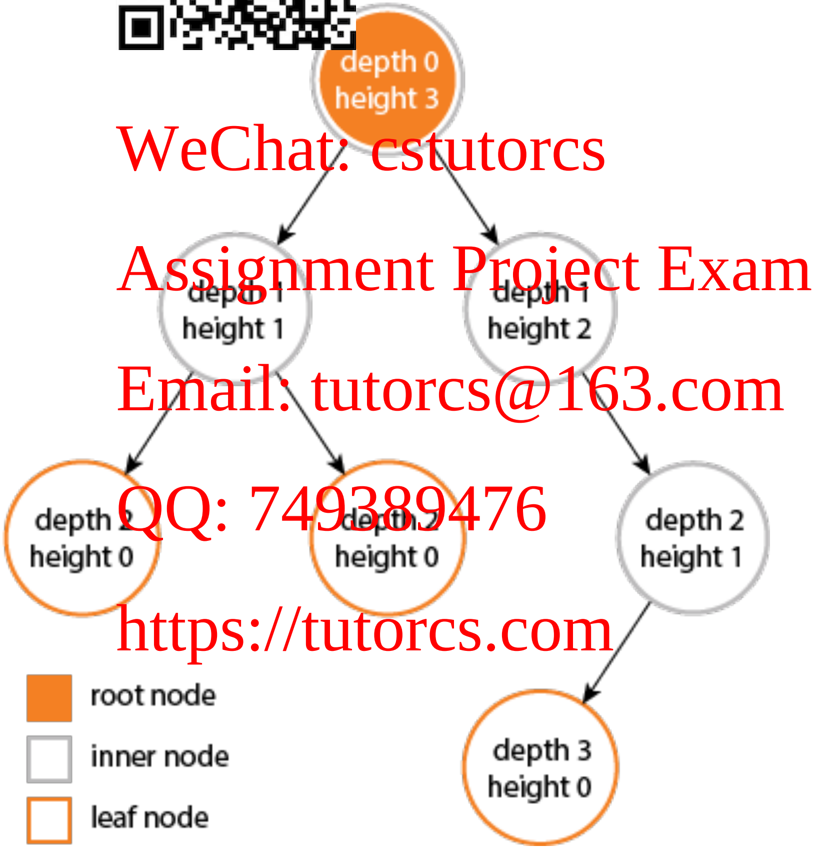
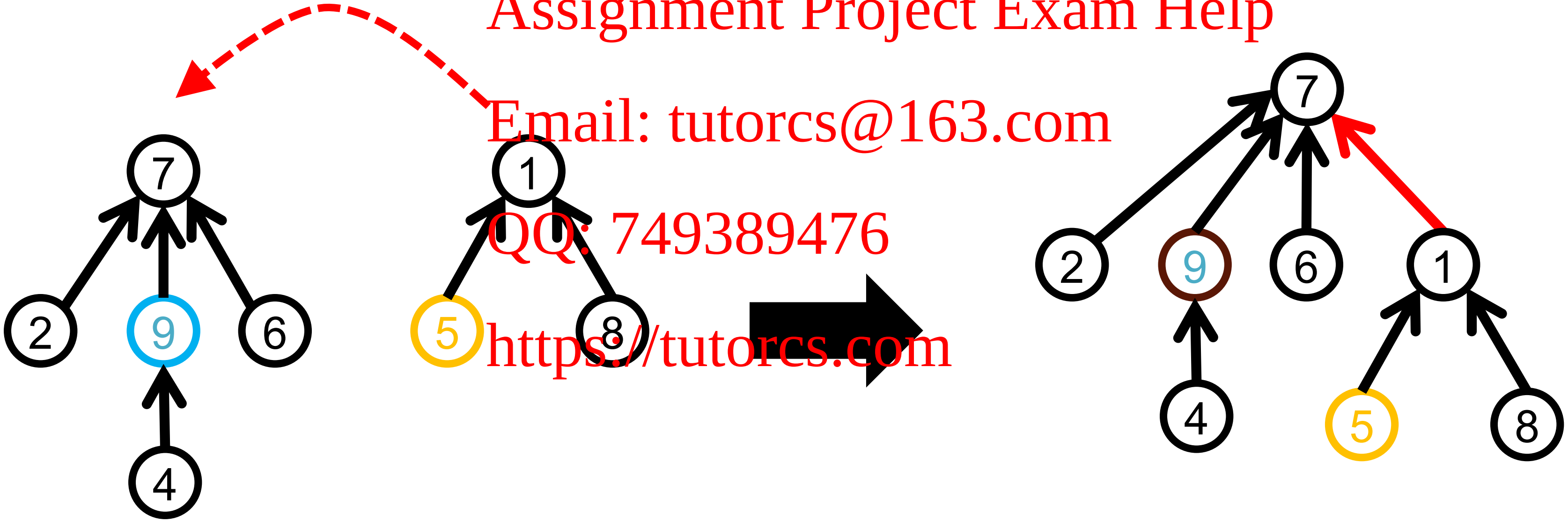


Image and definitions taken from:
https://stackoverflow.com/

Union – Heuristic – by size

Heuristic to control [illegible]nt to the trees after merging.

Idea: Merge tree wi[illegible]er number of nodes into the tree with the largest number of nodes (In practice, we can also use rank which is an upper bound on the height of nodes).

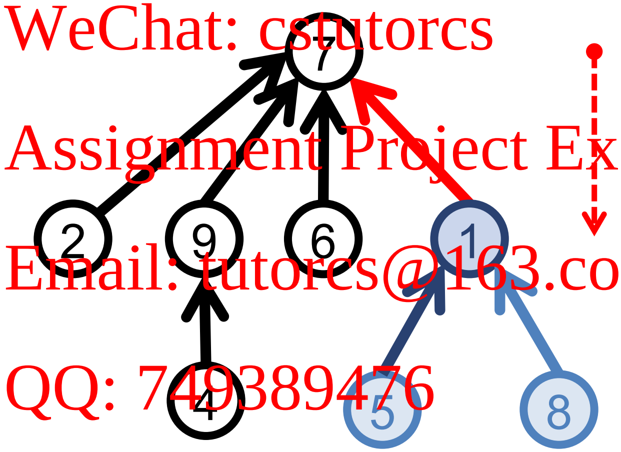


Union – Heuristic – by size

Claim: The depth of any node is at most log n.

Proof:

- If union causes the depth of a node to increase, then this node must belong to the smallest tree.



- Thus, when the depth increases, the size of the (merged) tree containing this node will at least double.
- But we can double the size of a tree at most log n times.

Union – Heuristic – by height

Idea: Merge tree wi[illegible]er height into tree with larger height.

Claim: The height of trees obtained by union-by-height is at most log n.

Corollary: An union-by-height tree of height h has at least $n_h \geq 2^h$ nodes.

Proof (Corollary):

- Base case: a tree of height 0 has one node.
- Induction: (hypothesis) $n_h \geq 2^h$. Show $n_{h+1} \geq 2^{h+1}$.

Heuristic – supplemental material

The base case k=0 is easy, since a tree of height 0 always has just 1=2^0 node (the root). Suppose [illegible] is true for h=k. Now consider a union-by height tree of height k[illegible] must have been a union that brought two trees together and in[illegible]e height of one of them from k to k+1. Let those two trees (at the time of that union) be T1 and T2. We know that both T1 and T2 were of height k before the union. [Why? If one of them were of height less than k, then union-by-height would have changed the root of that shorter one to make it point to the root of the taller one, and the height of the unioned tree would still be k. But its not; the unioned tree is of height k+1.] Now we can apply the induction hypothesis: the trees T1 and T2 each have at least 2^k nodes. Thus, the unioned tree has at least 2^k + 2^k = 2^(k+1) nodes.

Taken from Langer 2014

Union – running time

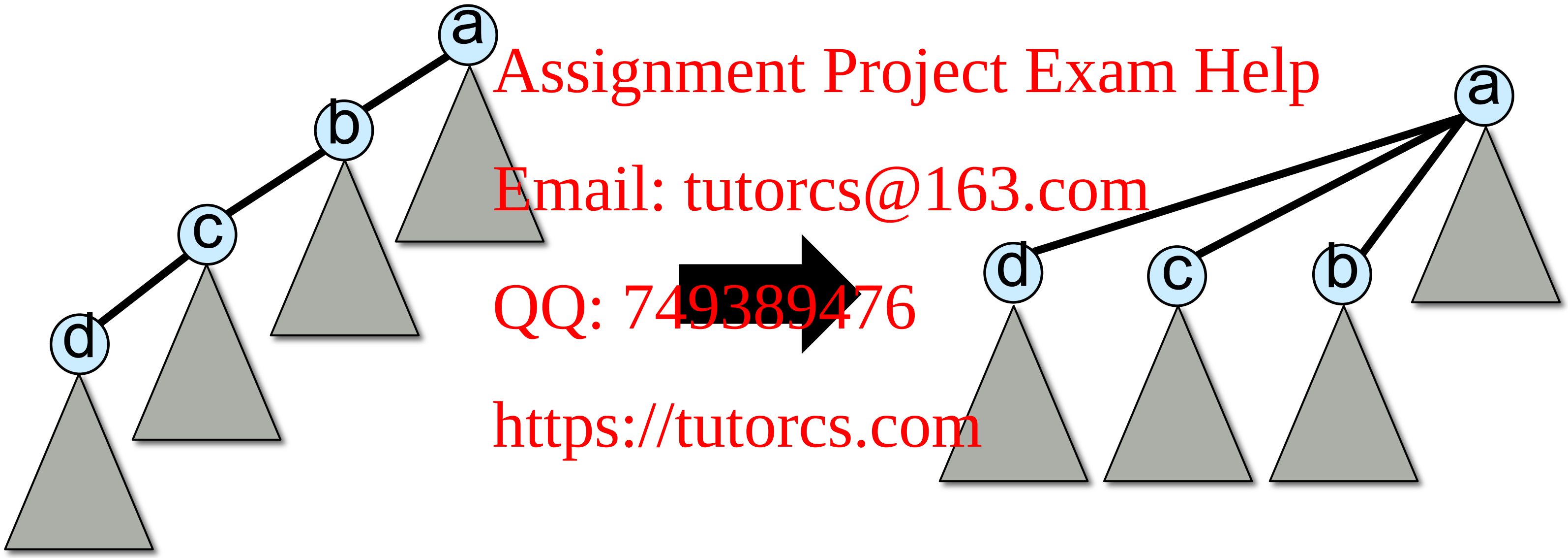
		find(i)	union(i,j)
		O(1)	O(n)
Quick Union	Union by size	O(log n)	O(log n)
	Union by height	O(log n)	O(log n)

↑ Quick union makes 2 calls to find.

Note: These are worst case complexities.

union-find – heuristics – path compression

- Find path = node… during the execution of find() on the trip to the…
- Make all nodes on the find path direct children of root.



union-find – heuristics – path compression

```
find
        i] == i {
            return i;
        } else {
            p[i] = find(p[i]);
            return p[i];
        }
}
```

union-find – running time

- Use union by size and path compression.
- **m union or find oper**[illegible]e O(m α(n)).

What is α(n) ?

n	α(n)
0 - 2	0
3	1
4 - 7	2
8 - 2047	3
2048 – $A_4(1)$	4

Where $A_4(1) >> 10^{80}$!!

union-find – running time

- Use union by size and path compression.
- Huge Practical proble [illegible]
  - 10^{10} edges conne [illegible] nodes.
  - The heuristics reduces time from 3.000 years to 1 minute
  - Supercomputer wont help much
  - Good algorithm makes solution possible
  - Good algorithms makes it possible to solve problems that could not otherwise be addressed.

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- Operations.

time for questions