CSCC63 Assignment 2

Reductions, Polytime Reductions, and NP

Due 11:59pm, July 14

Warning: *For this assignment you may work either alone or in pairs. Your electronic submission of a PDF to Crowdmark affirms that this assignment is your own work and that of your partner, and no one else's, and is also in accordanc[illegible] Toronto Code of Behaviour on Academic Matters, the Code of Student Conduct, an[illegible]iding plagiarism in CSCC63. Note that using Google or any other online resource is* [illegible]

1. (20 marks) Consi[illegible]

$$L_1 = \{\langle G_1, G_1' \rangle \mid G_1 \text{ and } G_1' \text{ are CFGs and no } x \in L(G_1) \text{ is a substring of any } y \in L(G_1')\}.$$

   Is L_1 decidable, recognizable, co-recognizable, or neither? Prove your answer.

2. (10 marks) Consider the language

$$L_2 = \{\langle M, N \rangle \mid L(M) \leqslant_m L(N)\}.$$

   Prove that $\text{ALL}_{\text{TM}} \leqslant_m L_2$.

3. (5 marks) Consider the language

$$\text{Fact-Range} = \{\langle n, a, b \rangle \mid n, a, b \in \mathbb{N} \text{ and } \exists k \in \mathbb{N}, a \leqslant k \leqslant b \text{ and } k \text{ divides } n\}.$$

   Now, consider the following program to solve Fact-Range:

   Find-Fact= "On $\langle n, a, b \rangle$:
   1. If $(0 < n < a)$ or $(b < a)$
   2. *Reject.*
   3. For $i = \max(a, 1)$ to b:
   4. If $n \% i == 0$:
   5. *Accept.*
   6. *Reject.*

   Is this a polytime algorithm? Why or why not?

4. (10 marks) If L is any language, let $T(L)$ the the language of strings x such that x has at least two different substrings in L.

   Prove that if L_2 is in NP, then so is $T(L_2)$.

5. (10 marks) Prove that if L_3 is in co-NP, then so is $T(L_3)$.

6. (15 marks) Let SQUARE-PARTITION$= \left\{ \langle S \rangle \;\middle|\; \begin{array}{l} S \text{ is a a multiset (it can have repeated elements),} \\ \text{and there is some } A \subseteq S \text{ such that } \sum_{x \in A} x = \left(\sum_{x \in S \setminus A} x\right)^2. \end{array} \right\}.$

   (a) (5 marks) Show that SQUARE-PARTITION $\in$ NP.

   (b) (10 marks) Assuming that SUBSET-SUM is NP-complete, show that SQUARE-PARTITION is NP-complete.

7. (15 marks) Let HITTING-PATH= $\left\{ \langle G=(V,E), S, s, t\rangle \;\middle|\; \begin{array}{l} G \text{ is a directed graph, } S \subseteq E,\ s,t \in V, \\ \text{and } G \text{ has a path from } s \text{ to } t \\ \text{that passes through every edge in } S. \end{array} \right\}$.

   (a) (5 marks) Show that HITTING-PATH $\in$ NP.

   (b) (10 marks) Assuming that HAM-PATH is NP-COMPLETE, show that HITTING-PATH is NP-COMPLETE.

8. (15 marks) Let G[illegible]nguage

$$\left\{ \langle G=(V_G,E_G), \text{[illegible]d } H \text{ are graphs, and there is an onto mapping } f: V_G \mapsto V_H \text{ that if } \{v,v'\} \in E_G, \text{ then } \{f(v),f(v')\} \in E_H. \text{ articular, if } \{v,v'\} \in E_G, \text{ then } f(v) \neq f(v').) \right\}.$$

   Note: In graph th[illegible]d a graph homomorphism.

   (a) (5 marks) Show that GRAPH-MAP $\in$ NP.

   (b) (10 marks) Assuming that 3COL is NP-COMPLETE, show that GRAPH-MAP is NP-COMPLETE.

9. **Bonus** (10 marks – your mark will be rounded to the nearest multiple of 2.5)

   In class we stated that a language L_B is in Σ_2 – that is, it can be recognized by a TM with access to a HALT oracle – iff it can be expressed in the form

$$L_B = \{\langle x\rangle \mid \exists y, \forall z, P(x,y,z)\},$$

   Where $P(x,y,z)$ is decidable with a regular (non-oracle) TM.

   Prove this assertion.