Solution to Homework Assignment 1

Solution to Problem 1:

(a)

$$G(f) = \begin{cases} e^{-j2\pi f t_0} & f \in [-f_0, f_0] \\ 0 & \text{elsewhere} \end{cases}.$$

By using the defini[illegible]

$$f(t) = \int [illegible] = \int_{-f_0}^{f_0} e^{j2\pi f(t-t_0)} df = \frac{1}{j2\pi(t-t_0)} e^{j2\pi f(t-t_0)}\Big|_{-f_0}^{f_0}$$

$$= \frac{\sin [illegible]}{\pi(t - t_0)} [illegible] f_0 \text{sinc}[2f_0(t-t_0)].$$

(b) $G(f) = \text{rect}\left(\frac{f}{2f_0}\right)$. Thus $g(t) = 2f_0 \text{sinc}(2f_0 t)$.

Or from the definition of inverse FT

$$f(t) = \int_{-f_0}^{f_0} e^{j2\pi f t} df = \frac{1}{j2\pi t} e^{j2\pi f t}\Big|_{-f_0}^{f_0} = \frac{\sin(2\pi f_0 t)}{\pi t} = 2f_0 \text{sinc}(2f_0 t).$$

Solution to Problem 2: From the plot,

$$g_1(t) = \sin t[u(t) - u(t-\pi)].$$

By definition:

$$\begin{aligned}
G_1(f) &= \int_0^{\pi} \sin(t) e^{-j2\pi f t} dt \\
&= -\int_0^{\pi} e^{-j2\pi f t} d\cos(t) \\
&= -\cos(t) e^{-j2\pi f t}\Big|_0^{\pi} + \int_0^{\pi} \cos(t) d e^{-j2\pi f t} \\
&= 1 + e^{-j2\pi^2 f} + (-j2\pi f) \int_0^{\pi} cos(t) e^{-j2\pi f t} dt \\
&= 1 + e^{-j2\pi^2 f} + (-j2\pi f) \int_0^{\pi} e^{-j2\pi f t} d\sin(t) \\
&= 1 + e^{-j2\pi^2 f} + (-j2\pi f) \left[\sin(t) e^{-j2\pi f t}\Big|_0^{\pi} - \int_0^{\pi} \sin(t) d e^{-j2\pi f t} \right] \\
&= 1 + e^{-j2\pi^2 f} - (-j2\pi f)^2 \int_0^{\pi} \sin(t) e^{-j2\pi f t} dt \\
&= 1 + e^{-j2\pi^2 f} + 4\pi^2 f^2 G_1(f)
\end{aligned}$$

After move the $4\pi^2 f^2 G_1(f)$ to the left of the equation, we get

$$G_1(f) - 4\pi^2 f^2 G_1(f) = 1 + e^{-j2\pi^2 f}.$$

Hence,

$$G_1(f) = \frac{1 + e^{-j2\pi^2 f}}{1 - 4\pi^2 f^2}.$$

By properties: Notice that

$$\ldots \sin(t-\pi)u(t-\pi).$$

$$\ldots \left(f - \frac{1}{2\pi}\right) - \delta\left(f + \frac{1}{2\pi}\right)\right] + \frac{1}{1-(2\pi f)^2}.$$

By using the time-s…

$$\sin(t-\pi)\,u(\ldots \left(f - \frac{1}{2\pi}\right) - \delta\left(f + \frac{1}{2\pi}\right)\right] + \frac{1}{1-(2\pi f)^2}\right\} e^{-j2\pi^2 f}.$$

By linearity of FT, we have

$$G(f) = \left\{\frac{\pi}{2j}\left[\delta\left(f - \frac{1}{2\pi}\right) - \delta\left(f + \frac{1}{2\pi}\right)\right] + \frac{1}{1-(2\pi f)^2}\right\}(1 + e^{-j2\pi^2 f}).$$

As $\delta\left(f \pm \frac{1}{2\pi}\right)(1 + e^{-j2\pi^2 f}) = \delta\left(f \pm \frac{1}{2\pi}\right)(1 + e^{\pm j\pi}) = 0$, we have

$$F(\omega) = \frac{1 + e^{-j2\pi^2 f}}{1 - 4\pi^2 f^2}.$$

For the second signal,

$$g_2(t) = e^{-at}[u(t) - u(t-T)].$$

By definition,

$$G_2(f) = \int_0^T e^{-at} e^{-j2\pi f t}\,dt = -\frac{1}{a + j2\pi f} e^{-(a+j2\pi f)t}\Big|_0^T = \frac{1 - e^{-(a+j2\pi f)T}}{a + j2\pi f}$$

Notice that

$$g_2(t) = e^{-at}u(t) - e^{-aT}e^{-a(t-T)}u(t-T).$$

$$e^{-at}u(t) \Longleftrightarrow \frac{1}{a + j2\pi f}.$$

By using the time-shifting property,

$$e^{-a(t-T)}u(t-T) \Longleftrightarrow \frac{1}{a + j2\pi f} e^{-j2\pi f T}.$$

By the linearity of FT, we have

$$F(\omega) = \frac{1 - e^{-(a+j2\pi f)T}}{a + j2\pi f}.$$

Solution to Problem 3:

(a) Since

$$g(t)\sin(2\pi f_c t) = \frac{1}{2j} g(t)\left[e^{j2\pi f_c t} - e^{-j2\pi f_c t}\right],$$

from frequency-shifting property and linearity,

$$g(t)\sin(2\pi f_c t) \rightleftharpoons \frac{1}{2j}[G(f - f_c) - G(f + f_c)].$$

(b) Since $2 + \cos(2\pi f_0 t) \rightleftharpoons 2\delta(f) + \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$, by using the property in (a), we have

$$S(\ldots) + \frac{1}{2}\delta(f - 100 - f_0) + \frac{1}{2}\delta(f - 100 + f_0)$$

$$\ldots 0) - \frac{1}{2}\delta(f + 100 - f_0) - \frac{1}{2}\delta(f + 100 + f_0)\Big].$$

The spectrum is as

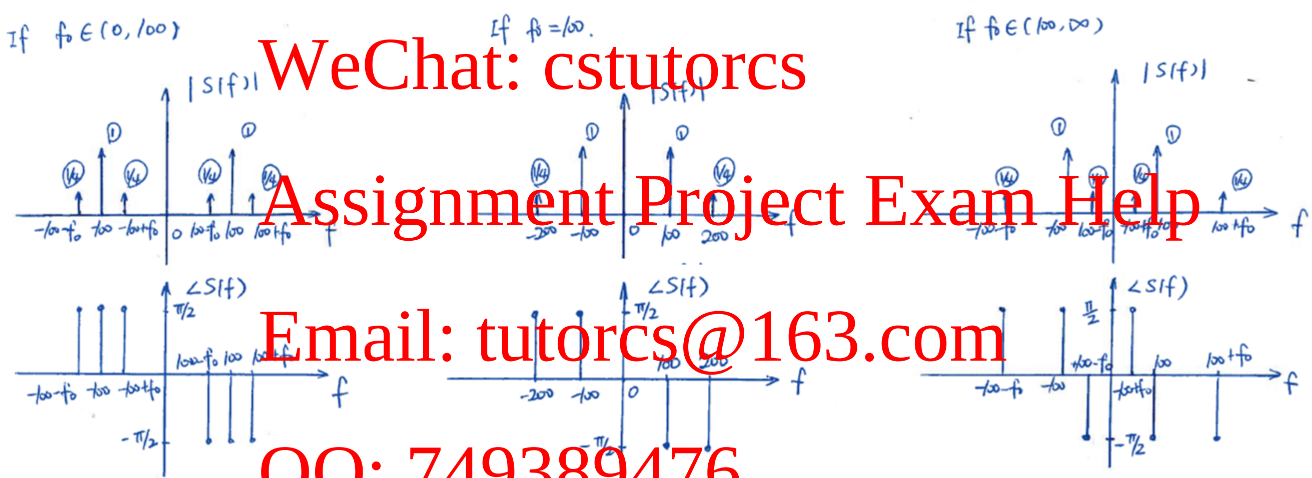
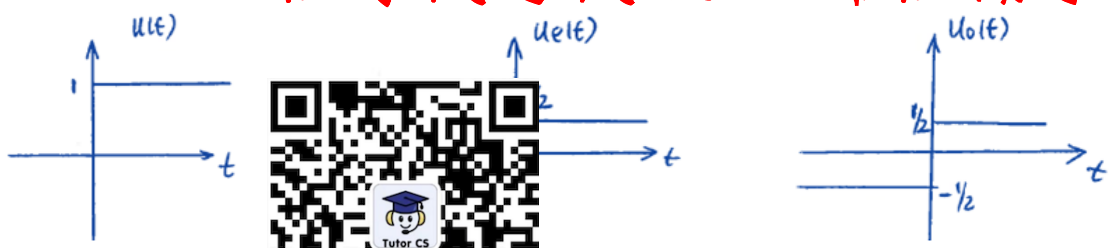


Figure 1: Spectra for Problem 3.

Solution to Problem 4:

(a) The signals $u(t), u_e(t), u_o(t)$ are represented by the following figure.



Signals for Problem 4.

(b)We have

$$u_e(t) = \frac{1}{2} \rightleftharpoons \frac{1}{2}\delta(f).$$

$$u_o(t) = \frac{1}{2}\mathrm{sgn}\,(t) \rightleftharpoons \frac{1}{j2\pi f}.$$

$$u(t) \rightleftharpoons \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}.$$