Computer Science/Mathematics 455
Lecture Notes
Lecture # 28

Numerical Integration – Romberg Integration

Need to compute

$$I = \int_a^b f(x)dx$$

where $f(x)$ has no known antiderivative.

Let $x_k = a + kh, k = 0, \ldots, n$ where $h = (b-a)/n$. The composite trapezoid rule is

$$T_0(h) = \frac{h}{2}[f(x_0) + f(x_n)] + h\sum_{k=0}^{n-1} f(x_k)$$

To halve the interval size, we need to evaluate the function f at the midpoints

$$x_{k+1/2} = [x_k + x_{k+1}]/2.$$

We obtain the formula

$$\begin{aligned} T_0(h/2) &= \frac{h}{4}[f(x_0) + f(x_n)] + \frac{h}{2}\sum_{k=0}^{n-1} f(x_k) + \frac{h}{2}\sum_{k=0}^{n-1} f(x_{k+1/2}) \\ &= \frac{1}{2}T_0(h) + \frac{h}{2}\sum_{k=0}^{n-1} f(x_{k+1/2}) \end{aligned}$$

Thus with the same function evaluations for $T_0(h/2)$, we can compute both $T_0(h)$ and $T_0(h/2)$. This can be very useful. In fact, for the same function evaluations that we need to compute $T_0(h/2^k)$ we can obtain $T_0(h), T_0(h/2), \ldots, T_0(h/2^k)$. That information may be used to obtain a more accurate approximation.

If $I = \int_a^b f(x)dx$ then assuming a sufficient number of continuous derivatives $(2n+2)$ we have

$$T_0(h) = I + a_2 h_2 + a_4 h^4 + \cdots + a_{2n} h^{2n} + O(h^{2n+2}).$$

Likewise, for $h/2$ we have

$$T_0(h/2) = I + a_2 h_2/4 + a_4 h^4/16 + \cdots + a_{2n} h^{2n}/4^n + O(h^{2n+2}).$$

If we let

$$T_1(h) = (4T_0(h/2) - T_0(h))/3$$

then

$$T_1(h) = I + a'_4 h^4 + a'_6 h^6 + \cdots + a'_{2n} h^{2n} + O(h^{2n+2})$$

where

$$a'_4 = -a_4/4, \quad a'_{2k} = -(1 - 1/4^{k-1})a_{2k}/3.$$

Thus all of these coefficients become smaller in magnitude.

$T_1(h)$ is the composite Simpson's rule. There is a formula for it, but this one is easier to use. We do not need to stop here.

If we have $T_0(h)$,$T_0(h/2)$, and $T_0(h/4)$ we can get $T_1(h)$ and $T_1(h/2)$. Since

$$T_1(h/2) = I + a'_4 h^4/16 + a'_6 h^6/64 + \cdots + a'_{2n} h^{2n}/4^n + O(h^{2n+2}).$$

We can do the same trick again and eliminate the h^4 term. Thus obtaining

$$T_2(h) = (16T_1(h/2) - T_1(h))/15 = I + a_6^{(2)} h^6 + O(h^8).$$

Now we have beginnings of an algorithm to obtain higher order approximations.

For instance, suppose we have $T_{k-1}(h)$ from $T_0(h)$, $T_0(h/2), \ldots, T_0(h/2^{k-1})$ that satisfies

$$T_{k-1}(h) = I + a_{2k}^{(k-1)} h^{2k} + a_{2k+2}^{(k-1)} h^{2k+2} + O(h^{2k+4}).$$

So that once we generate $T_0(h/2^k)$ followed by $T_1(h/2^{k-1}), \cdots T_{k-1}(h/2)$, the last of which satisfies

$$T_{k-1}(h/2) = I + a_{2k}^{(k-1)} h^{2k}/4^k + a_{2k+2}^{(k-1)} h^{2k+2}/4^{k+1} + O(h^{2k+4}).$$

Therefore we can compute

$$T_k(h) = (4^k T_{k-1}(h/2) - T_{k-1}(h))/(4^k - 1) = I + a_{2k+2}^{(k)} h^{2k+2} + O(h^{2k+4}). \quad (1)$$

This technique of using several low order approximations to obtain a high order approximation is called Richardson extrapolation. When applied to the Trapezoid rule, it is called *Romberg integration*.

The basic idea is to set up a table as follows.

$T_0(h)$	$T_1(h)$	$T_2(h)$	$T_3(h)$	$T_4(h)$
$T_0(h/2)$	$T_1(h/2)$	$T_2(h/2)$	$T_3(h/2)$	
$T_0(h/4)$	$T_1(h/4)$	$T_2(h/4)$		
$T_0(h/8)$	$T_1(h/8)$			
$T_0(h/16)$				

The rows (going to the right), the columns (going down), and the diagonals (up and to the right), all converge to the integral. However, the rows converge the most quickly.

In a code, you do not need to keep this entire table around. Instead you can get away with $T_0(h/2^k), T_1(h/2^{k-1}), \ldots, T_k(h)$ as needed.

The method comes with its own error estimate.

It is easily verified that

$$\begin{aligned} T_{k-1}(h) - T_k(h) &= a_{2k}^{(k-1)} h^{2k} + (a_{2k+2}^{(k-1)} - a_{2k+2}^{(k)}) h^{2k+2} + O(h^{2k+4}) \\ &= T_{k-1}(h) - I - a_{2k+2}^{(k)} h^{2k+2} + O(h^{2k+4}) \\ &= T_{k-1}(h) - I + O(h^{2k+4}) \end{aligned}$$

Thus $T_{k-1}(h) - T_k(h)$ captures the first order term of the error in $T_{k-1}(h)$ and a good stopping criterion is

$$|T_{k-1}(h) - T_k(h)| \le \epsilon$$

for some tolerence ϵ.

Incidentally, both the trapezoid method and Simpson's rule also have error estimates that may be gleaned from extrapolation. For Simpson's rule, suppose we have computed both $T_1(h)$ and $T_1(2h)$ (the second can be obtained at no additional cost).

$$T_1(2h) - T_1(h) = -15a_4' h^4 + O(h^4) = 15(I - T_1(h)) + O(h^6).$$

So a good error estimate is

$$Err = (T_1(2h) - T_1(h))/15.$$

If $|Err| < epsilon$ we generally accept

$$T_2(2h) = T_1(h) + Err$$

as the integral.

Verify for yourself that a good error estimate for the trapezoid rule is

$$Err = (T_0(2h) - T_0(h))/3 = I - T_0(h) + O(h^4).$$