You are not allowed to post the assignment questions anywhere; however, you are allowed to search the internet (just cite your resources if you find any). You are also allowed to bounce ideas off classmates and TAs, but at the end, you must write your own solutions.

Q1. Let $f(n) = a_k n^k + \ldots + a_1 n + a_0$. Prove that $f(n) = O(n^k)$. In other words, show that there exists n_0, M natural numbers such that $\forall n > n_0, |f(n)| \leq Mn^k$.

Hint: Use the triangle inequality and the fact that $n^j \geq n^i$ for all natural n when $j > i$.

Q2. Let L be some decision problem (you can think of it as a subset of $\mathbb{N}$, or as a subset of $\{0,1\}^*$). Prove that, there exists a polynomial time decider for L iff $L \in \bigcup_{k \in \mathbb{N}} TIME(n^k)$.

Q3. A *Hamiltonian path* is a path in the graph between two vertices which visits each vertex exactly once. A *Hamiltonian cycle* starts and ends at the same vertex and visits every vertex exactly once.

Define the following languages

$$HAMPATH = \{(G, s, t) : G \text{ is an undirected graph with a Hamiltonian path from } s \text{ to } t\}$$

$$HAMCYCLE = \{G : G \text{ is an undirected graph with a Hamiltonian cycle }\}$$

Prove that $HAMPATH \leq_p HAMCYCLE$.

Q4. The **3-colouring problem** is the following: Given a map consisting of connected countries, assign each country one of three colours (say red, blue, and green) so that no two bordering countries share the same colour.

The set of 3-colourable maps can be abstracted as the following language:

$$\begin{aligned} 3C = \{G : G = (V, E) &\text{ is an undirected loop-free graph and there is a function } f : V \to \{0, 1, 2\} \\ &\text{such that } (\forall v_1, v_2 \in V)[(v_1, v_2) \in E \Rightarrow f(v_1) \neq f(v_2)].\} \end{aligned}$$

Prove that $3C$ is NP-complete.

Q5. A **graph isomorphism** between two (undirected) graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a *bijective* function $f : V_1 \to V_2$ such that for all vertices $u, v \in V_1$,

$$(u, v) \in E_1 \iff (f(u), f(v)) \in E_2$$

If such an isomorphism exists, then we say that G_1 and G_2 are **isomorphic**. Intuitively, two graphs are isomorphic if they "look the same" after moving around the vertices.

We will define the **Graph Isomorphism problem** as

$$Iso = \{(G, H) : G \text{ and } H \text{ are isomorphic (undirected) graphs }\}$$

Prove that Iso is NP. Is Iso NP-complete?

Q6. Define the **Subgraph Isomorphism problem** as

$$SubIso = \{(G, H) : G \text{ and } H \text{ are (undirected) graphs, and}$$

$$H \text{ is isomorphic to some subgraph of } G\}$$

Prove that $SubIso$ is NP-complete.

Q7. (Bonus) On the topic of graph isomorphsim, consider the following special case of the graph isomorphism problem:

$$TIso_r = \{(T_1, T_2) \colon\ T_1 \text{ and } T_2 \text{ are ismoprhic trees of height at most } r\}.$$

We say that a tree graph has height at most r if there is no path (branch) in the tree that contains more than r edges. Assume that our trees can be infinite.

(a) Prove that $TIso_1$ is a Π_2^0 problem.

(b) Prove that $TIso_2$ is a Π_4^0 problem.

(c) Can you think of an oracle capable of deciding, for any two given trees of height 2, whether they are isomorphic or not?