## Solution to Homework Assignment 3



**Solution to Problem 1:** $T_0 = 3$ and $f_0 = 1/3$.

(a) Consider the periodic signal $x_{T_0}(t) = g_{T_0}(t) + 2$. Take one period of $x_{T_0}(t)$ for $0 \le t < 3$ and call it $x(t)$. We have $x(t) = 3\text{rect}\left(\frac{t-1}{2}\right)$. Thus

$$[illegible] = 6\,\text{sinc}(2f)e^{-j2\pi f}.$$

$$[illegible] \sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{2n}{3}\right) e^{-j\frac{2n\pi}{3}} \delta\left(f - \frac{n}{3}\right).$$

Since $g_{T_0}(t) = x_{T_0}(t) - 2$,

$$G_{T_0}(f) = X_{T_0}(f) - 2\delta(f) = 2 \sum_{n=-\infty, n\neq 0}^{\infty} \text{sinc}\left(\frac{2n}{3}\right) e^{-j\frac{2n\pi}{3}} \delta\left(f - \frac{n}{3}\right).$$



Thus, the power spectral density of $g_{T_0}(t)$ can be calculated as



$$S_{g_{T_0}}(f) = |G_{T_0}(f)|^2 = 4 \sum_{n=-\infty, n\neq 0}^{\infty} \text{sinc}^2\left(\frac{2n}{3}\right) \delta\left(f - \frac{n}{3}\right).$$



(b) From inverse Fourier transform, the autocorrelation function is

$$\begin{aligned} R_{g_{T_0}}(\tau) &= 4 \sum_{n=-\infty, n\neq 0}^{\infty} \text{sinc}^2\left(\frac{2n}{3}\right) e^{j\frac{2\pi n}{3}\tau} \\ &= 4\left[\sum_{n=-\infty}^{-1} \text{sinc}^2\left(\frac{2n}{3}\right) e^{j\frac{2\pi n}{3}\tau} + \sum_{n=1}^{\infty} \text{sinc}^2\left(\frac{2n}{3}\right) e^{j\frac{2\pi n}{3}\tau}\right] \\ &= 4\left[\sum_{n=1}^{\infty} \text{sinc}^2\left(\frac{2n}{3}\right) e^{-j\frac{2\pi n}{3}\tau} + \sum_{n=1}^{\infty} \text{sinc}^2\left(\frac{2n}{3}\right) e^{j\frac{2\pi n}{3}\tau}\right] \\ &= 8 \sum_{n=1}^{\infty} \text{sinc}^2\left(\frac{2n}{3}\right) \cos\left(\frac{2\pi n}{3}\tau\right) \end{aligned}$$



Notice that the imaginary part of $R_{g_{T_0}}(\tau)$ is zero. That is, $R_{g_{T_0}}(\tau)$ is a real valued function. The autocorrelation function can be drawn by Matlab in Figure 1 (by taking 60 terms of the summation, 30 for positive $n$ and 30 negative $n$). We can see that the autocorrelation function is a periodic triangular-plus-square wave with period 3.

**Solution to Problem 2:** Since

$$400\pi e^{-200\pi t}u(t) \rightleftharpoons \frac{400\pi}{200\pi + j2\pi f} = \frac{2}{1 + j\frac{f}{100}},$$

$$\cos(20,000\pi t) = \frac{1}{2}\left[\delta(f - 10,000) + \delta(f + 10,000)\right],$$

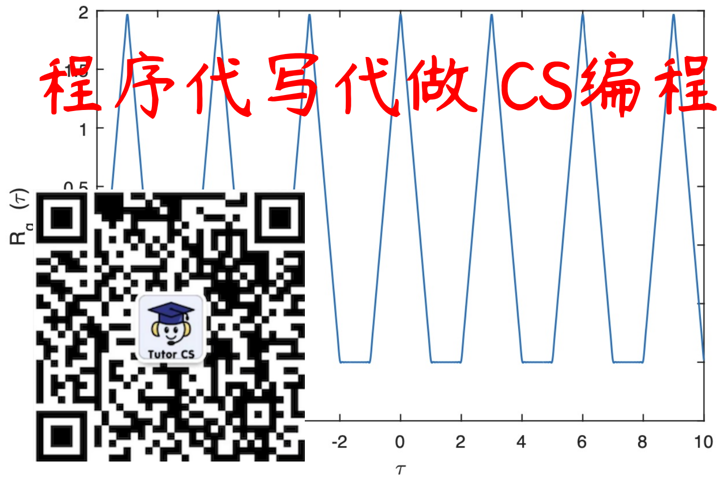


Figure 1: Solution to Problem 1.



we have

$$H_1(f) = \frac{2}{1 + j\frac{f}{100}}$$

$$H_2(f) = \mathbf{F}[h_2(t)] = \frac{1}{1 + j\frac{f-10{,}000}{100}} + \frac{1}{1 + j\frac{f+10{,}000}{100}}.$$

(b) Since $|H_1(f)|$ has 1 peak at 0Hz (maximum magnitude which is 2 at $f = 0$kHz and decreasing magnitude on both sides of the peak). It is a low-pass filter. For the 3-dB cut-off frequencies:

$$|H_1(f_{3dB})| = \frac{2}{\sqrt{2}} \Rightarrow \frac{2}{\sqrt{1 + f_{3dB}^2/10000}} = \frac{2}{\sqrt{2}} \Rightarrow f = \pm 100 \text{ HZ}.$$

The 3-dB bandwidth of the low-pass filter is $[0, 100]$Hz.

(c) $H_2(f)$ is the shifted version of $H_1(f)$ in the frequency domain. It has 2 peaks at 10kHz and -10kHz (maximum magnitude which is 1 at $f = \pm 10$kHz and decreasing magnitude on both sides of the two peaks). Hence, $H_2(t)$ is a band-pass filter centered at $f = \pm 10$kHz. The 3-dB frequency pass band of the filter is $[9,900, 10,100]$Hz. That is, the 3-dB cutoff frequencies are 9,900 Hz and 10,100 Hz.

**Solution to Problem 3:**

(a) Since the RC circuits are in cascade, the impulse response of the overall system can be obtained by convolving the impulse responses of all sub-systems. Thus, the transfer function of the overall system is

$$H(f) = \prod_{i=1}^{N} H_i(f) = \frac{1}{(1 + j2\pi f RC)^N}.$$

Hence, the magnitude response is

$$|H(f)| = [H(f)H^*(f)]^{1/2} = \frac{1}{[1 + 4\pi^2 (RC)^2 f^2]^{N/2}}.$$

(b)

$$|H(f)| = \frac{1}{[1+T^2f^2/N]^{N/2}} = \frac{1}{[1+T^2f^2/N]^{\frac{N}{T^2f^2}\frac{T^2f^2}{2}}} = e^{-f^2T^2/2}.$$



**Solution to Problem 4:** (a)

$$G(f) = \frac{2}{1+(2\pi f)^2},$$

Thus

$$|G(f)|^2 = \frac{4}{[1+(2\pi f)^2]^2}.$$

The energy spectral density of the output is

$$\Psi_y(f) = |H(f)|^2\Psi_{g_1}(f) = \frac{4}{1+(2\pi f)^2}.$$



Thus

$$E_y = \int_{-\infty}^{\infty} \frac{4}{1+(2\pi f)^2} df = \frac{2}{\pi} \arctan(2\pi f)\Big|_{-\infty}^{\infty} = 2.$$

(b) The energy spectral density of the output is



$$\Psi_y(f) = |H(f)|^2\Psi_g(f) = \begin{cases} \frac{4}{1+(2\pi f)^2} & 0 < |f| \le f_0 \\ 0 & \text{otherwise} \end{cases}.$$



Thus

$$E_y = 2\int_{-f_0}^{f_0} \frac{4}{1+(2\pi f)^2} df = \frac{4}{\pi}\arctan(2\pi f_0).$$



This solution demonstrates that different phase responses of LTI systems does not change the energy spectral density of the output. Rather, the magnitude responses effect output's energy spectral density.



**Solution to Problem 5:** For the input signal $x(t)$, from the figure, $H(f) = rect(f/4) + rect(f/2)$. So

$$\begin{aligned} \Psi_x(f) &= |X(f)|^2 = [rect(f/4) + recr(f/2)]^2 \\ &= rect^2(f/4) + rect^2(f/2) + 2rect(f/4)rect(f/2) \\ &= rect(f/4) + 3rect(f/2), \\ R_x(\tau) &= \mathcal{F}^{-1}[\Psi_x(f)] = 4sinc(4\tau) + 6sinc(2\tau), \\ E_x &= R_x(0) = 10. \end{aligned}$$

For the output signal $y(t) = x(t) * h(y)$, where $H(f) = rect(f/2)$.So,

$$\begin{aligned} Y(f) &= X(f)H(f) = 2rect(f/2), \\ \Psi_y(f) &= |Y(f)|^2 = 4recr(f/2), \\ R_Y(\tau) &= \mathcal{F}^{-1}[\Psi_y(f)] = 8sinc(2\tau) \\ E_y &= R_x(0) = 8 \end{aligned}$$