F Econometrics

Slides-05: Ti Analysis using ARMA models Part 2

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Plan.

Time Series Models (Mai ... ical Aspects)

- MA process
- AR process
  - Wold Decomposition
  - AF and PACF patterns
  - Impulse response function

- Moving average model
  - In Wold decomposi[illegible] as $i \to \infty$. A simple approximation to the GLP is to restricting

$$\theta_i = 0 \text{ for all } i > q.$$

  - The result is $MA(q)$ model:

$$y_t = \mu + \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q}, \ \epsilon_t \sim \text{ i.i.d } WN(0, \sigma^2),$$

where y_t is the "average" of the current shock and its q recent lags. The shock ϵ_t and its lags are unobservable.

  - Use lag operator L: $Lz_t = z_{t-1}$ to write $MA(q)$:

$$y_t = \mu + \Theta(L)\epsilon_t,$$

where

$$\Theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q.$$

MA(1) model

- $MA(1)$ model
  - $MA(1)$ model (as a data generating process)

$$y_t = \mu + \epsilon_t + \theta_1 \epsilon_{t-1}, \epsilon_t \sim \text{ i.i.d } WN(0, \sigma^2),$$

  - $MA(1)$:

$$y_t = \mu + \Theta(L)\epsilon_t,$$

where $\Theta(L) = 1 + \theta_1 L$.

- $MA(1)$ model: Charac
  - It is always stationa
    $E(y_t) = \mu$, $Var(y_t) = (1+\theta_1^2)\sigma^2$,

$$\gamma_j = Cov(y_t, y_{t-j}) = \begin{cases} \theta_1\sigma^2, & j = 1 \\ 0, & j > 1 \end{cases}$$

$$\rho_j = \frac{\gamma_j}{\gamma_0} = \begin{cases} \theta_1/(1+\theta^2), & j = 1 \\ 0, & j > 1 \end{cases}$$ (A cutoff at $j = 1$).

  - If the estimated $\hat{\rho}_j$ has a cutoff at $j = 1$, the time series may be fitted in an $MA(1)$ model.

MA(1) model: Conditional moments

- $MA(1)$ model: Conditional moments
  - Conditional on $\Omega_t = \{\epsilon_t, \epsilon_{t-1}, \cdots ; y_t, y_{t-1}, \cdots\}$

$$E\left(y_{t+h}|\Omega_t\right) = \left\{ \begin{array}{ll} \mu + \theta_1 \epsilon_t, & h = 1 \\ \mu, & h > 1 \end{array} \right\}$$

$$Var\left(y_{t+h}|\Omega_t\right) = \left\{ \begin{array}{ll} \sigma^2, & h = 1 \\ (1+\theta_1^2)\sigma^2 & h > 1 \end{array} \right\}.$$

- Conditional variance $\leq$unconditional variance (why?)

MA(1) model: Dynamic Behavior

- $MA(1)$ model: Impulse response function
  - the effect on y_{t+h} of a one-std-deviation increase in $]\epsilon_t$:

$$\sigma \frac{\delta y_{t+h}}{\delta \epsilon_t} = \left\{ \begin{array}{ll} \sigma\theta_1, & h = 1 \\ 0, & h > 1 \end{array} \right\}.$$

- $MA(1)$ model: Invertil
  - Can we back out a from:

$$\rho_j = \frac{\gamma_j}{\gamma_0} = \left\{ \begin{array}{ll} \theta_1/(1+\theta_1^2), & j = 1 \\ 0, & j > 1 \end{array} \right\}.$$

Can we get to know $\{\epsilon_t, \epsilon_{t-1}, \cdots\}$ based on $\{y_t, y_{t-1}, \cdots\}$?

  - Yes if MA is *invertible*

- The $MA(q)$ process $y_t = \mu + \Theta(L)\epsilon_t$ is invertible if the roots of $\Theta(z) = 0$ are all oytside the unit circle.

- For $MA(1)$, the root of $1 + \theta_1 z = 0$ is $z = -1/\theta_1$. Hence, $MA(1)$ is invertible when $|-1/\theta_1| > 1$ or $|\theta_1| < 1$.

- **Invertible** in the sense that $\Theta(L)^{-1}$ exists properly.

MA(1) model: Invertibility

- $MA(1)$ model: Invertibility
  - When MA is invertible, the shock may be recovered from the observable: $\epsilon_t = \Theta(L)^{-1}(y_t - \mu)$. For $MA(1)$, when invertible,

$$\Theta(L)^{-1} = (1+\theta_1 L)^{-1} = 1 + (-\theta_1)L + (-\theta_1)^2 L^2 + \cdots, \quad (1)$$

$$\epsilon_t = y_t - \mu + \sum_{i=1}^{\infty} (-\theta_1)^i (y_{t-i} - \mu) \quad (2)$$

$$= y_t + \sum_{i=1}^{\infty} (-\theta_1)^i y_{t-i} - \mu/(1+\theta_1). \quad (3)$$

Hint. Use expansion $1/(1-x) = 1 + x + x^2 + \cdots$

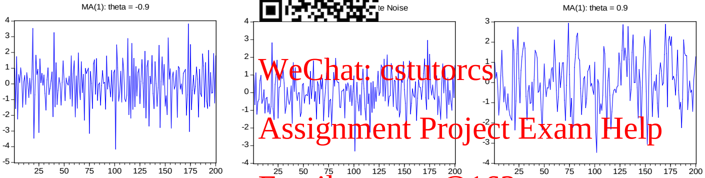
  - Parameters can be estimated by minimizing $\sum_{t=1}^{T} \epsilon_t^2$
  - The alternative expression: $y_t = \mu/(1+\theta_1) - \sum_{i=1}^{\infty} (-\theta_1)^i y_{t-i} + \epsilon_t$ indicates that the PAC function of invertible $MA(1)$ has no cutoffs and decays exponentially.

$MA(1)$ model: Example

$MA(1)$: simulated and fitted

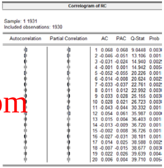
eg. time series plot … ated MA(1)

$\rho_1 = \theta_1/(1+\theta_1^2)$



eg. NYSE comp return

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.035311	0.02457	1.43729	0.1508
MA(1)	0.075177	0.02271	3.31031	0.0009


Correlogram of RC

Sample: 1 1931
Included observations: 1930

	AC	PAC	Q-Stat	Prob
1	0.068	0.068	9.0448	0.003
2	-0.046	-0.051	13.106	0.001
3	-0.031	-0.024	14.940	0.002
4	-0.001	0.001	14.942	0.005
5	-0.052	-0.055	20.226	0.001
6	-0.014	-0.008	20.624	0.002
7	-0.033	-0.037	22.761	0.002
8	0.011	0.012	22.992	0.003
9	0.033	0.028	25.155	0.003
10	0.028	0.021	26.723	0.003
11	-0.043	-0.044	30.332	0.001
12	0.054	0.061	35.987	0.000
13	0.015	0.004	36.403	0.001
14	-0.013	-0.009	36.720	0.001
15	-0.002	0.008	36.726	0.001
16	-0.027	-0.031	38.181	0.001
17	0.014	0.025	38.588	0.002
18	-0.007	-0.015	38.677	0.003
19	0.022	0.026	39.639	0.004
20	0.006	0.004	39.710	0.005

Dynamic Behaviour of a Moving Average Process $MA(q)$

An MA process is sim[illegible] combination of white noise error terms ?. These error terms can [illegible] **impulses** or **innovations** or **shocks** while the MA model describ[illegible]**amic impact** of these shocks on the series y_t.

The **impulse response function**, i.e. the dynamic impact of an impulse ϵ_t on $y_t, y_{t+1}, \cdots$ is given by

$$
\begin{aligned}
\delta y_t/\delta\epsilon_t &= 1 \\
\delta y_t/\delta\epsilon_t &= \theta_1 \\
&\vdots \\
\delta y_{t+q}/\delta\epsilon_t &= \theta_q \\
\delta y_{t+q+k}/\delta\epsilon_t &= 0, \text{ for } k > 0
\end{aligned}
$$

MA(q) model: Properties

■**General Properties of a Moving Average Process** $MA(q)$

- $E(y_t) = \mu$
- $\gamma_0 = (1 + \theta_1^2 + \theta_2^2 + \cdots$
- The ACF:

$$\begin{aligned}\gamma_k &= (\theta_k + \theta_{k+1}\theta_1 + \theta_{k+2}\theta_2 + \cdots + \theta_q\theta_{q-k})\sigma^2, \text{ fpr } k = 1, \cdots, q.\\ \gamma_k &= 0, \text{ for } k > q.\end{aligned}$$

- The PACF? $p_k \neq 0\ \forall k$ dies out slowly

■**Stationarity conditions** for an MA process:

- γ_0 is finite
- γ_k is finite

$\Longrightarrow$ a finite order MA process will always be stationary.

$MA(q)$ Conclusions

- As the **ACF cuts o**[illegible] **lags**, the order of an MA process can be determined f[illegible]spection of the sample ACF.
- It can be shown (see below) that the **PACF dies out slowly**.
- A finite order MA process is **stationary by construction**, as it is a weighted sum of a fixed number of white noise processes, i.e. the mean, variance and autocovariances don't depend on time!

Defining an Autoregressive Process

Let ε_t be a white noise process, then:

$$\begin{aligned} y_t &= \alpha_0 + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \ldots + \alpha_p y_{t-p} + \varepsilon_t \qquad (12) \\ &= \alpha_0 + \sum_{i=1}^{p} \alpha_i y_{t-i} + \varepsilon_t \end{aligned}$$

is an **autoregressive process of order** p, denoted AR(p).

$\rightarrow y_t$ depends on its own lagged values and on the current value of a white noise disturbance term ε_t.

The model can conveniently be rewritten in so-called **lag operator notation** as

$$\begin{aligned} y_t &= \alpha_0 + \sum_{i=1}^{p} \alpha_i L^i y_t + \varepsilon_t \quad \text{with} \quad L^i y_t = y_{t-i} \\ \alpha(L)\, y_t &= \alpha_0 + \varepsilon_t \qquad (13) \end{aligned}$$

where $\alpha(L) = 1 - \alpha_1 L - \alpha_2 L^2 - \ldots - \alpha_p L^p$ is a lag polynomial of order p

AR Process: Impulse response function

In an AR process, the value fc y a linear combination of past values plus a white noise $_t$. Again, these error terms can be seen as **impulse ions** or **shocks** while the AR model describes the **dynamic impact** of these shocks on the series y_t.

In order to trace out the dynamic impact of an impulse ε_t on $y_t, y_{t+1}, \ldots$, it is very convenient to first 'solve' the AR model in terms of the ε sequence. For notational convenience, first consider an AR(1) process

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \varepsilon_t$$

where ε_t is a white noise process.

AR Process: Impulse response function

The easiest way to express y_t [illegible] on of the ε sequence is b[illegible] **backward substitution**. Thi[illegible] bstituting

$$y_{t-1} = \alpha [illegible] + \varepsilon_{t-1}$$

in the equation for y_t to obtain

$$\begin{aligned} y_t &= \alpha_0 + \alpha_1 \left(\alpha_0 + \alpha_1 y_{t-2} + \varepsilon_{t-1} \right) + \varepsilon_t \\ &= \left(1 + \alpha_1 \right) \alpha_0 + \alpha_1^2 y_{t-2} + \alpha_1 \varepsilon_{t-1} + \varepsilon_t \end{aligned}$$

Next substitute

$$y_{t-2} = \alpha_0 + \alpha_1 y_{t-3} + \varepsilon_{t-2}$$

in the equation for y_t to obtain

$$y_t = \left(1 + \alpha_1 + \alpha_1^2 \right) \alpha_0 + \alpha_1^3 y_{t-3} + \alpha_1^2 \varepsilon_{t-2} + \alpha_1 \varepsilon_{t-1} + \varepsilon_t$$

AR Process: Impulse response function

After repeating this $t-1$ times, we obtain

$$
\begin{aligned}
y_t &= \left(1+\alpha_1+\ldots+\alpha_1^{t-1}\right)\alpha_0 + \alpha_1^t y_0 + \alpha_1^{t-1}\varepsilon_1+\ldots+\alpha_1\varepsilon_{t-1}+\varepsilon_t \\
&= \alpha_0 \sum_{i=0}^{t-1}\alpha_1^i + \alpha_1^t y_0 + \sum_{i=0}^{t-1}\alpha_1^i\varepsilon_{t-i} \qquad (14)
\end{aligned}
$$

where y_0 is the initial condition or the value for y in period 0.

The **impulse response function** can now easily be obtained

$$
\begin{aligned}
dy_t/d\varepsilon_t &= \alpha_1^0 = 1 \\
dy_{t+1}/d\varepsilon_t &= \alpha_1 \\
dy_{t+2}/d\varepsilon_t &= \alpha_1^2 \\
dy_{t+3}/d\varepsilon_t &= \alpha_1^3 \\
&\vdots
\end{aligned}
$$

AR Process: Convergence

Note that whether an AR(1) process is mean-reverting after being hit by a shock depends on the value for α_1. Two cases can be distinguished:

- The **convergence case** $|\alpha_1| < 1$
  A shock affects all future observations but with a decreasing effect, i.e. the AR(1) process is mean-reverting.

- The **non-convergence case** $|\alpha_1| \geq 1$
  A shock affects all future observations but with an equal impact ($\alpha_1 = 1$) or with an increasing impact ($\alpha_1 > 1$), i.e. the AR(1) series is not mean-reverting.

Properties of an AR(1)

Let $t \to \infty$ in eq. (14):

$$y_t = \alpha_0 \sum\nolimits_{i=0}^{\infty} \alpha_1^i + \alpha_1^{\infty} y_0 + \sum\nolimits_{i=0}^{\infty} \alpha_1^i \varepsilon_{t-i} \tag{15}$$

- The expected value of y_t is given by

$$E(y_t) = E\left(\left(1 + \alpha_1 + \alpha_1^2 + \ldots\right)\alpha_0 + \alpha_1^{\infty} y_0 + \sum\nolimits_{i=0}^{\infty} \alpha_1^i \varepsilon_{t-i}\right)$$

$$= E\left(\left(1 + \alpha_1 + \alpha_1^2 + \ldots\right)\alpha_0 + \alpha_1^{\infty} y_0\right)$$

$\to$ if $|\alpha_1| < 1$: $E(y_t)$ converges to $\dfrac{\alpha_0}{(1-\alpha_1)}$

$\to$ if $|\alpha_1| \geq 1$: $E(y_t)$ is time-dependent

Properties of $AR(1)$ Process: Unconditional Variance

- The variance of y_t is given by

$$
\begin{aligned}
V(y_t) &= E\left(y_t - E(y_t)\right)^2 \\
&= E\left(\sum_{i=0}^{\infty} \alpha_1^i \varepsilon_{t-i}\right)^2 \\
&= E\left(\varepsilon_t^2 + \alpha_1^2 \varepsilon_{t-1}^2 + \alpha_1^4 \varepsilon_{t-2}^2 + \ldots + \text{cross-products}\right) \\
&= E\left(\varepsilon_t^2\right) + \alpha_1^2 E\left(\varepsilon_{t-1}^2\right) + \alpha_1^4 E\left(\varepsilon_{t-2}^2\right) + \cdots \\
&= \left(1 + \alpha_1^2 + \alpha_1^4 + \ldots\right) \sigma^2
\end{aligned}
$$

$\rightarrow$ if $|\alpha_1| < 1$: $V(y_t)$ converges to $\dfrac{\sigma^2}{(1-\alpha_1^2)}$

$\rightarrow$ if $|\alpha_1| \geq 1$: $V(y_t)$ is time-dependent

- The autocovariances γ_k :

$$
\begin{aligned}
\gamma_1 = cov(y_t, y_{t-1}) &= E\left(\left(y_t - E(y_t)\right)\left(y_{t-1} - E(y_{t-1})\right)\right) \\
&= E\left(\left(\varepsilon_t + \alpha_1\varepsilon_{t-1} + \alpha_1^2\varepsilon_{t-2} + \dots\right)\left(\varepsilon_{t-1} + \alpha_1\varepsilon_{t-2} + \alpha_1^2\varepsilon_{t-3} + \dots\right)\right) \\
&= E\left(\alpha_1\varepsilon_{t-1}^2 + \alpha_1^3\varepsilon_{t-2}^2 + \alpha_1^5\varepsilon_{t-3}^2 + \dots + \text{cross-products}\right) \\
&= \alpha_1 E\left(\varepsilon_{t-1}^2\right) + \alpha_1^3 E\left(\varepsilon_{t-2}^2\right) + \alpha_1^5 E\left(\varepsilon_{t-3}^2\right) + \dots \\
&= \left(\alpha_1 + \alpha_1^3 + \alpha_1^5 + \dots\right)\sigma^2 \\
&= \alpha_1\left(1 + \alpha_1^2 + \alpha_1^4 + \dots\right)\sigma^2
\end{aligned}
$$

$\rightarrow$ if $|\alpha_1| < 1$: γ_1 converges to $\alpha_1 \dfrac{\sigma^2}{(1-\alpha_1^2)}$

$\rightarrow$ if $|\alpha_1| \geq 1$: γ_1 is time-dependent

$$
\begin{aligned}
\gamma_2 = cov\left(y_t, y_{t-2}\right) &= E\left(\left(y_t - E\left(y_t\right)\right)\left(y_{t-2} - E\left(y_{t-2}\right)\right)\right) \\
&= E\left(\left(\varepsilon_t + \alpha_1\varepsilon_{t-1} + \alpha_1^2\varepsilon_{t-2} + \ldots\right)\left(\varepsilon_{t-2} + \alpha_1\varepsilon_{t-3} + \alpha_1^2\varepsilon_{t-4} + \ldots\right)\right) \\
&= E\left(\alpha_1^2\varepsilon_{t-2}^2 + \alpha_1^4\varepsilon_{t-3}^2 + \alpha_1^6\varepsilon_{t-4}^2 + \ldots + \text{cross-products}\right) \\
&= \alpha_1^2 E\left(\varepsilon_{t-2}^2\right) + \alpha_1^4 E\left(\varepsilon_{t-3}^2\right) + \alpha_1^6 E\left(\varepsilon_{t-4}^2\right) + \ldots \\
&= \left(\alpha_1^2 + \alpha_1^4 + \alpha_1^6 + \ldots\right)\sigma^2 \\
&= \alpha_1^2\left(1 + \alpha_1^2 + \alpha_1^4 + \ldots\right)\sigma^2
\end{aligned}
$$

$\rightarrow$ if $|\alpha_1| < 1$: $\quad \gamma_2$ converges to $\alpha_1^2 \dfrac{\sigma^2}{\left(1-\alpha_1^2\right)}$

$\rightarrow$ if $|\alpha_1| \geq 1$: $\quad \gamma_2$ is time-dependent

$$\gamma_k = cov\left(y_t, y_{t-k}\right) = E\left(\left(y_t - E\left(y_t\right)\right)\left(y_{t-k} - E\left(y_{t-k}\right)\right)\right)$$

$\rightarrow$ if $|\alpha_1| < 1$: γ_k converges to $\alpha_1^k \dfrac{\sigma^2}{\left(1-\alpha_1^2\right)}$

$\rightarrow$ if $|\alpha_1| \geq 1$: γ_k is time-dependent

- The ACF (for stationary series!) is given by

$$\rho_1 = \gamma_1 / \gamma_0 = \alpha_1$$

$$\rho_2 = \gamma_2 / \gamma_0 = \alpha_1^2$$

$$\vdots$$

$$\rho_k = \gamma_k / \gamma_0 = \alpha_1^k$$

AR Process: Stationary Conditions

Stationarity conditions for rocess

- $\alpha_1^{\infty} = 0$
- $(1 + \alpha_1 + \alpha_1^2 + ...)$ is finite
- $(1 + \alpha_1^2 + \alpha_1^4 + \ldots)$ is finite
- $\alpha_1 \left(1 + \alpha_1^2 + \alpha_1^4 + \ldots\right)$ is finite
- $\ldots$

$\rightarrow$ an AR(1) process is stationary is $|\alpha_1| < 1$.

AR Process: Conclusions

- The PACF cuts off afte
- The ACF is infinite in e dies out for covariance stationary processes).
- The properties of an AR(1) process crucially depend on the value for α_1
  - If $|\alpha_1| < 1$ the AR(1) process can be written as a stable infinite MA process (the so-called **MA representation**):

$$y_t = \frac{\alpha_0}{1-\alpha_1} + \sum_{i=0}^{\infty} \alpha_1^i \varepsilon_{t-i}$$

    In this case the series is **stationary** as it has finite constant mean, variance and autocovariances.
  - If $|\alpha_1| \geq 1$ no stable MA representation exists. In this case the series is **non-stationary** as the mean, variance and autocovariances are time-varying.

$AR(1)$ Example: Simulated and Fitted

eg. time series plots ... d AR(1)

$\rho_j = \phi_1^j$

eg. NYSE comp return: c below is in fact $\mu = c/(1 - \phi_1)$

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.035159	0.024547	1.43235	0.1522
AR(1)	0.068401	0.022727	3.00976	0.0026