Computer Science/Mathematics 455
Lecture Notes
Lecture # 26

Numerical Integration

Need to compute

$$I = \int_a^b f(x)dx$$

where $f(x)$ has no known antiderivative.

Simple approach — integrate the interpolation polynomial. Let $p_n(x)$ interpolate f at $x_0, x_1, \ldots, x_n$ where

$$a \leq x_0 < x_1 < \cdots < x_n \leq b$$

Compute

$$\int_a^b p_n(x)dx \approx \int_a^b f(x)dx.$$

If $a = x_0$ and $b = x_n$, they are called *closed* formulas (our usual). If $a < x_0$ and $b > x_n$, they are called *open* formulas.

These formulas take the form

$$I_n(f) = \sum_{k=0}^{n} c_k f(x_k) \approx \int_a^b f(x)dx,$$

This formula should be exact for polynomials of degree n or less. Thus we have

$$\sum_{k=0}^{n} c_k x_k^j = \int_a^b x^j dx, \quad j = 0, \ldots, n.$$

This is the same as the linear system

$$\begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 \\ x_0 & x_1 & \cdots & \cdots & x_n \\ x_0^2 & x_1^2 & \cdots & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_0^n & x_1^n & \cdots & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \int_a^b dx \\ \int_a^b x dx \\ \vdots \\ \int_a^b x^n dx \end{pmatrix}.$$

This is the transpose of the Vandermonde system. Now for some particular instances and particular formulas. For simplicity, let $[a,b]=[0,1]$. If $f(\cdot)$ is a function on $[a,b]$, then

$$g(y) = f(a+(b-a)y)$$

is a function on $[0,1]$. Moroever by a change of variables,

$$\int_a^b f(x)dx = (b-a)\int_0^1 g(y)dy.$$

Thus it is easy to go back and forth.

For our purposes, we let $x_k = kh$ and $h = 1/n$.

n=1

$x_0 = 0$, $x_1 = 1$.

The coefficients c_0, c_1 solve

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \int_0^1 dx \\ \int_0^1 xdx \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}.$$

The solution is $c_0 = c_1 = 1/2$. Thus

$$I_1(f) = \frac{1}{2}[f(0)+f(1)].$$

This translates back to

$$I_1(f) = \frac{b-a}{2}[f(a)+f(b)].$$

This is called the *Trapezoid Rule.*

n=2

$x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$

Leads to

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1/2 & 1 \\ 0 & 1/4 & 1 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \int_0^1 dx \\ \int_0^1 xdx \\ \int_0^1 x^2dx \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ 1/3 \end{pmatrix}.$$

The solution is

$$c_0 = c_2 = 1/6, c_1 = 2/3.$$

Thus

$$I_2(f) = \frac{1}{6}[f(0) + 4f(1/2) + f(1)].$$

This translates back to

$$I_2(f) = \frac{(b-a)}{6}[f(a) + 4f([a+b]/2) + f(b)].$$

This is called *Simpson's* rule.

n=3

$x_0 = 0$, $x_1 = 1/3$,$x_2 = 2/3$,$x_3 = 1$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1/3 & 2/3 & 1 \\ 0 & 1/9 & 4/9 & 1 \\ 0 & 1/27 & 8/27 & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \int_0^1 dx \\ \int_0^1 x dx \\ \int_0^1 x^2 dx \\ \int_0^1 x^3 dx \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \end{pmatrix}.$$

The solution is

$$c_0 = 1/8, c_1 = 3/8, c_2 = 3/8, c_3 = 1/8.$$

Them method is

$$I_3(f) = \frac{1}{8}[f(0) + 3f(1/3) + 3f(2/3) + f(1)].$$

It translates back to

$$I_3(f) = \frac{(b-a)}{8}[f(a) + 3f((2a+b)/3) + 3f((a+2b)/3) + f(b)].$$

This is the Newton 3/8 rule.

If $x_0, x_1, \ldots, x_n$ are evenly spaced, this leads to the Newton-Cotes formula of order n. They are guaranteed exact for polynomials of degree n or less.

Both the Trapezoid method and Simpson's Rule have error formulas. For the Trapezoid rule,

$$\int_a^b f(x)dx = I_1(f) - \frac{(b-a)^2}{12} f''(\xi_2)$$

for some $\xi_2 \in (a, b)$.

For Simpson's rule,

$$\int_a^b f(x)dx = I_2(f) - \frac{(b-a)h^4}{180} f^{(4)}(\xi_4)$$

where $h = (b-a)/2$ and $\xi_4 \in (a, b)$.