CSEE6180 Modeling and Performance Evaluation : Take Home Exam, Fall 2023

1. This is a take home exam. You may consult your notes and other published material (articles, books). If you are using results from a published material, please cite it.

2. The exam is due Sunday, December 17th by midnight.

3. Show all your work. Partial credit is possible for an answer, but only if you show the intermediate steps in obtaining the answer. Please define your notation!

4. Please make your exam as neat as possible. Reading mathematics and handwriting is often not easy.

5. Last, but not least, the work that you turn in must be your own. It is OK to discuss homework problems with other students. It is not alright to discuss exam problems with other students. If you have a question, please direct it to me, either in person, by phone or by email. When you turn in your exam, include the following signed statement. All of the work reported here is my own. I have not used any other sources except class notes and articles/books cited here.

1. **Simple probability** (10 points)

   Mr. Thompson has baked a large chocolate cake for his family's weekend dessert. Eating more than a third of the cake at once can cause a stomach ache. While Mr. Thompson is out, his son, Tim, decides to have a slice of the cake. Later, unaware of Tim's indulgence, his daughter, Tina, also takes a slice of the remaining cake. Mr. Thompson is surprised to find that more than a third of the cake is gone upon his return.

   Assume that the size of each slice taken by Tim and Tina is random and uniformly distributed over the portion of the cake available at the time of their cutting. What is the probability that neither Tim nor Tina will have a stomach ache?

2. **M/G/1 with feedback** (20 points total)

   Consider an LLM modeled as an M/G/1 system with input rate of queries arriving at λ and service (response generation) time probability density function $b(x)$ with moments $E[X^k]$. However, whenever a user receives a response, she will, with probability p (p for politeness), return to the LLM for exactly 5 more seconds to say "Thank you" and the LLM responds with "It was my pleasure, feel free to return to ask more questions".

   Each user will do this at most once. Find the mean waiting time $E[W]$ in terms of λ, p and $E[X^k]$.

3. **Lazy** $M/M/1$ (20 points)

   Consider a lazy $M/M/1$ system when the system empties out, it won't start any service until there are k users in the system. Once service begins it proceeds until the system becomes empty again.

   (a) Draw the state diagram to represent this system

   (b) What is the average response time for this lazy $M/M/1$ system?

4. **Stochastic Differential Equations** (20 points)

   Consider an on-off Markov modulated Poisson process. The rate transition matrix of the continuous time Markov chain describing the on-off process is given by

$$Q = \begin{bmatrix} -\mu_0 & \mu_0 \\ \mu_1 & -\mu_1 \end{bmatrix}$$

   When in state 0, no arrivals are generated, and when in state 1, arrivals are generated with Poisson rate λ.

   (a) What is the arrival rate of this process?

   (b) Let $\{N_0(t)\}$ and $\{N_1(t)\}$ be two Poisson counters with rate μ_0 and μ_1 respectively. Let $X(t) \in \{0, 1\}$ denote the state of this process. Write a stochastic differential equation to describe $dX(t)$. Use this equation to derive $E[X]$ where $X = \lim_{t\to\infty} X(t)$. Finally, use this expression to derive $P(X = i), i = 0, 1$. Verify your result with π_i obtained from Q.

(c) Let $M(t)$ be the process that counts the number of arrivals in $[0, t]$. Let $\{N(t)\}$ be a Poisson counter with rate λ. Write a stochastic differential equation to describe the behavior of $M(t)$, i.e. $dM(t)$. Use this to derive an expression for $\lim_{t\to\infty} dM(t)/dt$. How do you interpret the result?

5. **Load Balancing** (30 points total)

Consider a coffee shop with two baristas. Each barista has room for only two customers (including the one getting the coffee). Service time is exponential for both servers with service rate μ. Arrivals to the service station are Poisson with rate λ. We consider two policies wherein policy A, customers are routed to either queue with equal probability. Policy B is to do load balancing wherein customers are routed to the barista with shorter queue. Ties are broken randomly. If the selected queue is full, the customer walks away.

(a) (10pts) For policy A, compute the average throughput and delay (for the accepted customers).

(b) (20pts) For policy B, draw the state transition diagram, then compute the throughput and delay (for the accepted customers).