# ECON3206/5 ... ew of Linear regression model

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- Suppose we are interested i[illegible] in the United states. Wages vary across workers and can be described using a pr[illegible]ribution.
- Formally, we view the wage of an individual worker as a random variable $wage$ with **probability distribution**

$$F(u) = Pr(wage \leq u)$$

- A person wage is random: do not know the wage before it is measured. Observed wages are realizations from the distribution $F$
- We usually do not know $F$: we can learn about the distribution from many realizations of the *wage* variable.




Figure 3.1: Wage Distribution and Density. All full-time U.S. workers

- Important measures of cent[illegible] are the median and the mean. The median $m$ of a continuous distribution $F$ is [illegible] solution to

$$F(m) = \frac{1}{2}$$

The median U.S. wage in 2009 is \$19.23.



- A convenient measure (but not robust) of central tendency is the **mean** or **expectation**.
- The expectation of a random variable $y$ with density $f$ is



$$\mu = E(y) = \int_{-\infty}^{\infty} y f(y)\, dy$$



We use the single character $y$ to denote the random variable, rather than the more cumbersome label $wage$

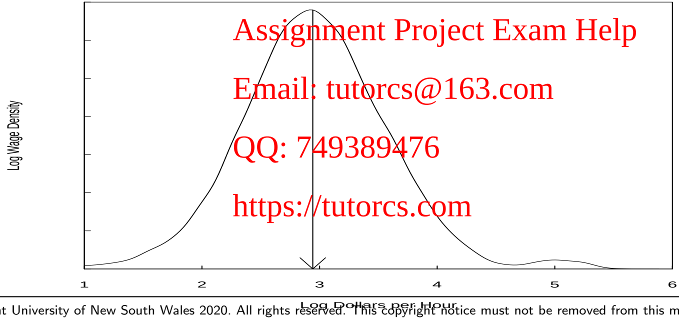


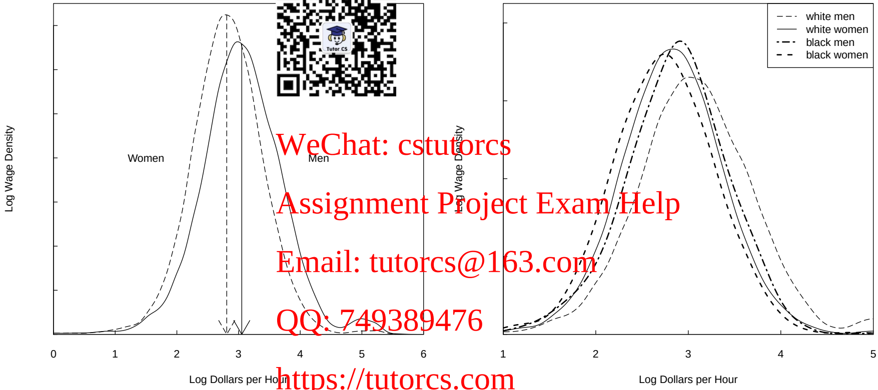
The mean wage in this example is \$23.90. The mean is not robust in the presence of substantial skewness or thick tails, which are both features of the wage distribution.

- In this context it is useful to transform the data by taking natural logarithm.

  The mean of the random variable $log(wage)$ also denoted $log(y)$ is $2.95.
- The density of log wages is [illegible]ewed and fat-tailed than the density of the level of wages, so its mean $E(log(y$ [illegible] a much better measure of central tendency of the distribution.
- In fact, the geometric mean [illegible] $y))) = \$19.11$ is a robust measure of central tendency of $y$!!






Log Wage Density
Women
Men
0
1
2
3
4
5
6
Log Dollars per Hour
Log Wage Density
white men
white women
black men
black women
1
2
3
4
5
Log Dollars per Hour

- Is the wage distribution the [illegible] workers, or does the wage distribution vary across subpopulations?
- the plots above displays the [illegible] log wages in the U.S. men and women with their means (3.05 and 2.81).
- the means displayed are called the **conditional means** (or **conditional expectations**) of log wages given gender:

$$E(log(wage)|gender = man) = 3.05$$

$$E(log(wage)|gender = woman) = 2.81$$

- Here the conditioning variable $gender$ is a random variable from the viewpoint of econometric analysis.
- We can use more than one variable in the conditioning of the expectation:

$$E(log(wage)|gender = man, race = white) = 3.07$$

- In many cases it is convenie[illegible]y notation by writing variables using single characters, typically $y$, $x$, a[illegible]
- Typically in econometrics it [illegible]nal to denote the dependent variable by the letter $y$ and the conditioning variabl[illegible]ter $x$, and multiple conditioning by the subscripted letters $x_1, x_2, \cdots, x_k$.
- Conditional expectation can be written with the generic notation



$$E(y|x_1, x_2, \cdots, x_k) = m(x_1, x_2, \cdots, x_k)$$



- This is called the **conditional expectation function**. For example, the conditional expectation of $y = log(wage)$ given $(x_1, x_2) = (gender, race)$ is given by



| | men | woman |
|---|---|---|
| white | 3.07 | 2.82 |
| black | 2.86 | 2.73 |
| other | 3.03 | 2.86 |

- An econometrician has obse… …a

$$\ldots (x_2, y_2), \cdots, (x_n, y_n)\}$$

- If the data are cross-section… …nable to assume they are mutually independent
- If the data are randomly gathered, it is reasonable to model each observation as a random draw from the same probability distribution. In this case the data are **independent and identically distributed**, or **iid**.



- To study how the distribution of $y_i$ varies with $x_i$, we can focus on the conditional density of $y_i$ given $x_i$ and its conditional mean $m(\mathbf{x}_i)$.



- The conditional mean function is the regression function.



$$y_i = E[y_i|x_i] + (y_i - E[y_i|x_i]) = E[y_i|x_i] + \mu_i$$



- $E[\mu_i|x_i] = 0$.
- $\mu$ is called the conditional expectation function error.

- While the conditional mean $m(\mathbf{x})$ is the best predictor of $y$ among all functions of $\mathbf{x}$ , its functional form is typically [illegible]
- For empirical implementatio[illegible]ation, it is typical to replace $m(\mathbf{x})$ with an approximation.
- Most commonly, this appro[illegible]near in $\mathbf{x}$.
- It is convenient to augment the regressor vector $x$ by listing the number 1 as an element. We call this the **constant** or **intercept** term.



$$m(\mathbf{x}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k = \mathbf{x}'\beta$$



where



$$\mathbf{x} = (1, x_1, x_2, \cdots, x_K)'$$

$$\beta = (\beta_0, \beta_1, \beta_2, \cdots, \beta_K)'$$



- Boldface letter indicates a column vector. In the case of one regressor $x$ and a constant term: $\beta = (\beta_0, \beta_1)$ and $\mathbf{x} = (1, x)'$, and $\mathbf{x}'\beta = \beta_0 + \beta_1 x$.



(Wisdom: Models should have a constant term unless the theory says they should not.)

### Assumption

*MLR.1 Linearity : The population model is linear in the parameters:*

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k + \mu, \qquad (1)$$

*where $\beta_i$, $i = 0, \cdots, k$ are the unknown (constants) parameters of interest, $x_i$'s are the regressors which can be assumed to be either fixed or random, and $\mu$ the random error.*



If the linearity assumption is violated then the regression model is misspecified. This is known as *functional form misspecification* (although this is still linear in $\beta$'s )

- The model does not account for some important nonlinearities;
- Omitting important variables is also model misspecification;
- Generally functional form misspecification causes biases in the remaining parameter estimators.

## Example

Suppose that the correct specification of the wage equation is:

$$log(wage) = \beta_0 + \beta_1 educ + \beta_2 exper + \beta_3 (exper)^2 + \mu, \quad (2)$$

then the return for an extra year of experience is



$$\frac{\partial wage}{\partial exper} = wage \times [\beta_2 + 2\beta_3 exper].$$



If the estimated model is instead:



$$log(wage) = \beta_0 + \beta_1 educ + \beta_2 exper + \mu, \quad (3)$$

then use of the biased (upward) OLS estimator of $\beta_2$ can be misleading.
If the estimated model is instead:





$$wage = \beta_0 + \beta_1 educ + \beta_2 exper + \beta_3 (exper)^2 + \mu, \quad (4)$$

$$\partial wage / \partial exper = \beta_2 + 2\beta_3 exper \quad (5)$$

## Assumption

*MLR2. Random Sampling:*

*We have a random sample of $n$ observations, $\{(x_{i1}, x_{i2}, \cdots, x_{ik}, y_i) : i = 1, 2, \cdots, n\}$, following the population model in Assumption 1.*

*Nonrandom sampling causes OLS estimator to be biased and inconsistent.*

Scenarios where Assumption 2 does not hold include:

- Missing Data
- Nonrandom Samples
- Outliers

# Assumption MLR3: No Perfect Collinearity



## Assumption

*MLR3. No Perfect Collinearity:*
*In the sample and in the population, none of the independent variables is constant, and there are no exact linear relationships among the independent variables.*



Scenarios where Assumption 3 is violated include:

- One independent variable is a linear combination of one or more other regressors. It is not a problem to include nonlinear functions of the same variables
  - For example include consumption, investment and income on the right hand side of the regression equation. In national accounts, national income is the sum of consumption and investment
  - Including all seasonal dummies and the constant term in the regression

# Assumption MLR4: No Perfect Collinearity



## Assumption

*MLR4. Zero Conditional Mean:*

*The error term $\mu$ has a conditional expected value of zero given any values of the independent variables,*



$$E(\mu | x_1, \cdots, x_K) = 0$$



This assumption fails for many reasons, these include:

- Misspecification of the functional form.
- Omitting important factors correlated with any of the regressors: omitted variables bias.
- Measurement error in the explanatory variables (more later, W. Ch. 15).
- Endogeneity and Simultaneity: some explanatory variables are determined jointly with the dependent variable

## Theorem

*Unbiasedness*

*Under Assumptions MLR1-MLR4, the ordinary least squares (OLS) estimator,* $\widehat{\beta}_j$, $j = 0, \cdots, K$ *is unbiased. That is its expected value is equal to the population parameter,*



$$E\left(\widehat{\beta}_j\right) = \beta_j, \quad \text{for } j = 0, \cdots, K$$



- OLS estimator minimizes the sum of squared residuals. For the simple case of one regressor $x_1$, $\widehat{\beta} = (\widehat{\beta}_0, \widehat{\beta}_1)$ minimizes,



$$SSR(\beta) = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_1)^2$$

Consider the case of multiple regressors:

$$y_i = \beta_0 + \beta_1 x_{i1} + \mu_i \tag{6}$$

The population regression coefficients $\beta_0$ and $\beta_1$ are defined by solving:

$$\beta_0, \beta_1 = \arg\min_{b_0, b_1} E\left[(y_i - b_0 - b_1 x_i)^2\right]$$

The first order conditions,

$$\frac{\partial E\left[(y_i - \beta_0 - \beta_1 x_i)^2\right]}{\partial \beta_0} = E\left[-2(y_i - \beta_0 - \beta_1 x_i)\right] = 0 \tag{7}$$

$$\frac{\partial E\left[(y_i - \beta_0 - \beta_1 x_i)^2\right]}{\partial \beta_1} = E\left[-2x_i(y_i - \beta_0 - \beta_1 x_i)\right] = 0 \tag{8}$$

Solving for $\beta_0$ and $\beta_1$:

$$\beta_1 = \frac{Cov(y_i, x_i)}{V(x_i)} \tag{9}$$

$$\beta_0 = E[y_i] - \beta_1 E[x_i] \tag{10}$$

Consider the case of multiple regressors:

$$y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_k x_{iK} + \mu_i \tag{11}$$

**Matrix notation!**
Let $\mathbf{x}_i = (1, x_{i1}, \cdots, x_{iK})'$ be the vector of regressors (including the constant term) and $\beta = (\beta_0, \beta_1, \cdots, \beta_K)$, then:



$$y_i = \mathbf{x}_i'\beta + \mu_i \tag{12}$$



- Useful representation! The population regression coefficients are defined by:



$$\beta_k = \frac{Cov(y_i, \overline{x}_{ki})}{V(\overline{x}_{ki})}, \tag{13}$$



  where $\overline{x}_{ki}$ is the residual from **a regression of $x_{ki}$ on all other variables.**



- Each coefficient in a multivariate regression is the bivariate slope coefficient for the corresponding regressor, after "**partialling out**" all the other variables in the model.

## Assumption

*MLR5. Homoskedasticity*

*The error term has the same variance given any values of the explanatory variables:*



$$Var(\mu_i|x_{i1},\cdots,x_{iK})=\sigma^2$$



- Homoskedasticity : the variance of the error term does not depend on the explanatory variables,



- When is this a bad assumption?



  If omitted variables are not correlated with the included variables, but have a different order of magnitude for (groups of) observations.

# Gauss Markov Theorem



**Theorem**

*Gauss Markov*
*Under Assumptions MLR1-MLR5, OLS estimator is* **BLUE**.

- What happens to OLS estimator if one/all of these assumptions does not hold?

- Suppose the correct model of variables,

$$X_1\beta_1 + X_2\beta_2 + \epsilon$$

- Compute least squares omitting $X_2$. Denote this estimator by $\widetilde{\beta_1}$ Some easily proved results:
  - $V(\widetilde{\beta_1})$ is smaller than $V(\widehat{\beta_1})$, i.e., you get a smaller variance when you omit $X_2$. (One interpretation: Omitting $X_2$ amounts to using extra information ($\beta_2 = 0$). Even if the information is wrong (see the next result), it reduces the variance. (This is an important result.)



- **(No free lunch)**



$$E[\widetilde{\beta_1}] = \beta_1 + (X_1'X_1)^{-1}X_1'X_2\beta_2 \neq \beta_1.$$



So, $\widetilde{\beta_1}$ is **biased**.

The bias can be huge. Can reverse the sign of a price coefficient in a "demand equation."



$\widetilde{\beta_1}$ may be more "**precise**." Smaller variance but positive bias. If bias is small, may still favor the short regression.

- (Free lunch?) Suppose $X_1'X$ ... the bias goes away. Interpretation, the information is not "right,"it is irrelevant ... ame as $\widehat{\beta_1}$.
- It can be shown that

$$V(\widehat{\beta}_1) = \frac{\sigma^2}{SST_1(1 - R_1^2)}$$



where $SST_1$ is the total variation in $X_1$ and $R_1$ is the $R - squared$ from the regression of $X_1$ on $X_2$. Furthermore,



$$V(\widetilde{\beta}_1) = \frac{\sigma^2}{SST_1}$$



- when $\beta_2 \neq 0$, $\widetilde{\beta}_1$ is biased, and $V(\widetilde{\beta}_1) < V(\widehat{\beta}_1)$;
- when $\beta_2 = 0$, both $\widetilde{\beta}_1$ and $\widehat{\beta}_1$ are unbiased, and $V(\widetilde{\beta}_1) < V(\widehat{\beta}_1)$;

- The variance of the OLS es , conditional on the sample values of the independent variables is

$$= \frac{\sigma^2}{SST_j(1-R_j^2)} \tag{14}$$

where $SST_j = \sum_{i=1}^{n}(X_{ij} - \bar{X}_j)^2$ is the total sample variation in $X_j$ and $R_j^2$ is the R-squared from the regression of $X_j$ on all other independent variables including constant term.



- The larger $\sigma^2$, the larger is the variance of OLS estimator. More noise means difficult to estimate the partial effect of any variable.
- The larger the total variation in $X_j$, the smaller is the variance of $\widehat{\beta}_j$. To increase the in sample variation of $X_j$, one can increase the sample size!

- The variance of an estimated coefficient will tend to be larger if there are other $X$'s in the model that can predict $X_j$. This is reflected by a high $R_j^2$ in equation 14;



- The standard error of prediction will also tend to be larger if there are unnecessary or redundant $X$'s in the model.

This is a variant on linear regression that downplays the influence of outliers

- First performs the original OLS regression
- Drops observations with Cook's distance $> 1$
- Calculates weights for each observation based on their residuals
- Performs weighted least squares regression using these weights.