CS Discussion 7

TA: Albert Ji

Friday 17th November, 2023

Information

- HW3
  - Due Nov 27, 11:59 pm

Sentential Decision Diagrams (SDDs)

- We want to rewrite $(p_1 \wedge s_1) \vee (p_2 \wedge s_2) \vee \ldots \vee (p_n \wedge s_n)$
- Partition on variables $(\mathbf{X}, \mathbf{Y})$
- Prime:
  - $p_1, \ldots, p_n$ over variables in $\mathbf{X}$
  - Mutually exclusive: $p_i \wedge p_j = \bot$ for $i \neq j$
  - Exhaustive: $p_1 \vee p_2 \vee \ldots \vee p_n = \top$
  - $p_i \neq \bot$
- Sub:
  - $s_1, \ldots, s_n$ over variables in $\mathbf{Y}$
  - $s_i = \Delta|p_i$

- $\Delta = A \vee B$
- $X = \{A\}, Y = \{B\}$
-

prime	sub
A	$\Delta \vert A$
$\neg A$	$\Delta \vert \neg A$

SDD

- $\Delta = A \vee B$
- $X = \{A\}, Y = \{B\}$
-

prime	sub
A	$\top$
$\neg A$	B

- $\Delta = [(A) \wedge (\top)] \vee [(\neg A) \wedge (B)]$

SDD

- $\Delta = A \vee B$
- $X = \{A\}, Y = \{B$

prime	sub
A	$\top$
$\neg A$	B

- $\Delta = [(A) \wedge (\top)] \vee [(\neg A) \wedge (B)]$

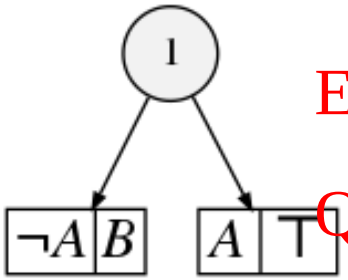

Figure: SDD

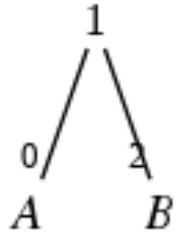

Figure: vtree

SDD: Compression

- Subs:
  - $s_1, \ldots, s_n$ over $\mathbf{Y}$
  - $s_i \neq s_j$ for $i \neq j$
- A formula Δ has a unique compressed $(\mathbf{X}, \mathbf{Y})$-partition.

SDD: Compression

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  - $s_1, \ldots, s_n$ over $\mathbf{Y}$
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- A formula Δ has a unique compressed $(\mathbf{X}, \mathbf{Y})$-partition.

Canonicity

- An SDD is trimmed by traversing it bottom up, replacing decompositions $\{(\top, \alpha)\}$ and $\{(\alpha, \top), (\neg\alpha, \bot)\}$ with α.

- Let α and β be compressed and trimmed SDDs of the same vtree. Then α and β are equivalent iff $\alpha = \beta$

SDD: Polytime Apply

- $(\mathbf{X}, \mathbf{Y})$-partition of Δ:
  - $(p_1, s_1), \ldots, (p_n, s_n)$
- $(\mathbf{X}, \mathbf{Y})$-partition of Γ:
  - $(q_1, t_1), \ldots, (q_m, t_m)$

SDD: Polytime Apply

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- $(\mathbf{X}, \mathbf{Y})$-partition of Γ:
  - $(q_1, t_1), \ldots, (q_m, t_m)$
- $(\mathbf{X}, \mathbf{Y})$-partition of $\Delta \circ \Gamma$:
  - $(p_i \wedge q_j, s_i \circ t_j) \mid p_i \wedge q_j \neq \bot$

SDD: Polytime Apply

- $(\mathbf{X}, \mathbf{Y})$-partition of α:
  - $(p_1, s_1), \ldots, (p_n, s_n)$
- $(\mathbf{X}, \mathbf{Y})$-partition of Γ:
  - $(q_1, t_1), \ldots, (q_m, t_m)$
- $(\mathbf{X}, \mathbf{Y})$-partition of $\alpha \circ \Gamma$:
  - $(p_i \wedge q_j, s_i \circ t_j) \mid p_i \wedge q_j \neq \bot$
- Polytime apply requires the same vtree
- Compression is not guaranteed

- $\Delta = (C \vee D) \wedge (A$
- $X = \{C, D\}, Y = \{A, B\}$

prime	sub
$C \wedge D$	$\Delta \vert C, D$
$C \wedge \neg D$	$\Delta \vert C, \neg D$
$\neg C \wedge D$	$\Delta \vert \neg C, D$
$\neg C \wedge \neg D$	$\Delta \vert \neg C, \neg D$

SDD

- $\Delta = (C \vee D) \wedge (A$
- $X = \{C, D\}, Y = \{A, B\}$

prime	sub
$C \wedge D$	$A \vee B$
$C \wedge \neg D$	$A \vee B$
$\neg C \wedge D$	$A \vee B$
$\neg C \wedge \neg D$	$\bot$

- $\Delta = [(C \vee D) \wedge (A \vee B)] \vee [(\neg C \wedge \neg D) \wedge (\bot)]$

SDD

- $\Delta = (C \vee D) \wedge (A$
- $X = \{C, D\}, Y =$
- $\Delta = [(C \vee D) \wedge (A$ $(\neg C \wedge \neg D) \wedge (\bot)]$

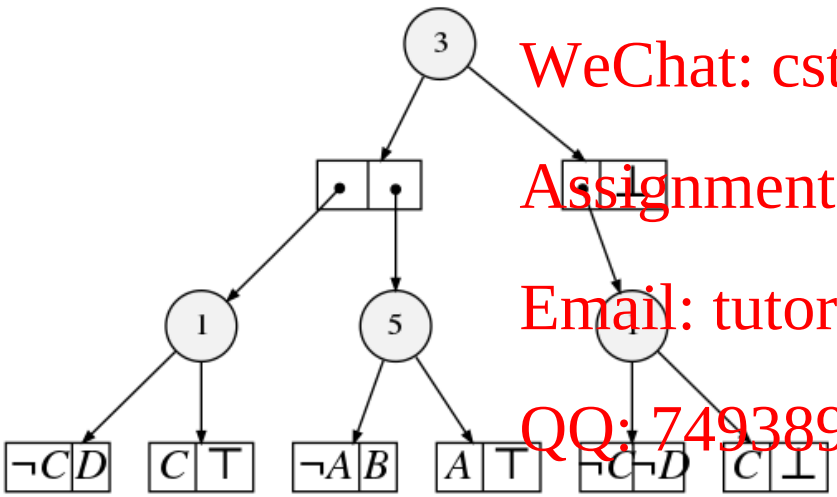


Figure: SDD

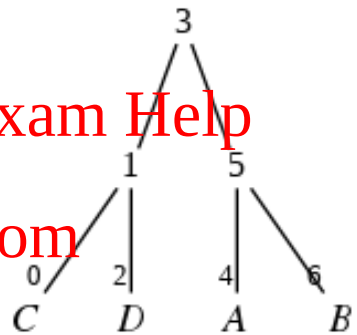
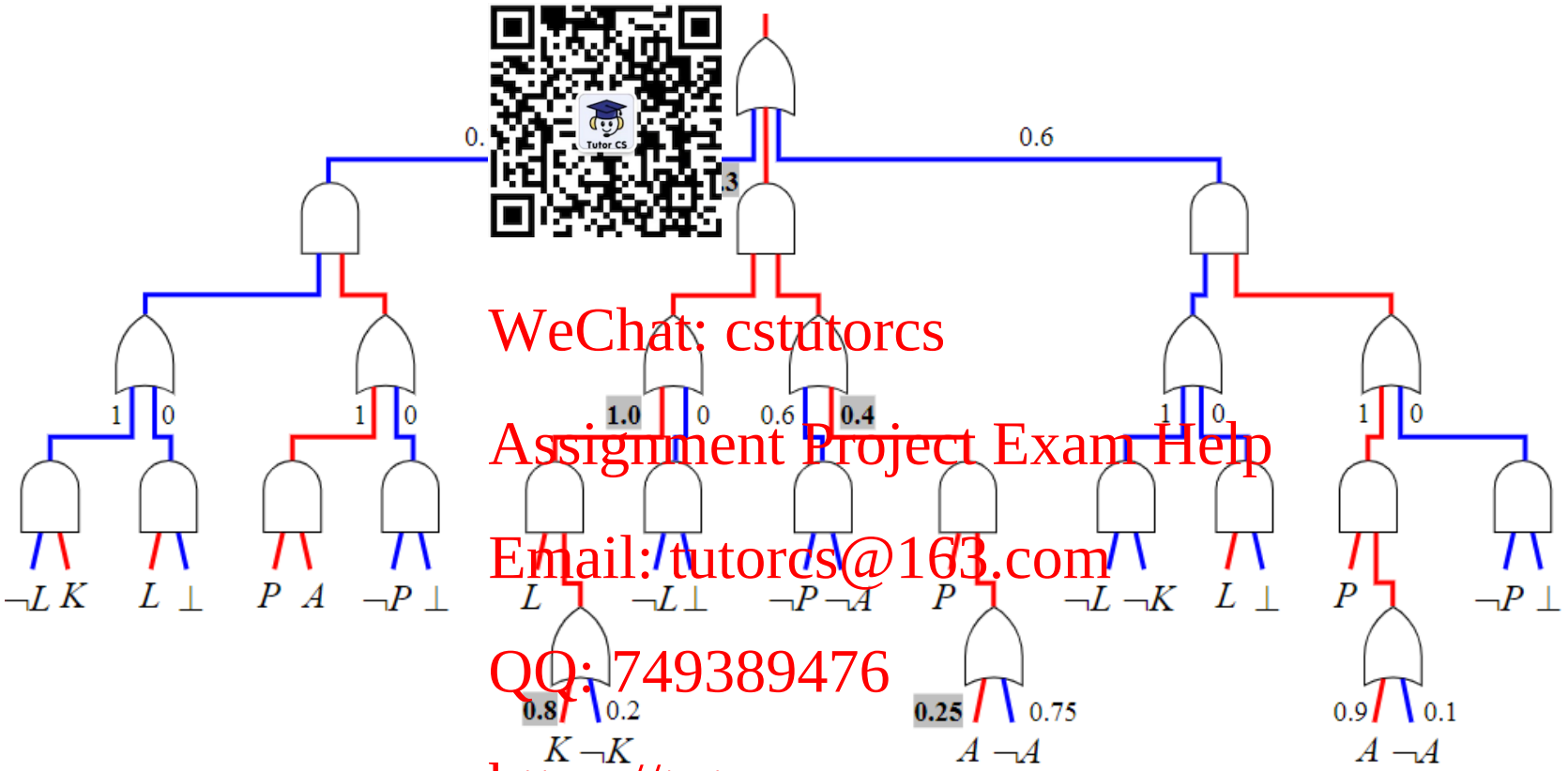


Figure: v-tree

PSDDs



Input: L, K, P, A $\Pr(L,K,P,A) = 0.3 \times 1.0 \times 0.8 \times 0.4 \times 0.25 = 0.024$

Bottom-Up Compilation

- How to construct 

L-SDD for a literal?
  - Simple

Bottom-Up Compilation

- How to construct an OBDD/SDD for a literal?
  - Simple
- How to construct an OBDD/SDD for $A \vee B$?
  - Construct for both literals
  - Disjoin

Bottom-Up Compilation

- How to construct an OBDD/SDD for a literal?
  - Simple
- How to construct an OBDD/SDD for $A \vee B$?
  - Construct for both literals
  - Disjoin
- How to construct an OBDD/SDD for $A \wedge B$?
  - Construct for both literals
  - Conjoin

Two Types of Compilation

Bottom-up compila

- Start from literals
- Keep doing the Apply operation

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Bottom-up compila

- Start from literals
- Keep doing the Apply operation

Top-down compilation

- Run Exhaustive-DPLL
- Compile the trace into a circuit

Top-Down Compilation vs. Bottom-Up Compilation

Incremental compila

- Assume that we h / compiled a given CNF Δ. What will happen if we add some clauses to Δ?
- Bottom-up compilation: just do Apply
- Top-down compilation: doesn't work.

Top-Down Compilation vs. Bottom-Up Compilation

Incremental compila

- Assume that we h / compiled a given CNF Δ. What will happen if we add some clauses to Δ?
- Bottom-up compilation: just do Apply
- Top-down compilation: doesn't work.

	Top-down	Bottom-up
Input	CNF-only	Any Boolean formula
Space performance	Acceptable	Poor
Incremental compilation	No	Yes

HW2 Solution: Q1

Q1:

- Identify a minimal unsatisfiable core of the following CNF:
- $\Delta = (X \vee Y) \wedge (X \vee Z) \wedge (\neg Y \vee W) \wedge (\neg Z \vee \neg W) \wedge$
  $(X \vee \neg Y) \wedge (\neg X \vee W) \wedge (\neg X \vee \neg W \vee V) \wedge$
  $(\neg X \vee \neg W \vee \neg V) \wedge (Z \vee \neg X) \wedge (\neg Z \vee W)$

HW2 Solution: Q1

Q1:

- Identify a minimal unsatisfiable core of the following CNF:
- $\Delta = (X \vee Y) \wedge (X \vee Z) \wedge (\neg Y \vee W) \wedge (\neg Z \vee \neg W) \wedge$
  $(X \vee \neg Y) \wedge (\neg X \vee W) \wedge (\neg X \vee \neg W \vee V) \wedge$
  $(\neg X \vee \neg W \vee \neg V) \wedge (Z \vee \neg X) \wedge (\neg Z \vee W)$

- One of the minimal UNSAT cores:
  $(X \vee Y) \wedge (X \vee \neg Y) \wedge (Z \vee \neg X) \wedge (\neg Z \vee \neg W) \wedge (\neg X \vee W)$

HW2 Solution: Q2

Q2

- Consider the CNF, $\Delta = (\neg X) \wedge (X \vee Y) \wedge (X \vee Z) \wedge (\neg Y \vee \neg Z)$
- Assuming that every clause has weight 1, construct a table that shows the costs of all worlds. What is the optimal solution for this MAX-SAT problem?

HW2 Solution: Q2

Q2

- Consider the CNF $(X \vee Y) \wedge (X \vee Z) \wedge (\neg Y \vee \neg Z)$
- Assuming that every clause has weight 1, construct a table that shows the costs of all truth assignments. What is the optimal solution for this MAX-SAT problem?

-

X	Y	Z	Cost
0	0	0	2
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	2

- All the solutions that have cost 1 are optimal. One of them is $\neg X, Y, Z$.

HW2 Solution: Q2

Q2

- Use MAX-SAT resolution to derive an empty clause (show trace)

- Variable order: X, Y, Z

- $\Delta = (\neg X) \wedge (X \vee Y) \wedge (X \vee Z) \wedge (\neg Y \vee \neg Z)$

- Resolve $\neg X$ and $X \vee Y$.

HW2 Solution: Q2

Q2

- Use MAX-SAT resolution to derive an empty clause (show trace)

- Variable order: X, Y, Z
- $\Delta = (\neg X) \wedge (X \vee Y) \wedge (X \vee Z) \wedge (\neg Y \vee \neg Z)$
- Resolve $\neg X$ and $X \vee Y$.
  - Y
  - $\neg X \vee \neg Y$
- $\Delta = (Y) \wedge (\neg X \vee \neg Y) \wedge (X \vee Z) \wedge (\neg Y \vee \neg Z)$

HW2 Solution: Q2

Q2

- Use MAX-SAT resolution to derive an empty clause (show trace)

- Variable order: X, Y, Z

- $\Delta = (Y) \wedge (\neg X \vee \neg Y) \wedge (X \vee Z) \wedge (\neg Y \vee \neg Z)$.

- Resolve $\neg X \vee \neg Y$ and $X \vee Z$

HW2 Solution: Q2

Q2

- Use MAX-SAT res derive an empty clause (show trace)

- Variable order: X, Y, Z
- $\Delta = (Y) \wedge (\neg X \vee \neg Y) \wedge (X \vee Z) \wedge (\neg Y \vee \neg Z)$.

- Resolve $\neg X \vee \neg Y$ and $X \vee Z$
  - $\neg Y \vee Z$
  - $\neg X \vee \neg Y \vee \neg Z$
  - $X \vee Z \vee Y$

- $\Delta = (Y) \wedge (\neg Y \vee Z) \wedge (\neg X \vee \neg Y \vee \neg Z) \wedge (X \vee Z \vee Y) \wedge (\neg Y \vee \neg Z)$

HW2 Solution: Q2

Q2

- Use MAX-SAT resolution to derive an empty clause (show trace)

- Variable order: X, Y, Z
- $\Delta = (Y) \wedge (\neg Y \vee Z) \wedge (\neg X \vee Y \vee \neg Z) \wedge (X \vee Z \vee Y) \wedge (\neg Y \vee \neg Z)$
- Resolve Y and $\neg Y \vee Z$
  - Z
  - $Y \vee \neg Z$
- $\Delta = Z \wedge (Y \vee \neg Z) \wedge (\neg X \vee \neg Y \vee \neg Z) \wedge (X \vee Z \vee Y) \wedge (\neg Y \vee \neg Z)$

HW2 Solution: Q2

Q2

- Use MAX-SAT resolution to derive an empty clause (show trace)

- Variable order: X, Y, Z
- $\Delta = Z \wedge (Y \vee \neg Z) \wedge (\neg X \vee \neg Y \vee \neg Z) \wedge (X \vee Z \vee Y) \wedge (\neg Y \vee \neg Z)$
- Resolve $Y \vee \neg Z$ and $\neg Y \vee \neg Z$
  - $\neg Z$
- $\Delta = Z \wedge \neg Z \wedge (\neg X \vee \neg Y \vee \neg Z) \wedge (X \vee Z \vee Y)$

HW2 Solution: Q2

Q2

- Use MAX-SAT resolution to derive an empty clause (show trace)

- Variable order: X, Y, Z
- $\Delta = Z \wedge \neg Z \wedge (\neg X \vee \neg Y \vee \neg Z) \wedge (X \vee Z \vee Y)$

- Resolve Z and $\neg Z$
  - $\bot$

- $\Delta = \bot \wedge (\neg X \vee \neg Y \vee \neg Z) \wedge (X \vee Z \vee Y)$

HW2 Solution: Q2

Q2

- Use MAX-SAT resolution to derive an empty clause (show trace)

- Variable order: X, Y, Z
- $\Delta = Z \wedge \neg Z \wedge (\neg X \vee \neg Y \vee \neg Z) \wedge (X \vee Z \vee Y)$
- Resolve Z and $\neg Z$
  - $\bot$
- $\Delta = \bot \wedge (\neg X \vee \neg Y \vee \neg Z) \wedge (X \vee Z \vee Y)$
- The minimal cost is 1. The residual clauses are $(\neg X \vee \neg Y \vee \neg Z), (X \vee Y \vee Z)$

HW2 Solution: Q3

Q3

- Consider the CNF $\Delta = (A \vee B \vee C) \wedge (\neg A \vee D \vee E)$, and assume that each clause has weight 1.
- What is the CNF which results from applying Max-Sat resolution to Δ?

HW2 Solution: Q3

Q3

- Consider the CNF $\Delta = (A \vee B \vee C) \wedge (\neg A \vee D \vee E)$, and assume that each clause has weight 1.
- What is the CNF which results from applying Max-Sat resolution to Δ?

- Apply MAX-SAT resolution with variable order A, B, C, D, E
- Resolve $(A \vee B \vee C)$ and $(\neg A \vee D \vee E)$
  - $B \vee C \vee D \vee E$
  - $A \vee B \vee C \vee \neg D$
  - $A \vee B \vee C \vee D \vee \neg E$
  - $\neg A \vee D \vee E \vee \neg B$
  - $\neg A \vee D \vee E \vee B \vee \neg C$
- $\Delta = (B \vee C \vee D \vee E) \wedge (A \vee B \vee C \vee \neg D) \wedge (A \vee B \vee C \vee D \vee \neg E) \wedge (\neg A \vee D \vee E \vee \neg B) \wedge (\neg A \vee D \vee E \vee B \vee \neg C)$

HW2 Solution: Q3

Q3

- Consider the CNF $\Delta = (A \vee B \vee C) \wedge (\neg A \vee D \vee E)$, and assume that each clause has weight 1.
- What is the CNF that results from applying Max-Sat resolution to Δ?

- Apply MAX-SAT resolution with variable order A, B, C, D, E
- Resolve $(A \vee B \vee C)$ and $(\neg A \vee D \vee E)$
  - $B \vee C \vee D \vee E$
  - $A \vee B \vee C \vee \neg D$
  - $A \vee B \vee C \vee D \vee \neg E$
  - $\neg A \vee D \vee E \vee \neg B$
  - $\neg A \vee D \vee E \vee B \vee \neg C$
- $\Delta = (B \vee C \vee D \vee E) \wedge (A \vee B \vee C \vee \neg D) \wedge (A \vee B \vee C \vee D \vee \neg E) \wedge (\neg A \vee D \vee E \vee \neg B) \wedge (\neg A \vee D \vee E \vee B \vee \neg C)$
- No resolutions can be done on A. No resolutions can be done on B, C, D, E. Done.

HW2 Solution: Q4

Q4

- $\Delta = (\neg D \vee \neg E \vee B) \wedge (\neg B \vee E \vee \neg A) \wedge (\neg D \vee C \vee \neg B) \wedge (\neg B \vee C \vee E)$
- What is the solution of the MAJ-SAT problem on this CNF? Justify your answer

HW2 Solution: Q4

Q4

- $\Delta = (\neg D \vee \neg E \vee A) \wedge (\neg D \vee E \vee \neg A) \wedge (\neg D \vee C \vee \neg B) \wedge (\neg B \vee C \vee E)$
- What is the solution of the MAJ-SAT problem on this CNF? Justify your answer

- $B = \text{false}$, $\neg D \vee \neg E = \text{true}$: 12 models.

- $B = \text{true}$, $C = \text{true}$, $\neg A \vee E = \text{true}$: 6 models

- This non-exhaustive count gives us 18 models. $18 > \frac{32}{2}$.

- The answer to the MAJ-SAT problem is yes.
- You can also solve this problem using a truth table or CDPLL. Δ has 20 models. $20 > \frac{32}{2}$.

HW2 Solution: Q4

Q4

- $\Delta = (\neg D \vee \neg E \vee A) \wedge (\neg D \vee E \vee \neg A) \wedge (\neg D \vee C \vee \neg B) \wedge (\neg B \vee C \vee E)$
- Using the split $X = \{A, B, C\}, Y = \{D, E\}$, what is the solution of the E-MAJ-SAT problem on this CNF?

- Consider a variable setting for X: $\{A = \text{true}, B = \text{false}, C = \text{true}\}$. Δ is satisfied as long as $\neg D \vee \neg E$ is true. Since $3 > \frac{4}{2}$, the majority of the truth-assignments for $Y = \{D, E\}$ satisfy Δ for this instance of X.

- The answer to the E-MAJ-SAT problem is yes.

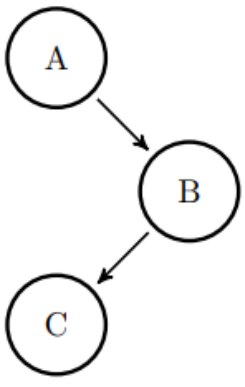
HW2 Solution: Q4

Q4

- $\Delta = (\neg D \vee \neg E \vee A) \wedge (\neg D \vee E \vee \neg A) \wedge (\neg D \vee C \vee \neg B) \wedge (\neg B \vee C \vee E)$
- Using the split $X = \{A, B, C\}$, $Y = \{D, E\}$, what is the solution of the MAJ-MAJ-Sat problem on this CNF?

- For all X with $B = \text{false}$, the majority of Y satisfy Δ. $2^2 = 4$.
- For $X = \{A = \text{false}, B = \text{true}, C = \text{true}\}$, $\Delta|\neg A, B, C = \top$, so the majority of Y satisfy Δ. 1
- $4 + 1 = 5, 5 > \frac{8}{2}$
- The answer to the MAJ-MAJ-SAT problem is yes.

HW2 Solution: Q5



A	B
T	0.8
F	0.9

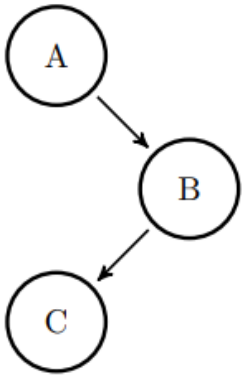
B	C	
	T	F
T	0.25	0.75
F	0.3	0.7

A	
T	F
0.6	0.4

Q5

- Compute $Pr(A = \text{true}, C = \text{false})$ by computing weighted model count.

HW2 Solution: Q5



	C	
B	T	F
F	0.3	0.7
T	0.25	0.75

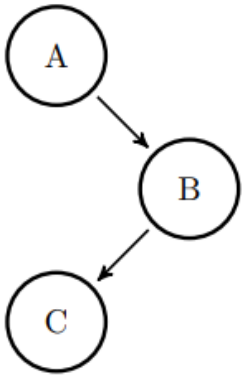
A	
T	F
0.6	0.4

Q5

- Compute $Pr(A = \text{true}, C = \text{false})$ by computing weighted model count.

- $\Delta = (A \Leftrightarrow P_A), (\neg A \Leftrightarrow P_{\neg A}),$
  $(A \wedge B \Leftrightarrow P_{B|A}), (A \wedge \neg B \Leftrightarrow P_{\neg B|A}),$
  $(\neg A \wedge B \Leftrightarrow P_{B|\neg A}), (\neg A \wedge \neg B \Leftrightarrow P_{\neg B|\neg A}),$
  $(B \wedge C \Leftrightarrow P_{C|B}), (B \wedge \neg C \Leftrightarrow P_{\neg C|B}),$
  $(\neg B \wedge C \Leftrightarrow P_{C|\neg B}), (\neg B \wedge \neg C \Leftrightarrow P_{\neg C|\neg B}),$

HW2 Solution: Q5



	C	
B	T	F
F	0.3	0.7
T	0.25	0.75

A	
T	F
0.6	0.4

Q5

- Compute $Pr(A = \text{true}, C = \text{false})$ by computing weighted model count.

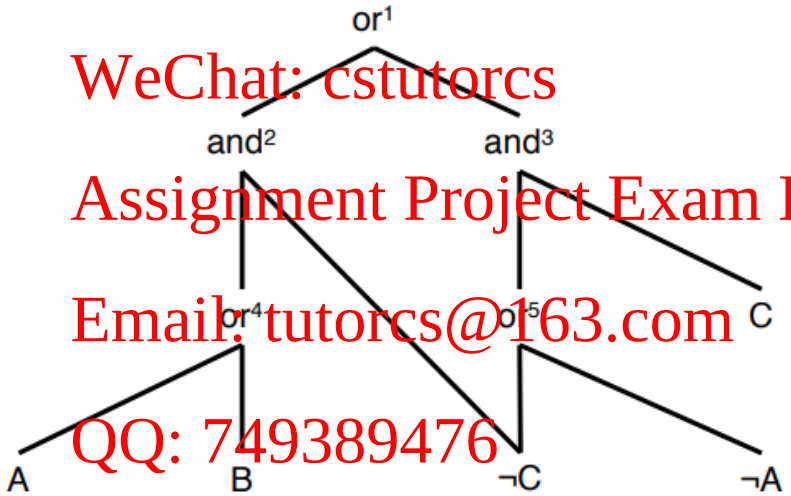
- Set $w(A) = w(B) = w(C) = 1$, $w(P_{x|y}) = \theta_{x|y}$, and $w(\neg P_{x|y}) = 1$.
- $\mathbf{P}(A = T, C = F) = wmc(\Delta \wedge (A = T, C = F))$

$$
\begin{aligned}
&= w(A, B, \neg C, P_A, P_{B|A}, P_{\neg C|B}, \ldots) \\
&\quad + w(A, \neg B, \neg C, P_A, P_{\neg B|A}, P_{\neg C|\neg B}, \ldots,) \\
&= \theta_A \theta_{B|A} \theta_{\neg C|B} + \theta_A \theta_{\neg B|A} \theta_{\neg C|\neg B} \\
&= 0.6 \cdot 0.8 \cdot 0.75 + 0.6 \cdot 0.2 \cdot 0.7 \\
&= 0.444
\end{aligned}
$$

HW2 Solution: Q6

Q6

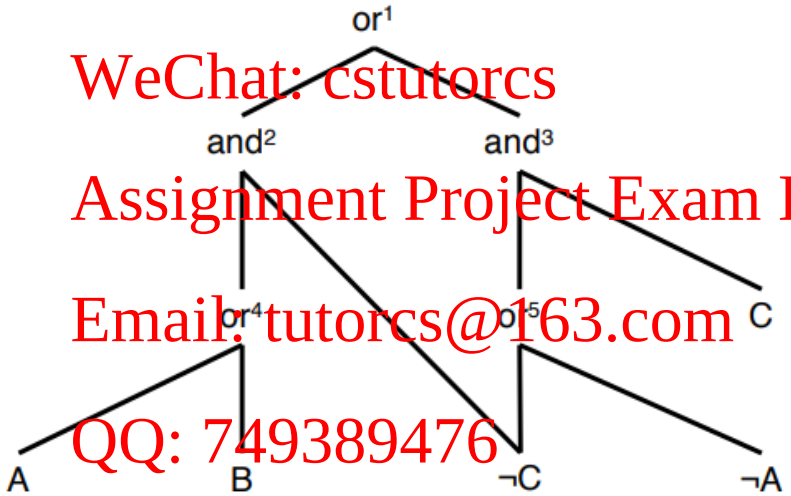
- Please identify and explain whether the NNF circuit is decomposable and/or deterministic.



HW2 Solution: Q6

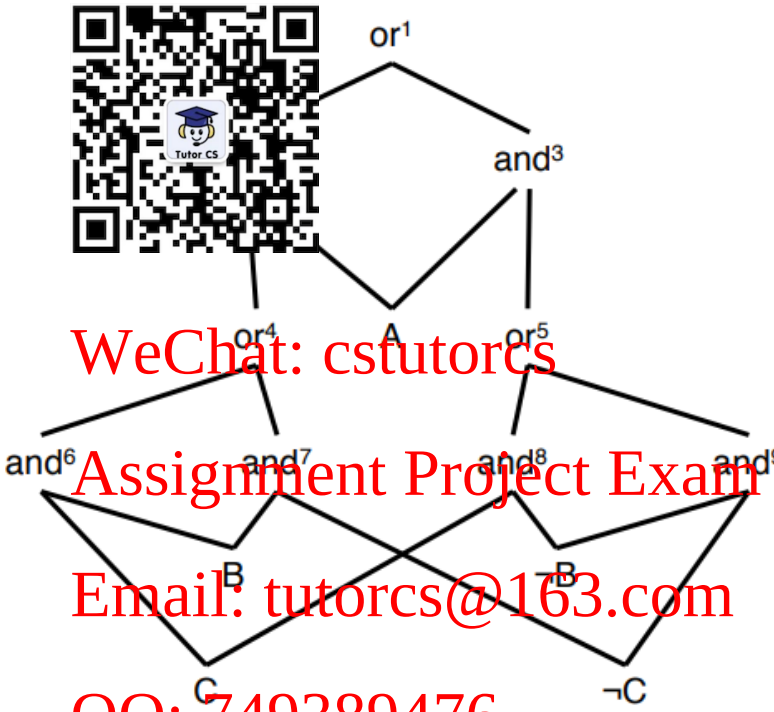
Q6

- Please identify and explain whether the NNF circuit is decomposable and/or deterministic.



- Not decomposabe: and^3 mentions C and $\neg C$.
- Not deterministic: or^4 mentions A and B, which are not mutually exclusive.

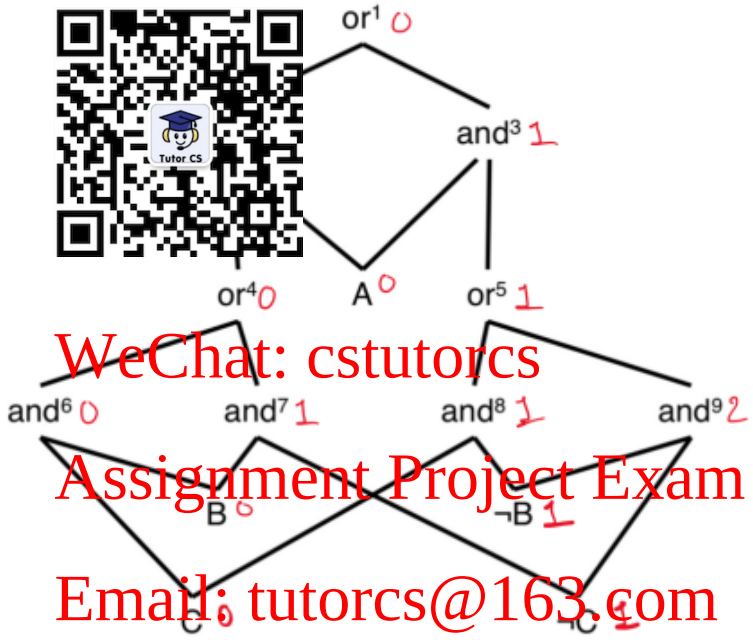
HW2 Solution: Q7



Q7

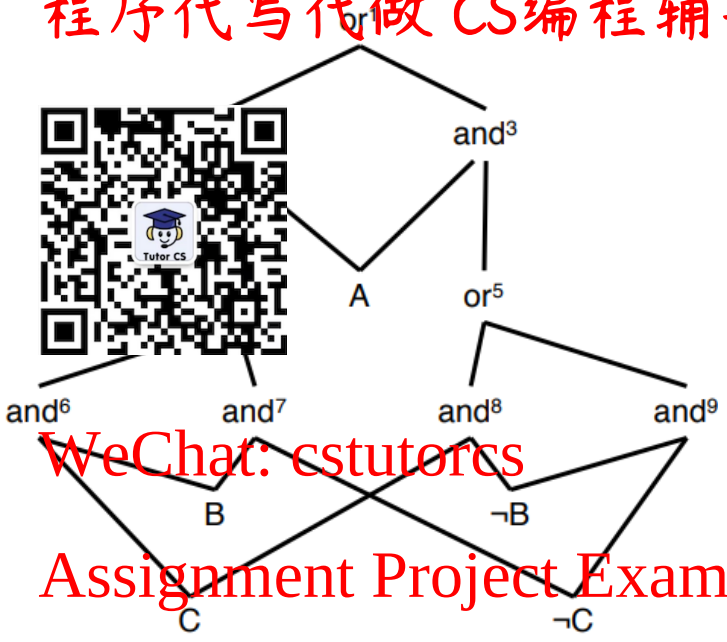
- Run the minimum cardinality query on this circuit

HW2 Solution: Q7



- The minimum cardinality (the number of variables set to false) is 0.
- Corresponding assignment: $A = \text{true}, B = \text{true}, C = \text{true}$
- Or you can define the minimum cardinality as the number of variables set to true. Then the minimum cardinality is 1.

HW2 Solution: Q7

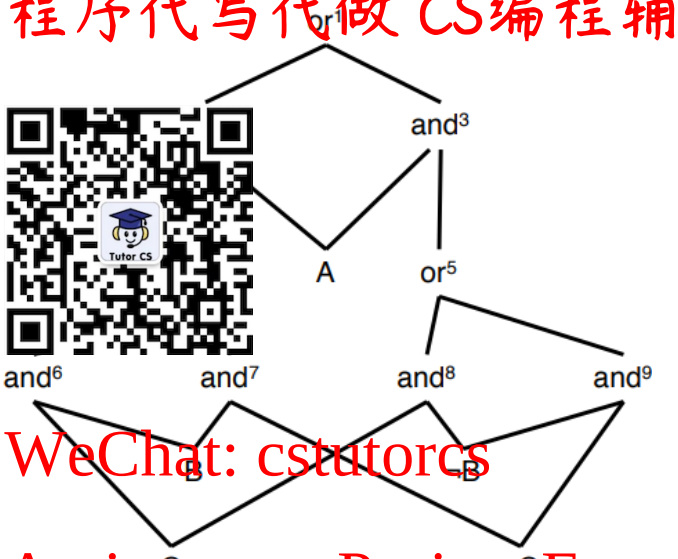


Q7

- Explain how to check whether the clause $(A \vee B)$ is entailed.

- The DNNF Δ entails $A \vee B$ if and only if $\Delta \wedge \neg(A \vee B)$ is inconsistent.
- After we set $A =$ false, $B =$ false, the DNNF returns false.
- Thus this DNNF entails $(A \vee B)$ by the refutation theorem.

HW2 Solution: Q7



Q7

- Explain how to existential quantify variable C, and show the resulting circuit.

- Remember that existential quantification distributes in DNNFs.
- To existentially quantify C from the DNNF, we simply set both C and C to true.
- This is because $\exists C \cdot C = \exists C \cdot \neg C = \top$

HW2 Solution: Q7

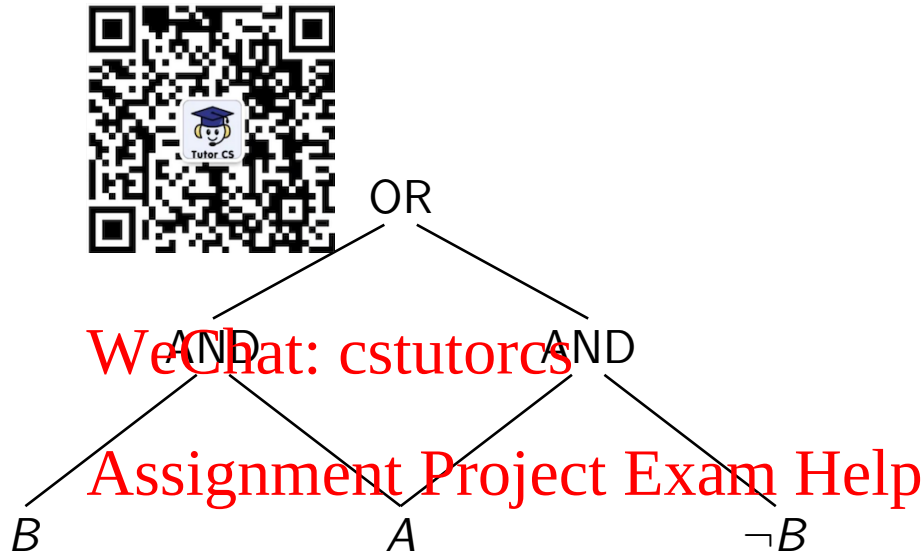
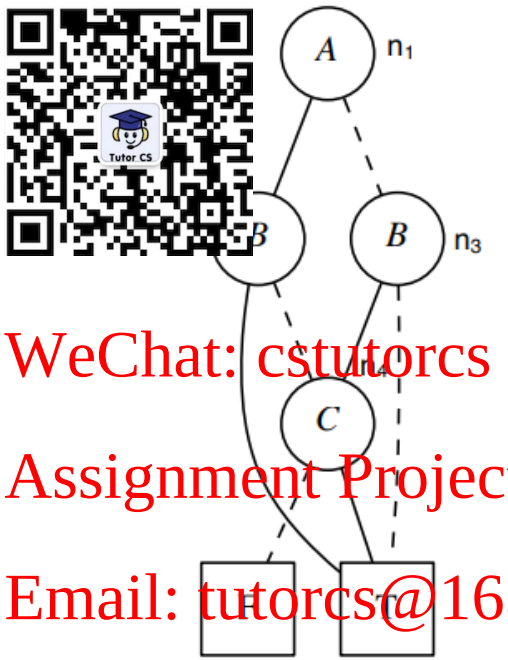


Figure: The circuit after quantification

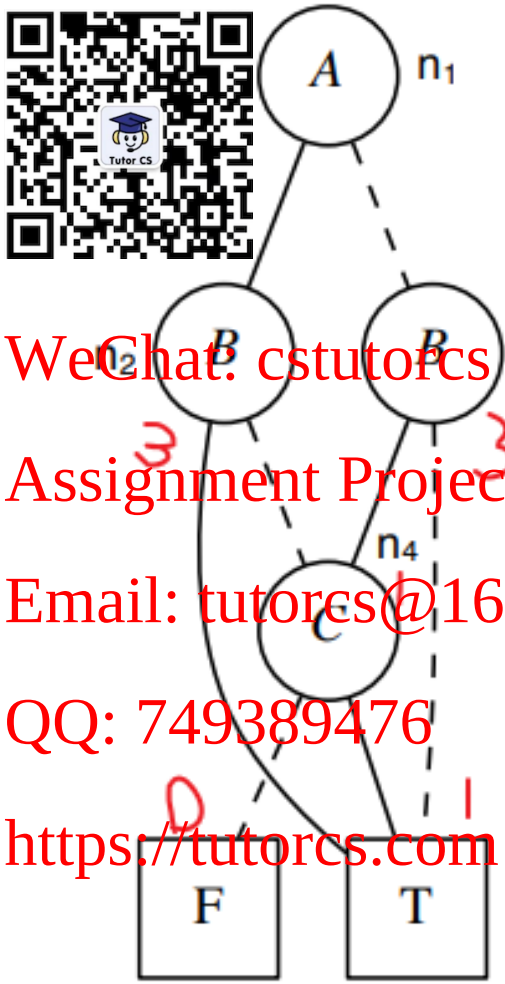
HW2 Solution: Q8



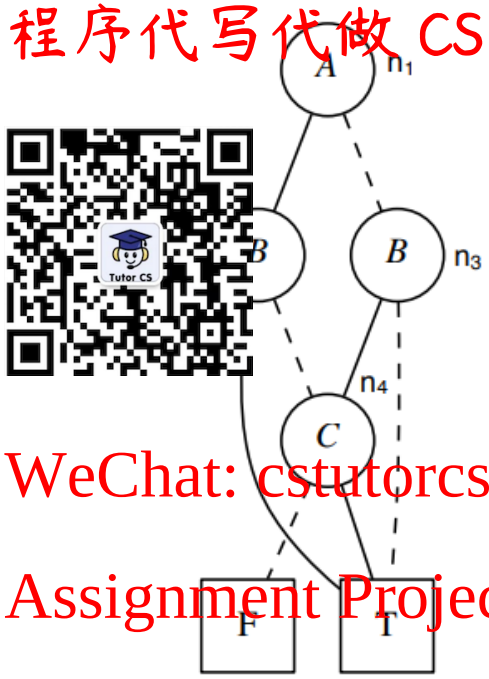
Q8

- Label each OBDD node with its model count (the number of models for the sentence represented by that node).

HW2 Solution: Q8



HW2 Solution: Q8



Q8

- Does the OBDD entail the clause $(A \vee \neg B)$?

- Remember the refutation theorem.
- Condition on $A =$ false and $B =$ true, the remaining circuit is the C node.
- No contradiction, so the OBDD doesn't entail $A \vee \neg B$

HW2 Solution: Q9

Q9

- The cardinality of a world equals the number of variables it sets to true.
- Consider a knowledge base Δ over variables $X_1, \ldots, X_N$ whose models consist of all worlds with odd cardinality.
- Given a variable order $X_1, \ldots, X_N$, what are the sub-functions after setting variables $X_1, \ldots, X_{N/2}$?

HW2 Solution: Q9

Q9

- Consider a knowledge base Δ over variables $X_1, \ldots, X_N$ whose models consist of all worlds with odd cardinality.
- Given a variable order $X_1, \ldots, X_N$, what are the sub-functions after setting variables $X_1, \ldots, X_{N/2}$?

- Let the given function be $f(X_1, \ldots, X_N)$, and α be a truth-assignment for variables $X_1, \ldots, X_{N/2}$
- $f(X_1, \ldots, X_N)|\alpha =$

$$\begin{cases} f(X_{N/2+1}, \ldots, X_N) & \text{if the cardinality of } \alpha \text{ is even} \\ \neg f(X_{N/2+1}, \ldots, X_N) & \text{if the cardinality of } \alpha \text{ is odd} \end{cases}$$

HW2 Solution: Q9

Q9

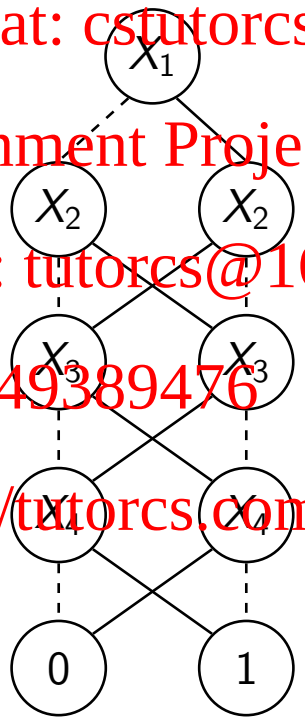
- Consider a knowledge base Δ over variables $X_1, \ldots, X_N$ whose models consist of all worlds with odd cardinality.
- How many nodes are in the reduced OBDD representing Δ with the variable order $X_1, \ldots X_N$?

- One node for X_1, two nodes of each of $X_2, \ldots, X_N$, two sinks.
- The answer is $2N + 1$.

HW2 Solution: Q9

Q9

- Consider a knowle[illegible] over variables $X_1, \ldots, X_N$ whose models consist of [illegible]with odd cardinality.
- Draw the OBDD [illegible]with the variable order X_1, X_2, X_3, X_4.



HW2 Solution: Q10

Q10

- Describe a property on NNF circuits which allows one to universally quantify variables in time linear in the circuit size.

HW2 Solution: Q10

Q10

- Describe a property on NNF circuits which allows one to universally quantify variables in time linear in the circuit size.

- Remember that we can existentially quantify DNNFs in linear time.

- This is because $\exists X(\alpha \vee \beta) = (\exists X \cdot \alpha) \vee (\exists X \cdot \beta)$ and $\exists X \cdot (\alpha \wedge \beta) = (\exists X \cdot \alpha) \wedge (\exists X \cdot \beta)$ for DNNFs.

- Hence, existentially quantify DNNFs is just quantifying the variable nodes (leaves).

HW2 Solution: Q10

Q10

- Describe a propert[illegible] circuits which allows one to universally quantify variables [illegible]ear in the circuit size.

- We proved in HW1 that $\forall X \cdot (\alpha \wedge \beta) = (\forall X \cdot \alpha) \wedge (\forall X \cdot \beta)$
- If $\forall X \cdot (\alpha \vee \beta) = (\forall X \cdot \alpha) \vee (\forall X \cdot \beta)$, then we can quantify circuits in linear time.
- Property: disjunctions don't share variables.
- $$\begin{aligned} \forall X \cdot (\alpha \vee \beta) &= (\alpha|X \vee \beta|X) \wedge (\alpha|\neg X \vee \beta|\neg X) \\ &= (\alpha|X \vee \beta) \wedge (\alpha|\neg X \vee \beta) \\ &= (\alpha|X \wedge \alpha|\neg X) \vee \beta \\ &= (\forall X \cdot \alpha) \vee \beta \\ &= (\forall X \cdot \alpha) \vee (\forall X \cdot \beta) \end{aligned}$$