Prove that each of the following problems is NP-hard.

**1** Given an undirected graph $G$, does $G$ contain a simple path that visits all but 374 vertices?

## Solution:

We prove this problem is NP-hard by a reduction from the undirected Hamiltonian path problem. Given an arbitrary graph $G$, let $H$ be the graph obtained from $G$ by adding 374 isolated vertices. Call a path in $H$ ***almost-Hamiltonian*** if it visits all but 374 vertices. I claim that $G$ contains a Hamiltonian path if and only if $H$ contains an almost-Hamiltonian path.

- $\Rightarrow$ Suppose $G$ has a Hamiltonian path $P$. Then $P$ is an almost-Hamiltonian path in $H$, because it misses only the 374 isolated vertices.
- $\Leftarrow$ Suppose $H$ has an almost-Hamiltonian path $P$. This path must miss all 374 isolated vertices in $H$, and therefore must visit every vertex in $G$. Every edge in $H$, and therefore every edge in $P$, is also an edge in $G$. We conclude that $P$ is a Hamiltonian path in $G$.

Given $G$, we can easily build $H$ in polynomial time by brute force.

**2** Given an undirected graph $G$, does $G$ have a spanning tree with at most 374 leaves?



## Solution:

We prove this problem is NP-hard by a reduction from the undirected Hamiltonian path problem.[1] Given an arbitrary graph $G$, let $H$ be the graph obtained from $G$ by adding the following vertices and edges:



- First we add a vertex $z$ with edges to every other vertex in $G$.
- Then we add 373 vertices $\ell_1, \dots, \ell_{373}$, each with edges to $z$ and nothing else.



Call a spanning tree of $H$ ***almost-Hamiltonian*** if it has at most 374 leaves. I claim that $G$ contains a Hamiltonian path if and only if $H$ contains an almost-Hamiltonian spanning tree.

- $\Rightarrow$ Suppose $G$ has a Hamiltonian path $P$. Suppose $P$ starts at vertex $s$ and ends at vertex $t$. Let $T$ be subgraph of $H$ obtained by adding the edge $tz$ and all possible edges $z\ell_i$. Then $T$ is a spanning tree of $H$ with exactly 374 leaves, namely $s$ and all 373 new vertices $\ell_i$.
- $\Leftarrow$ Suppose $H$ has an almost-Hamiltonian spanning tree $T$. Every node $\ell_i$ is a leaf of $T$, so $T$ must consist of the 373 edges $z\ell_i$ and a simple path from $z$ to some vertex $s$ of $G$. Let $t$ be the only neighbor of $z$ in $T$ that is not a leaf $\ell_i$, and let $P$ be the unique path in $T$ from $s$ to $t$. This path visits every vertex of $G$; in other words, $P$ is a Hamiltonian path in $G$.

Given $G$, we can easily build $H$ in polynomial time by brute force.

**3** Recall that a 5-coloring of a graph $G$ is a function that assigns each vertex of $G$ a "color" from the set $\{0, 1, 2, 3, 4\}$, such that for any edge $uv$, vertices $u$ and $v$ are assigned different "colors". A 5-coloring is ***careful*** if the colors assigned to adjacent vertices are not only distinct, but differ by more than 1 (mod 5). Prove that deciding whether a given graph has a careful 5-coloring is NP-hard.

## Solution:

We prove that careful 5-coloring is NP-hard by reduction from the standard 5COLOR problem.

Given a graph $G$, we construct a new graph $H$ by replacing each edge in $G$ with a path of length three. I claim that $H$ has a careful 5-coloring if and only if $G$ has a (not necessarily careful) 5-coloring.

$\Longleftarrow$ Suppose $G$ has a 5-coloring. Consider a single edge $uv$ in $G$, and suppose $\mathit{color}(u) = a$ and $\mathit{color}(v) = b$. We color the path from $u$ to $v$ in $H$ as follows:

- If $b = (a + 1) \bmod 5$, use colors $(a,\ (a + 2) \bmod 5,\ (a - 1) \pmod 5,\ b)$.
- If $b = (a - 1) \bmod 5$, use colors $(a,\ (a - 2) \bmod 5,\ (a + 1) \pmod 5,\ b)$.
- Otherwise, use colors $(a,\ b,\ a,\ b)$.

In particular, every vertex in $G$ retains its color in $H$. The resulting 5-coloring of $H$ is careful.

$\Longrightarrow$ On the other hand, suppose $H$ has a careful 5-coloring. Consider a path $(u, x, y, v)$ in $H$ corresponding to an arbitrary edge $uv$ in $G$. Without loss of generality, say $\mathit{color}(u) = 0$; there are exactly eight careful colorings of this path with $\mathit{color}(u) = 0$, namely: $(0, 2, 0, 2)$, $(0, 2, 0, 3)$, $(0, 2, 4, 1)$, $(0, 2, 4, 2)$, $(0, 3, 0, 3)$, $(0, 3, 0, 2)$, $(0, 3, 1, 3)$, $(0, 3, 1, 4)$. It follows immediately that $\mathit{color}(u) \neq \mathit{color}(v)$. Thus, if we color each vertex of $G$ with its color in $H$, we obtain a valid 5-coloring of $G$.

Given $G$, we can clearly construct $H$ in polynomial time.