Describe recursive backtracking algorithms for the following problems. *Don't worry about running times.*

1 Given an array $A[1\,..\,n]$ of integers, compute the length of a ***longest increasing subsequence***.

Solution:

[#1 of ∞] Add a sentinel value $A[0] = -\infty$. Let $LIS(i,j)$ denote the length of the longest increasing subsequence of $A[j\,..\,n]$ where every element is larger than $A[i]$. This function obeys the following recurrence:

$$LIS(i,j) = \begin{cases} 0 & \text{if } j > n \\ LIS(i, j+1) & \text{if } j \le n \text{ and } A[i] \ge A[j] \\ \max\{LIS(i,j+1), 1 + LIS(j, j+1)\} & \text{otherwise} \end{cases}$$

We need to compute $LIS(0,1)$.

Solution:

[#2 of ∞] Add a sentinel value $A[n+1] = -\infty$. Let $LIS(i,j)$ denote the length of the longest increasing subsequence of $A[1\,..\,j]$ where every element is smaller than $A[j]$. This function obeys the following recurrence:

$$LIS(i,j) = \begin{cases} 0 & \text{if } i < 1 \\ LIS(i-1, j) & \text{if } i \ge 1 \text{ and } A[i] \ge A[j] \\ \max\{LIS(i-1,j), 1 + LIS(i-1, i)\} & \text{otherwise} \end{cases}$$

We need to compute $LIS(n, n+1)$.

Solution:

[#3 of ∞] Let $LIS(i)$ denote the length of the longest increasing subsequence of $A[i\,..\,n]$ that begins with $A[i]$. This function obeys the following recurrence:

$$LIS(i) = \begin{cases} 1 & \text{if } A[j] \le A[i] \text{ for all } j > i \\ 1 + \max\left\{LIS(j) \;\middle|\; j > i \text{ and } A[j] > A[i]\right\} & \text{otherwise} \end{cases}$$

(The first case is actually redundant if we define $\max \varnothing = 0$.) We need to compute $\max_i LIS(i)$.

Solution:

[#4 of ∞] Add a sentinel value $A[0] = -\infty$. Let $LIS(i)$ denote the length of the longest increasing subsequence of $A[i\,..\,n]$ that begins with $A[i]$. This function obeys the following recurrence:

$$LIS(i) = \begin{cases} 1 & \text{if } A[j] \le A[i] \text{ for all } j > i \\ 1 + \max\left\{LIS(j) \;\middle|\; j > i \text{ and } A[j] > A[i]\right\} & \text{otherwise} \end{cases}$$

(The first case is actually redundant if we define $\max \varnothing = 0$.) We need to compute $LIS(0) - 1$; the -1 removes the sentinel $-\infty$ from the start of the subsequence.

Solution:

[#5 of ∞] Add sentinel values $A[0] = -\infty$ and $A[n+1] = \infty$. Let $LIS(j)$ denote the length of the longest increasing subsequence of $A[1\,..\,j]$ that ends with $A[j]$. This function obeys the following recurrence:

$$LIS(j) = \begin{cases} 1 & \text{if } j = 0 \\ 1 + \max\left\{ LIS(i) \;\middle|\; i < j \text{ and } A[i] < A[j] \right\} & \text{otherwise} \end{cases}$$

We need to compute $LIS(n+1) - 2$; the -2 removes the sentinels $-\infty$ and ∞ from the subsequence.

2 Given an array $A[1\,..\,n]$ of integers, compute the length of a ***longest decreasing subsequence***.

Solution:

[one of many] Add a sentinel value $A[0] = \infty$. Let $LDS(i, j)$ denote the length of the longest decreasing subsequence of $A[j\,..\,n]$ where every element is smaller than $A[i]$. This function obeys the following recurrence:

$$LDS(i,j) = \begin{cases} 0 & \text{if } j > n \\ LDS(i, j+1) & \text{if } j \le n \text{ and } A[i] \le A[j] \\ \max\{LDS(i, j+1), 1 + LDS(j, j+1)\} & \text{otherwise} \end{cases}$$

We need to compute $LDS(0, 1)$.

Solution:

[clever] Multiply every element of A by -1, and then compute the length of the longest increasing subsequence using the algorithm from problem 1.

3 Given an array $A[1\,..\,n]$ of integers, compute the length of a ***longest alternating subsequence***.

Solution:

[one of many] We define two functions:

- Let $LAS^{+}(i, j)$ denote the length of the longest alternating subsequence of $A[j\,..\,n]$ whose first element (if any) is larger than $A[i]$ and whose second element (if any) is smaller than its first.
- Let $LAS^{-}(i, j)$ denote the length of the longest alternating subsequence of $A[j\,..\,n]$ whose first element (if any) is smaller than $A[i]$ and whose second element (if any) is larger than its first.

These two functions satisfy the following mutual recurrences:

$$LAS^{+}(i,j) = \begin{cases} 0 & \text{if } j > n \\ LAS^{+}(i, j+1) & \text{if } j \le n \text{ and } A[j] \le A[i] \\ \max\left\{LAS^{+}(i, j+1), 1 + LAS^{-}(j, j+1)\right\} & \text{otherwise} \end{cases}$$

$$LAS^{-}(i,j) = \begin{cases} 0 & \text{if } j > n \\ LAS^{-}(i, j+1) & \text{if } j \le n \text{ and } A[j] \ge A[i] \\ \max\left\{LAS^{-}(i, j+1), 1 + LAS^{+}(j, j+1)\right\} & \text{otherwise} \end{cases}$$

To simplify computation, we consider two different sentinel values $A[0]$. First we set $A[0] = -\infty$ and let $\ell^{+} = LAS^{+}(0, 1)$. Then we set $A[0] = +\infty$ and let $\ell^{-} = LAS^{-}(0, 1)$. Finally, the length of the longest alternating subsequence of A is $\max\{\ell^{+}, \ell^{-}\}$.

Solution:

[one of many] We define two functions:

- Let $LAS^{+}(i)$ denote the length of the longest alternating subsequence of $A[i \,..\, n]$ that starts with $A[i]$ and whose second element (if any) is larger than $A[i]$.
- Let $LAS^{-}(i)$ denote the length of the longest alternating subsequence of $A[i \,..\, n]$ that starts with $A[i]$ and whose second element (if any) is smaller than $A[i]$.

These two functions satisfy the following mutual recurrences:

$$LAS^{+}(i) = \begin{cases} 1 & \text{if } A[j] \leq A[i] \text{ for all } j > i \\ 1 + \max\left\{LAS^{-}(j) \;\middle|\; j > i \text{ and } A[j] > A[i]\right\} & \text{otherwise} \end{cases}$$

$$LAS^{-}(i) = \begin{cases} 1 & \text{if } A[j] \geq A[i] \text{ for all } j > i \\ 1 + \max\left\{LAS^{+}(j) \;\middle|\; j > i \text{ and } A[j] < A[i]\right\} & \text{otherwise} \end{cases}$$

We need to compute $\max_i \max\left\{LAS^{+}(i), LAS^{-}(i)\right\}$.

To think about later:

1 Given an array $A[1 \,..\, n]$ of integers, compute the length of a longest ***convex*** subsequence of A.

Solution:

Let $LCS(i, j)$ denote the length of the longest convex subsequence of $A[i \,..\, n]$ whose first two elements are $A[i]$ and $A[j]$. This function obeys the following recurrence:

$$LCS(i, j) \;=\; 1 + \max\left\{LCS(j, k) \;\middle|\; j < k \leq n \text{ and } A[i] + A[k] > 2A[j]\right\}$$

Here we define $\max \varnothing = 0$; this gives us a working base case. The length of the longest convex subsequence is $\max_{1 \leq i < j \leq n} LCS(i, j)$.

Solution:

[with sentinels] Assume without loss of generality that $A[i] \geq 0$ for all i. (Otherwise, we can add $|m|$ to each $A[i]$, where m is the smallest element of $A[1 \,..\, n]$.) Add two sentinel values $A[0] = 2M + 1$ and $A[-1] = 4M + 3$, where M is the largest element of $A[1 \,..\, n]$.

Let $LCS(i, j)$ denote the length of the longest convex subsequence of $A[i \,..\, n]$ whose first two elements are $A[i]$ and $A[j]$. This function obeys the following recurrence:

$$LCS(i, j) \;=\; 1 + \max\left\{LCS(j, k) \;\middle|\; j < k \leq n \text{ and } A[i] + A[k] > 2A[j]\right\}$$

Here we define $\max \varnothing = 0$; this gives us a working base case.

Finally, we claim that the length of the longest convex subsequence of $A[1 \,..\, n]$ is $LCS(-1, 0) - 2$.

Proof: First, consider any convex subsequence S of $A[1 \,..\, n]$, and suppose its first element is $A[i]$. Then we have $A[-1] - 2A[0] + A[i] = 4M + 3 - 2(2M + 1) + A[i] = A[i] + 1 > 0$, which implies that $A[-1] \cdot A[0] \cdot S$ is a convex subsequence of $A[-1 \,..\, n]$. So the longest convex subsequence of $A[1 \,..\, n]$ has length at most $LCS(-1, 0) - 2$.

On the other hand, removing $A[-1]$ and $A[0]$ from any convex subsequence of $A[-1 \,..\, n]$ laves a convex subsequence of $A[1 \,..\, n]$. So the longest subsequence of $A[1 \,..\, n]$ has length at least $LCS(-1, 0) - 2$. ■

2 Given an array $A[1..n]$, compute the length of a longest ***palindrome*** subsequence of A.

Solution:

[naive] Let $LPS(i, j)$ denote the length of the longest palindrome subsequence of $A[i\,..\,j]$. This function obeys the following recurrence:

$$LPS(i,j) = \begin{cases} 0 & \text{if } i > j \\ 1 & \text{if } i = j \\ \max\left\{ \begin{array}{c} LPS(i+1,j) \\ LPS(i,j-1) \end{array} \right\} & \text{if } i < j \text{ and } A[i] \neq A[j] \\ \max\left\{ \begin{array}{c} 2 + LPS(i+1,j-1) \\ LPS(i+1,j) \\ LPS(i,j-1) \end{array} \right\} & \text{otherwise} \end{cases}$$

We need to compute $LPS(1, n)$.

Solution:

[with greedy optimization] Let $LPS(i, j)$ denote the length of the longest palindrome subsequence of $A[i\,..\,j]$. Before stating a recurrence for this function, we make the following useful observation.

Claim 0.1. *If $i < j$ and $A[i] = A[j]$, then $LPS(i, j) = 2 + LPS(i + 1, j - 1)$.*

Proof: Suppose $i < j$ and $A[i] = A[j]$. Fix an arbitrary longest palindrome subsequence S of $A[i\,..\,j]$. There are four cases to consider:

- If S uses neither $A[i]$ nor $A[j]$, then $A[i] \bullet S \bullet A[j]$ is a palindrome subsequence of $A[i\,..\,j]$ that is longer than S, which is impossible.
- Suppose S uses $A[i]$ but not $A[j]$. Let $A[k]$ be the last element of S. If $k = i$, then $A[i] \bullet A[j]$ is a palindrome subsequence of $A[i\,..\,j]$ that is longer than S, which is impossible. Otherwise, replacing $A[k]$ with $A[j]$ gives us a palindrome subsequence of $A[i\,..\,j]$ with the same length as S that uses both $A[i]$ and $A[j]$.
- Suppose S uses $A[j]$ but not $A[i]$. Let $A[h]$ be the first element of S. If $h = j$, then $A[i] \bullet A[j]$ is a palindrome subsequence of $A[i\,..\,j]$ that is longer than S, which is impossible. Otherwise, replacing $A[h]$ with $A[i]$ gives us a palindrome subsequence of $A[i\,..\,j]$ with the same length as S that uses both $A[i]$ and $A[j]$.
- Finally, S might include both $A[i]$ and $A[j]$.

In all cases, we find either a contradiction or a longest palindrome subsequence of $A[i\,..\,j]$ that uses both $A[i]$ and $A[j]$. ∎

Claim 1 implies that the function LPS satisfies the following recurrence:

$$LPS(i,j) = \begin{cases} 0 & \text{if } i > j \\ 1 & \text{if } i = j \\ \max\left\{LPS(i+1,j),\ LPS(i,j-1)\right\} & \text{if } i < j \text{ and } A[i] \neq A[j] \\ 2 + LPS(i+1,j-1) & \text{otherwise} \end{cases}$$

We need to compute $LPS(1, n)$.