Here are several problems that are easy to solve in $O(n)$ time, essentially by brute force. Your task is to design algorithms for these problems that are significantly faster.

1 Suppose we are given an array $A[1..n]$ of n distinct integers, which could be positive, negative, or zero, sorted in increasing order so that $A[1] < A[2] < \cdots < A[n]$.

1.A. Describe a fast algorithm that either computes an index i such that $A[i] = i$ or correctly reports that no such index exists.

> ### Solution:
>
> Suppose we define a second array $B[1..n]$ by setting $B[i] = A[i] - i$ for all i. For every index i we have
> $$B[i] \;=\; A[i] - i \;\leq\; (A[i+1] - 1) - i \;=\; A[i+1] - (i+1) \;=\; B[i+1],$$
> so this new array is sorted in increasing order. Clearly, $A[i] = i$ if and only if $B[i] = 0$. So we can find an index i such that $A[i] = i$ by performing a binary search in B. We don't actually need to compute B in advance; instead, whenever the binary search needs to access some value $B[i]$, we can just compute $A[i] - i$ on the fly instead!
>
> Here are two formulations of the resulting algorithm, first recursive (keeping the array A as a global variable), and second iterative.
>
> ```
> // Return any index i such that ℓ ≤ i ≤ r and A[i] = i
> FindMatch(ℓ, r):
>     if ℓ > r
>         return NONE
>     mid ← (ℓ + r)/2
>     if A[mid] = mid              // B[mid] = 0
>         return mid
>     else if A[mid] < mid         // B[mid] < 0
>         return FindMatch(mid + 1, r)
>     else                         // B[mid] > 0
>         return FindMatch(ℓ, mid − 1)
> ```
>
> ```
> FindMatch(A[1..n]):
>     hi ← n
>     lo ← 1
>     while lo ≤ hi
>         mid ← (lo + hi)/2
>         if A[mid] = mid          // B[mid] = 0
>             return mid
>         else if A[mid] < mid     // B[mid] < 0
>             lo ← mid + 1
>         else                     // B[mid] > 0
>             hi ← mid − 1
>     return NONE
> ```
>
> In both formulations, the algorithm **is** binary search, so it runs in $O(\log n)$ time.

1.B. Suppose we know in advance that $A[1] > 0$. Describe an even faster algorithm that either computes an index i such that $A[i] = i$ or correctly reports that no such index exists. (**Hint:** This is **really** easy.)

Solution:

The following algorithm solves this problem in $O(1)$ time:

$$\underline{\textbf{FindMatchPos}(A[1\mathinner{.\,.}n])\text{:}}$$
> **if** $A[1] = 1$
> > **return** 1
>
> **else**
> > **return** NONE

Again, the array $B[1\mathinner{.\,.}n]$ defined by setting $B[i] = A[i] - i$ is sorted in increasing order. It follows that if $A[1] > 1$ (that is, $B[1] > 0$), then $A[i] > i$ (that is, $B[i] > 0$) for every index i. $A[1]$ cannot be less than 1.

2 Suppose we are given an array $A[1\mathinner{.\,.}n]$ such that $A[1] \geq A[2]$ and $A[n-1] \leq A[n]$. We say that an element $A[x]$ is a ***local minimum*** if both $A[x-1] \geq A[x]$ and $A[x] \leq A[x+1]$. For example, there are exactly six local minima in the following array:

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| 9 | 7 | 7 | 2 | 1 | 3 | 7 | 5 | 4 | 7 | 3 | 3 | 4 | 8 | 6 | 9 |

Describe and analyze a fast algorithm that returns the index of one local minimum. For example, given the array above, your algorithm could return the integer 9, because $A[9]$ is a local minimum. (**Hint:** With the given boundary conditions, any array **must** contain at least one local minimum. Why?)

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Solution:

The following algorithm solves this problem in $O(\log n)$ time:

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$$\underline{\textbf{LocalMin}(A[1\ldots n])\text{ :}}$$
> **if** $n < 100$
> > find the smallest element in A by brute force
>
> $m \leftarrow \lfloor n/2 \rfloor$
> **if** $A[m] < A[m+1]$
> > **return** **LocalMin**$(A[1\ldots m+1])$
>
> **else**
> > **return** **LocalMin**$(A[m\ldots n])$

If n is less than 100, then a brute-force search runs in $O(1)$ time. There's nothing special about 100 here; any other constant will do.

Otherwise, if $A[n/2] < A[n/2+1]$, the subarray $A[1\ldots n/2 + 1]$ satisfies the precise boundary conditions of the original problem, so the recursion fairy will find local minimum inside that subarray.

Finally, if $A[n/2] > A[n/2 + 1]$, the subarray $A[n/2\ldots n]$ satisfies the precise boundary conditions of the original problem, so the recursion fairy will find local minimum inside that subarray.

The running time satisfies the recurrence $T(n) \leq T(\lceil n/2 \rceil + 1) + O(1)$. Except for the $+1$ and the ceiling in the recursive argument, which we can ignore, this is the binary search recurrence, whose solution is $T(n) = O(\log n)$.

Alternatively, we can observe that $\lceil n/2 \rceil + 1 < 2n/3$ when $n \geq 100$, and therefore $T(n) \leq T(2n/3) + O(1)$, which implies $T(n) = O(\log_{3/2} n) = O(\log n)$.

3   Suppose you are given two sorted arrays $A[1 .. n]$ and $B[1 .. n]$ containing distinct integers. Describe a fast algorithm to find the median (meaning the nth smallest element) of the union $A \cup B$. For example, given the input

$$A[1..8] = [0, 1, 6, 9, 12, 13, 18, 20] \qquad B[1..8] = [2, 4, 5, 8, 17, 19, 21, 23]$$

your algorithm should return the integer 9. (**Hint:** What can you learn by comparing one element of A with one element of B?)

Solution:

The following algorithm solves this problem in $O(\log n)$ time:

> $\underline{\text{Median}(A[1 .. n], B[1 ... n]) :}$
> **if** $n < 10^{100}$
>   use brute force
> **else if** $A[n/2] > B[n/2]$
>   **return** $\text{Median}(A[1 .. n/2], B[n/2 + 1 .. n])$
> **else**
>   **return** $\text{Median}(A[n/2 .. n], B[1 .. n/2])$

Suppose $A[n/2] > B[n/2]$. Then $A[n/2 + 1]$ is larger than all n elements in $A[1 .. n/2] \cup B[1 .. n/2]$, and therefore larger than the median of $A \cup B$, so we can discard the upper half of A. Similarly, $B[n/2 - 1]$ is smaller than all n elements of

$A[n/2 .. n] \cup B[n/2 .. n]$, and therefore smaller than the median of $A \cup B$, so we can discard the lower half of B. Because we discard the same number of elements from each array, the median of the remaining subarrays is the median of the original $A \cup B$.

To think about later:

4   Now suppose you are given two sorted arrays $A[1 .. m]$ and $B[1 .. n]$ and an integer k. Describe a fast algorithm to find the kth smallest element in the union $A \cup B$. For example, given the input

$$A[1 .. 8] = [0, 1, 6, 9, 12, 13, 18, 20] \qquad B[1 .. 5] = [2, 5, 7, 17, 19] \qquad k = 6$$

your algorithm should return the integer 7.

Solution:

The following algorithm solves this problem in $O(\log \min\{k, m + n - k\}) = O(\log(m + n))$ time:

> $\underline{\text{Select}(A[1 .. m], B[1 .. n], k) :}$
> **if** $k < (m + n)/2$
>   **return** $\text{Median}(A[1 .. k], B[1 .. k])$
> **else**
>   **return** $\text{Median}(A[k - n .. m], B[k - m .. n])$

Here, MEDIAN is the algorithm from problem 3 with one minor tweak. If MEDIAN wants an entry in either A or B that is outside the bounds of the original arrays, it uses the value $-\infty$ if the index is too low, or ∞ if the index is too high, instead of creating a core dump.