# Chapter 2. Fourier Representation of Signals and Systems – *Review*



*Summary:* *Time- and fr[illegible]omain representations of signals, properties of signals, ti[illegible]equency-domain analysis of systems (signal transmission), F[illegible]sform, Fourier series.*

*Textbook coverage:*



2.1 Fourier Transform **(Haykin & Moher 2.1)**

2.2 Properties of Fourier Transform **(Haykin & Moher 2.2)**



2.3 Fourier Series and Fourier Transform of Periodic Signals **(Haykin & Moher 2.4 and 2.5)**



2.4 Transmission of Signals through Linear Time-Invariant Systems **(Haykin & Moher 2.6)**



2.5 Filters **(Haykin & Moher 2.7 and more)**

2.6 Energy Spectral Density and Autocorrelation Function for Energy Signals **(Haykin & Moher 2.8)**



2.7 Power Spectral Density and Autocorrelation Function for Power Signal **(Haykin & Moher 2.9 and more)**

**Signal:** A set of data that is functions of one or more independent variables.

- Examples: speech sign[illegible]sure as a function of time, etc.
- Focus on single-variab[illegible]ous signal: <u>A function of time g($t$)</u>.

*<u>Fundamental signals:</u>*

- Dirac delta function: $\delta(t) = 0 \; for \; t \neq 0 \; and \; \int_{-\infty}^{\infty} \delta(t)dt = 1$

$$\int_{-\infty}^{\infty} g(t)\delta(t - t_0)dt = g(t_0)$$

- Signum function: $\operatorname{sgn}(t) = \begin{cases} 1 & , t > 0 \\ 0 & , t = 0 \\ -1 & , t < 0 \end{cases}$

- Unit step function: $u(t) = \begin{cases} 1 & , t > 0 \\ 1/2 & , t = 0 \\ 0 & , t < 0 \end{cases}$

*Fundamental signals (cont):*



- Unit rectangular function: $(t) = \begin{cases} 1, & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0, & t < -\frac{1}{2} \text{ or } t > \frac{1}{2} \end{cases}$

- Unit triangle function: $\begin{cases} 0 & |t| \ge \frac{1}{2} \\ 1 - 2|t| & |t| < \frac{1}{2} \end{cases}$



- Sinc function: $\text{sinc}(t) = \dfrac{\sin(\pi t)}{\pi t}$



- Complex exponential function: $x(t) = e^{jw_0 t} = \cos(w_0 t) + j\sin(w_0 t)$

*Periodic* signal: $g(t) = g(t + nT)$
- $T$: fundamental perio lest period) of signal $g(t)$

*Symmetric* signal
- Even signal: $g(t) = g$ - Odd signal: $g(t) = -g(-t)$



Q: is $\mathrm{x(t)} = e^{jw_0 t}$ a periodic signals?

**Signal:** A set of data that is functions of one or more independent variables.

- Examples: speech sign[illegible]sure as a function of time, etc.
- Focus on single-variab[illegible]ous signal: <u>A function of time g($t$)</u>.

Signal energy & power

- Energy: $E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt$
- Power: $P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 dt = \lim_{T\to\infty} (E_g/T)$
- Energy signal: $0 < E_g < \infty$
- Power signal: $0 < P_g < \infty$

Q1: Are periodic signals are energy signals?

Q2: Is there any signal that is both energy signal and power signal?

Q3: Is there any signal that is neither energy signal nor power signal?

**System:** An entity that processes its inputs to produce outputs.

input $g(t) \longrightarrow \boxed{\text{System}} \longrightarrow$ output $y(t)$

- Processing the input s[illegible]odify it or to extract information from it.
- Focus on <u>single-input</u>[illegible]<u>put, *linear time-invariant (LTI)* systems</u>.

A system is <u>**linear**</u> if <u>the principle of superposition</u> holds, i.e., its response to the weighted sum of a number of inputs is equal to the weighted sum of its responses when each input is applied individually.

If $g_1(t) \rightarrow y_1(t), g_2(t) \rightarrow y_2(t)$,

then for any $a_1, a_2$, $a_1 g_1(t) + a_2 g_2(t) \rightarrow a_1 y_1(t) + a_2 y_2(t)$.

Q: Are following systems linear?

(a) $g(t) \rightarrow e^{g(t)}$

(b) $g(t) \rightarrow g(t) + 1$

A system is **time invariant** if its response to a time-shifted input by any amount of time is equal to the time-shift of its response to the input with the same amount of time.



If $g(t) \to y(t)$, then for a[illegible] $- T) \to y(t - T)$.

Q: Are following systems [illegible]ant? $x(t)$ and $y(t)$ represent input and output of the system.

(a) $g(t) \to g(t) + 1$

(b) $g(t) \to \sin(t)g(t)$



A system is linear time-invariant (LTI) system if both conditions hold.



example: $g(t) \to g(t) + 1$ is not a LTI system as it is non-linear.

**Consider linear time-invariant (LTI) systems.**



The **unit impulse response** of a system, $h(t)$, is its output when the input is the unit impulse fun

$\delta$ )

For an LTI system, its o y input $g(t)$ is the **convolution** of the input and its unit impulse response:



$$y(t) = g(t) * h(t) = \int_{-\infty}^{\infty} g(\tau)h(t-\tau)d\tau$$



A system is **Causal** if it does not respond before the excitation is applied.



- For a causal LTI system, $h(t) = 0$ for $t < 0$.



A system is **stable** if the output is bounded for all bounded input.



- For a stable LTI system, $\int_{-\infty}^{\infty} |h(t)| dt < \infty$

input $g(t)$ $\rightarrow$ LT $\rightarrow$ output $y(t) = g(t) * h(t)$

**Why frequency-domain representation/analysis?**



- Understand a signal via its frequency components (Fourier transform, Fourier series)



- Understand a system's response for different frequency components (Frequency response)

**Why frequency-domain representation/analysis?**



- Understand a signal v[illegible]ency components (Fourier transform, Fourier series)
- Understand a system’[illegible] for different frequency components (Frequency response).



Example from R. Nowak “Intro to digital image processing”.

## <u>2.1 Fourier Transform</u>



Represent a non-periodic signal as a continuous sum of infinitesimal 'simple' exponential functions

**Def.** The <u>Fourier transform</u> of a nonperiodic signal $g(t)$ (if Fourier transformable) is:

$$G(f) = \int_{-\infty}^{\infty} g(t) \exp(-j2\pi f t) dt.$$



Given the Fourier transform $G(f)$, the original time-domain signal can be recovered by the following <u>inverse Fourier transform</u>:



$$g(t) = \int_{-\infty}^{\infty} G(f) \exp(j2\pi f t) df.$$



$g(t)$ and $G(f)$ constitute a **<u>Fourier transform pair</u>.**



$f$ : frequency in Hertz (Hz).

$\omega = 2\pi f$ : frequency in radian/second (rad/sec).

$$G(f) = \int_{-\infty}^{\infty} g(t) \exp(-j2\pi f t) dt. \rightleftharpoons g(t) = \int_{-\infty}^{\infty} G(f) \exp(j2\pi f t) df.$$



Notation: $G(f) = \mathbf{F}[$[illegible]$= \mathbf{F}^{-1}[G(f)] \quad g(t) \rightleftharpoons G(f)$

Lower case letter: time [illegible]

Upper case letter: freq[illegible]tion



Q1: Please compute F[$\delta(t)$].

Q2: Given $G(f) = \delta(f)$, $g(t) = ?$

$$G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi ft)dt. \rightleftharpoons g(t) = \int_{-\infty}^{\infty} G(f)\exp(j2\pi ft)df.$$



Table [illegible] Fourier transform pairs

| Time-domain function $g(t)$ | Fourier transform $G(f)$ |
|---|---|
| $1$ | $\delta(f)$ |
| $\delta(t)$ | $1$ |
| $u(t)$ | $\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$ |
| $\text{rect}\left(\frac{t}{T}\right)$ | $T\text{sinc}(fT)$ |
| $2W\text{sinc}(2Wt)$ | $\text{rect}\left(\frac{f}{2W}\right)$ |
| $\Delta\left(\frac{t}{T}\right)$ | $\frac{T}{2}\text{sinc}^2\left(\frac{fT}{2}\right)$ |
| $e^{-at}u(t) \quad (a > 0)$ | $\frac{1}{a+j2\pi f}$ |
| $e^{-a\|t\|} \quad (a > 0)$ | $\frac{2a}{a^2+(2\pi f)^2}$ |
| $\sin(2\pi f_c t)$ | $\frac{1}{2j}[\delta(f - f_c) - \delta(f + f_c)]$ |
| $\cos(2\pi f_c t)$ | $\frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$ |

$G(f)$: Generally, a complex function of frequency $f$.

$$G(f) = |G(f)| \exp[j\theta(f)] = |G(f)| \angle \theta(f)$$

**Amplitude spectrum** (magnitude spectrum): $|G(f)|$

**Phase spectrum:** $\angle\theta(f)$



**Prove:** for a real-value function $g(t)$,

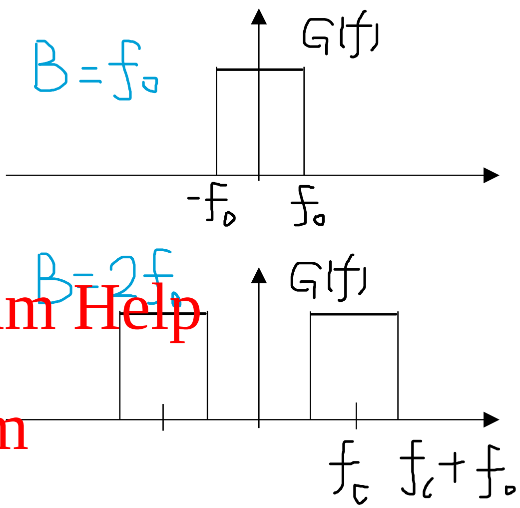
- $G(-f) = G^*(f)$ → conjugate symmetry
- $|G(-f)| = |G(f)|$ → even function of $f$
- $\theta(-f) = -\theta(f)$ → odd-symmetric w.r.t. $f$

**Bandwidth of a signal:** The difference between the highest and lowest frequencies of the spectral components of a signal.

- It is convention to state the bandwidth as the range of positive frequencies.
- A signal cannot be st[illegible]d in both time and frequency.

For band-limited signals:

- Baseband signal: energy centered around zero-frequency.
- Bandpass signal: energy centered around a frequency far away from zero.



**Bandwidth of a signal:** The difference between the highest and lowest frequencies of the spectral components of a signal.



- It is convention to state the bandwidth as the range of positive frequencies.
- A signal cannot be strictly limited in both time and frequency.

For <u>pulse-like signals</u>:

- main lobe bandwidth for symmetric signals
- 3-dB bandwidth
- 95% essential bandwidth of a signal: Range of the frequency that contains 95% of the energy.

## 2.2 Properties of Fourier Transform



1. **Linearity:** $c_1 g_1(t) + c_2 g_2(t) \rightleftharpoons c_1 G_1(f) + c_2 G_2(f)$
2. **Time scaling:** If $g(t) \rightleftharpoons G(f)$ then $g(at) \rightleftharpoons \frac{1}{|a|} G(\frac{f}{a})$ (Dilation) $\boxed{a \neq 0}$
3. **Conjugation:** $g^*(t) \rightleftharpoons G^*(-f)$
4. **Duality:** $g(t) \rightleftharpoons G(f)$ then $G(t) \rightleftharpoons g(-f)$
5. **Time shifting:** $g(t - t_0) \rightleftharpoons e^{-j2\pi f t_0} G(f)$
6. **Frequency shifting:** $g(t) \exp(j2\pi f_c t) \rightleftharpoons G(f - f_c)$ for any constant $f_c$.

7. **Areas under $g(t)$ and $G(f)$:**



$$\int_{-\infty}^{+\infty} g(t)dt = G(0) \text{ and } \int_{-\infty}^{+\infty} G(f)df = g(0)$$

8. **Differentiation and Integration:**

$$\frac{d}{dt}g(t) \rightleftharpoons j2\pi f G(f) \quad \text{and} \quad \int_{-\infty}^{t} g(\tau)d\tau \rightleftharpoons \frac{1}{j2\pi f}G(f) + \frac{1}{2}G(0)\delta(f)$$



9. **Convolution and Modulation:**



$$g_1(t) * g_2(t) \rightleftharpoons G_1(f) \cdot G_2(f) \text{ and } g_1(t) \cdot g_2(t) \rightleftharpoons G_1(f) * G_2(f)$$



10. **Parseval's theorem:**



$$E_g = \int_{-\infty}^{+\infty} |g(t)|^2 dt = \int_{-\infty}^{+\infty} |G(f)|^2 df$$

# Energy Spectral Density & Autocorrelation Function for Energy Signals



For an energy signal $g(t)$, i.e. $E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt < \infty$, its **energy spectral density** is de[illegible]

$$\Psi_g(f) \text{ [illegible]}, \text{ where } g(t) \rightleftharpoons G(f).$$



**Wiener-Khitchine Relation:** energy signal's autocorrelation function and energy spectral density function form a Fourier transform pair:



$$R_g(\tau) \rightleftharpoons \Psi_g(f),$$

where the **autocorrelation function** $R_g(\tau) = \int_{-\infty}^{\infty} g(t)g^*(t-\tau)dt$.





Energy spectral density tells how signal energy spreads over frequencies, so we have the Parseval's Theorem:



$$E_g = \int_{-\infty}^{\infty} |G(f)|^2 df = \int_{-\infty}^{\infty} \Psi_g(f) df = R_g(0) = \int_{-\infty}^{\infty} |g(t)|^2 dt$$

## Parseval's Power theorem & Power Spectral Density of Power Signals



For a power signal g(t), i.e., $P_g = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |g(t)|^2 dt < \infty$, define

$$g_T(t) = \begin{cases} g(t) & -\frac{T}{2} \le t \le T/2 \\ 0. & otherwise \end{cases}$$



Then $g_T(t)$ is an energy signal.



From $g_T(t) \rightleftharpoons G_T(f)$, **Parseval's power theorem** is



$$P_g = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |g(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} |G_T(f)|^2 df$$



$$= \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{T} |G_T(f)|^2\, df = \int_{-\infty}^{\infty} S_g(f) df$$

Define **power spectral density (PSD)** of $g(t)$ as:

$$\lim_{T\to\infty} \frac{1}{T}|G_{T_0}(f)|^2$$

PSD tells how signal ...eads over frequencies.



The **auto-correlation function** of power signal $g(t)$ is



$$R_g(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} g(t)g^*(t-\tau)dt$$



Similarly, we have $R_g(\tau) \rightleftarrows S_g(f)$ and $P_g = \int_{-\infty}^{\infty} S_g(f)df = R_g(0)$

## 2.3 Fourier Series and Fourier Transform of Periodic Signals



$g_{T_0}(t)$: A periodic signal with period $T_0$.

**Fundamental frequency:** $f_0 = \frac{1}{T_0}$

A periodic signal $g_{T_0}(t)$ can be represented as a sum of complex exponential (complex exponential Fourier series):



$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} c_n \exp(j2\pi n f_0 t)$$



where $c_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} g_{T_0}(t) e^{-j2\pi n f_0 t} dt$.



Fourier transform of $g_{T_0}(t)$ is:



$$G_{T_0}(f) = \sum_{n=-\infty}^{+\infty} c_n \delta(f - n f_0)$$

For periodic signal $g_{T_0}(t)$ with period $T_0$

Fundamental freq. $f_0 = \frac{1}{T_0}$

FS: $g_{T_0}(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_0 t}$, $C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_{T_0}(t) e^{-j2\pi n f_0 t}\, dt$

$\Rightarrow$ FT: $G_{T_0}(f) = \sum_{n=-\infty}^{\infty} C_n \delta(f - nf_0)$

To find $C_n$ for $\cos(2\pi f_0 t)$

$$C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \cos(2\pi f_0 t)\, e^{-j2\pi n f_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \frac{1}{2}\left[e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right] e^{-j2\pi n f_0 t}\, dt$$

$$= \frac{1}{2T_0}\int_{-T_0/2}^{T_0/2} e^{j2\pi f_0 (1-n) t} + e^{-j2\pi f_0 (n+1) t}\, dt$$

$$= \frac{1}{2T_0}\left[\int_{-T_0/2}^{T_0/2} e^{j2\pi f_0 (1-n) t}\, dt + \int_{-T_0/2}^{T_0/2} e^{-j2\pi f_0 (n+1) t}\, dt\right]$$

$$= \frac{1}{2T_0}\left[\frac{e^{j2\pi f_0 (1-n) t}}{j2\pi f_0 (1-n)}\Bigg|_{-T_0/2}^{T_0/2} + \frac{e^{-j2\pi f_0 (n+1) t}}{-j2\pi f_0 (n+1)}\Bigg|_{-T_0/2}^{T_0/2}\right]$$

$$= \frac{1}{2T_0}\left[\frac{e^{j2\pi f_0 (1-n)\frac{T_0}{2}} - e^{-j2\pi f_0 (1-n)\frac{T_0}{2}}}{j2\pi f_0 (1-n)} - \frac{e^{-j2\pi f_0 (n+1) T_0/2} - e^{j2\pi f_0 (n+1) T_0/2}}{j2\pi f_0 (n+1)}\right]$$

$f_0 = \frac{1}{T_0} \Rightarrow f_0 T_0 = 1$

$$= \frac{1}{2T_0}\left[\frac{e^{j\pi(1-n)} - e^{-j\pi(1-n)}}{j2\pi f_0 (1-n)} - \frac{e^{-j\pi(n+1)} - e^{j\pi(n+1)}}{j2\pi f_0 (n+1)}\right]$$

$$= \frac{1}{2T_0}\left[\frac{2j\sin[\pi(1-n)]}{j2\pi f_0 (1-n)} + \frac{2j\sin[\pi(n+1)]}{j2\pi f_0 (n+1)}\right]$$

$$= \frac{\sin[\pi(1-n)]}{2\pi(1-n)} + \frac{\sin[\pi(n+1)]}{2\pi(n+1)}$$

$$= \begin{cases} 0 & n \neq \pm 1 \\ \frac{\sin[\pi(1-n)]}{2\pi(1-n)}\Big|_{n=1} + 0 & n = 1 \\ 0 + \frac{\sin[\pi(1+n)]}{2\pi(1+n)} & n = -1 \end{cases}$$

$$= \begin{cases} 0 & n \neq \pm 1 \\ \frac{-\pi\cos[\pi(1-n)]}{-2\pi}\Big|_{n=1} & n = 1 \\ \frac{\pi\cos[\pi(1+n)]}{2\pi}\Big|_{n=-1} & n = -1 \end{cases}$$

$$= \begin{cases} 0 & n \neq \pm 1 \\ \frac{1}{2} & n = 1 \\ \frac{1}{2} & n = -1 \end{cases}$$

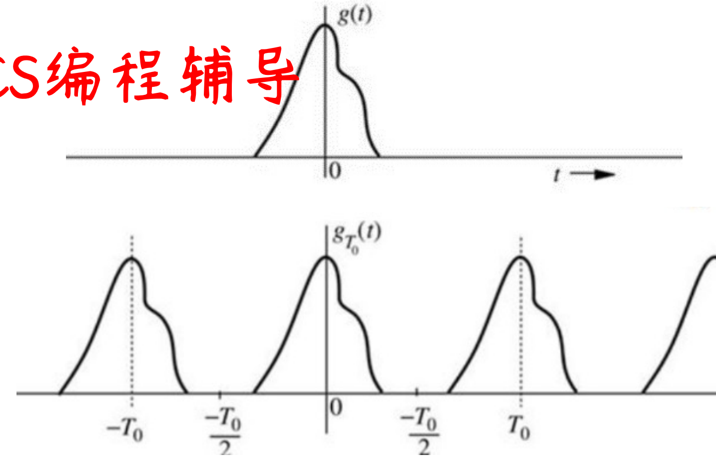
$$\Rightarrow \cos(2\pi f_0 t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi f_0 n t} = \frac{1}{2}e^{j2\pi f_0 t} + \frac{1}{2}e^{-j2\pi f_0 t}$$

$$\Rightarrow \mathcal{F}[\cos(2\pi f_0 t)] = \sum_{n=-\infty}^{\infty} C_n \delta(f - nf_0) = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0) \quad //.$$

## Relation between FT and FS



Define two signals $g(t)$ and $g_{T_0}(t)$ :




$$g_{T_0}(t) = \sum_{m=-\infty}^{\infty} g(t - mT_0)$$

$$g(t) = \begin{cases} g_{T_0}(t) & -\frac{T_0}{2} \le t \le \frac{T_0}{2} \\ 0 & \text{otherwise} \end{cases}.$$



Let $g(t) \rightleftharpoons G(f) \quad g_{T_0}(t) \rightleftharpoons G_{T_0}(f)$,



we have $c_n = \frac{1}{T_0} G(nf_0) = f_0 G(nf_0)$ .



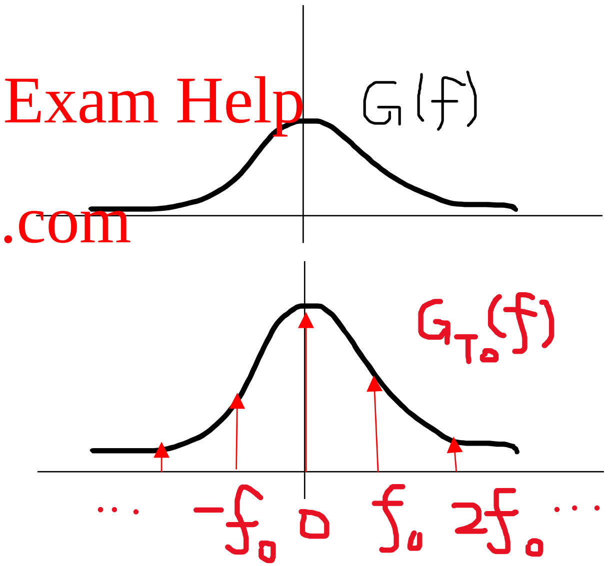
That is,



$$G_{T_0}(f) = f_0 \sum_{n=-\infty}^{\infty} G(nf_0)\delta(f - nf_0).$$



Repetition in the time domain results in sampling in the frequency domain.



**Power spectral density for periodic signal** $g(t)$ with period $T_0$.



$$G_{T_0}(f) = f_0 \sum_{n=-\infty}^{\infty} G(nf_0)\delta(f - nf_0). \qquad f_0 = \frac{1}{T_0}$$

Then its power spectral density is

$$S_{g_{T_0}}(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - nf_0) = \sum_{n=-\infty}^{\infty} f_0^2 |G(nf_0)|^2 \delta(f - nf_0)$$



**Parseval's power theorem for periodic signal:**

$$\frac{1}{T}\int_{-T/2}^{T/2} |g(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

# 2.4 Transmission of signals through LTI systems



**System:** An entity tha[illegible]s its inputs to produce outputs.


- Processing the input s[illegible]dify it or to extract information from it.
- Focus on <u>single-input-single-output, *linear time-invariant (LTI)* system.</u>



The **<u>unit impulse response</u>** of a system, $h(t)$, is its output when the input is the unit impulse function $\delta(t)$.



$$\delta(t) \rightarrow h(t)$$



For an LTI system, its output to any input $g(t)$ is the **convolution** of the input and its unit impulse response:



$$y(t) = g(t) * h(t) = \int_{-\infty}^{\infty} g(\tau)h(t-\tau)d\tau$$

For an LTI system, its <u>frequency response</u> is the Fourier transform of its unit impulse response:

$$H(f) = \int_{-\infty}^{\infty} h(t)\exp(-j2\pi f t)dt. \qquad h(t) \rightleftharpoons H(f)$$

- Magnitude response $|H(f)|$
- Phase response $\theta(f), or\ \angle H(f)$



The output of an LTI system with any input $g(t)$ in freq. domain:

$$Y(f) = H(f)G(f) \rightleftharpoons y(t) = g(t) * h(t)$$



Given any LTI system's input-output pair $g(t) \to y(t)$, the impulse response of the system can be specified in two steps;

- Step 1: $H(f) = \frac{Y(f)}{G(f)}$
- Step 2: $h(t) = \mathcal{F}^{-1}(H(f)$

## Filters



- Filters are systems that are designed to remove some unwanted components or features from a signal.
    - Low-pass filter
    - High-pass filter
    - Band-pass filter
    - Band-reject or notch filters



- Cutoff frequencies: frequencies beyond which the filter will not pass the signal components.

- 3-dB frequencies: the frequency at which the power-transfer ratio of the filter drops to half of the maximum, i.e., the energy spectral density (or PSD) at this frequency is half (3dB drop) of its maximum.

## Effect of filtering (LTI system) on energy spectral density



$$g(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$Y(f) = H(f)G(f) \quad \Psi_y(f) = |H(f)|^2\Psi_g(f).$$

The energy spectral density of the output equals the energy spectral density of the input multiplied by the squared amplitude of frequency response of the LTI system.



## Effect of filtering (LTIC system) on power spectral density

$$g(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$S_y(f) = |H(f)|^2 S_g(f)$$