Financial Econometrics

lides-02

r Regression

Rev pplications in Finance

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Economics[1]

Linear Regression

- A model where one … is linearly explained by a group of variables $(X_{1t}, \cdots, X_{kt}), t =$ …
  - Easy to Implement
  - Versatile for financial data analysis
  - Foundation for more advanced models

- General formulation
  - $Y_t = \beta_1 + \beta_2 X_{t1} + \beta_3 X_{t2} + \cdots + \beta_K X_{tK} + \mu_t, t = 1 \cdots, T$
  - Y_t: dependent variable
  - $X_{t1}, \cdots X_{tK}$: explanatory variables, regressors
  - $\beta_1, \beta_2, \cdots, \beta_K$: parameters (to be estimated)
  - μ_t: error term
  - T: number of observations

Application 1: Capital Asset Pricing Model *aka CAPM*

One of the most importa[illegible]s of modern financial economics is the **quantification of the tr[illegible]een risk and expected return**. Common sensesuggests risky inves[illegible]ck market) will generally yield higher returns than investments[illegible]!

- Markowitz (1995) casts the investor's portfolio selection problem in terms of expected return and variance of the return.

→ Investors optimally hold a mean-variance efficient portfolio: a portfolio with the highest expected return for a given level of variance.

⟹ **The Efficient Frontier & Capital Market Line**

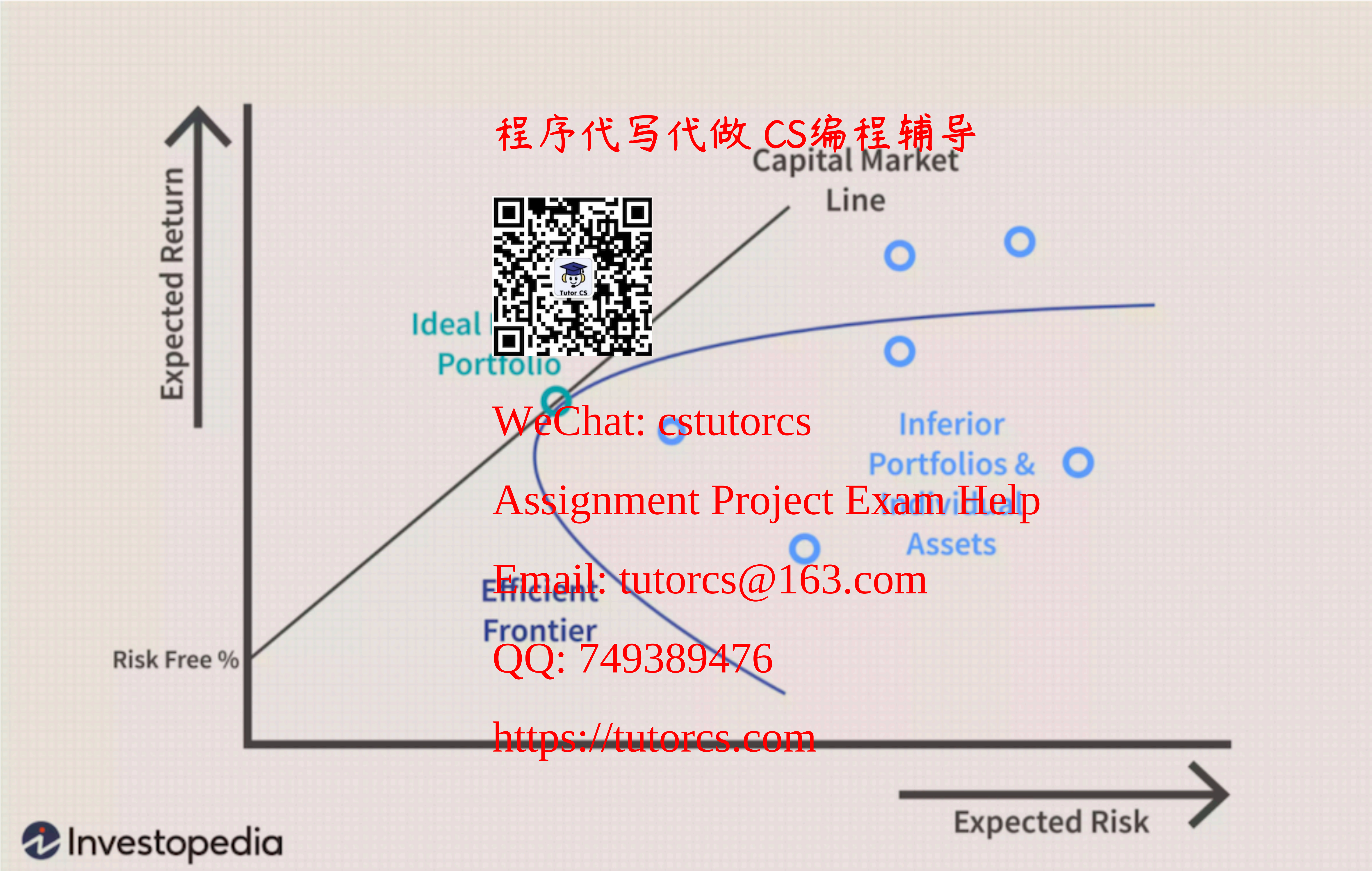
- Capital Asset Pricing Model is concerned with the pricing of assets in equilibrium. In equilibrium, all assets must be held by someone.

⟶ How investors determine the expected returns—and thereby asset prices—as a function of risk.

⟹ **The Security Market Line**

CAPM

- Given that: some risk can be diversified, diversification is easy and costless, and rational investors diversify
- There should be no premium associated with diversifiable risk.
- The question becomes: What is the equilibrium relation between systematic risk and expected return in the capital markets?
- The CAPM is the best known and most widely used equilibrium model of the risk/return (systematic risk/return) relation.



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CAPM Intuition

What would be a "fair" ... eturn on any stock?

- $E(R_{it}) = R_{ft}$ (risk ... Premium
- Risk free assets earn the risk-free rate (think of this as a rental rate on capital). The risk free compensate for time.
- If the asset is risky, we need to add a risk premium.
- The size of the risk premium depends on the amount of systematic risk for the asset (stock, bond, or investment project) and the price per unit risk.
- $R_{it} - R_{ft}$: Excess return

CAPM Intuition Formalized

$$E\left[R_{it}\right] = \frac{Cov\left(R_{it}, R_{mt}\right)}{Var\left(R_{mt}\right)}\left[E\left[R_{mt}\right] - R_{ft}\right]$$

$$E\left[R_{it}\right] = R_{ft} + \beta_i\left[E\left[R_{mt}\right] - R_{ft}\right]$$

The expression above is referred to as the "Security Market Line".

- $E\left[R_{mt}\right] - R_{ft}$ Market Risk premium (compensation for risk) or the price per unit of risk
- β_i: number of units of systematic risk
  - $\beta_i > 1$ (or < 1): the asset is more (less) risky than the market portfolio
  - $\beta_i < 0$: the asset is a hedge against the market portfolio
  - β_i how sensitive the asset to **market movement**

CAPM Formalized

Three inputs are require

1. An estimate of the r... erest rate. The current yield on short term treasury bills is one ... titioners tend to favor the current yield on longer-term treasury bonds but this may be a fix for a problem we don't fully understand.

2. An estimate of the market risk premium, $E\left[R_{mt}\right] - R_{ft}$. Expectations are not observable. Generally use a historically estimated value.

   The market is defined as a portfolio of all wealth including real estate, human capital, etc. In practice, a broad based stock index, such as the S&P 500 or the portfolio of all NYSE stocks, is generally used.

3. An estimate of beta.

CAPM: Econometric model

Let $X_{mt} = R_{mt} - R_{ft}$ and $X_{it} = R_{it} - R_{ft}$ and consider the econometric model:

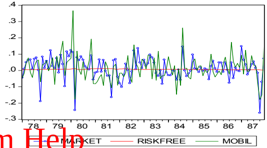
$$X_{it} = \alpha_i + \beta_i X_{mt} + \mu_{it}$$

- The CAPM can be examined by testing $H_0 : \alpha_i = 0$
- If $\alpha_i > 0$, asset i beats the market by earning more than $\beta_i E\left[X_{mt}\right]$
- This has been used to test the performance of mutual funds (application in the Brooks textbook)

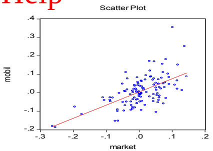
CAPM: Application

What detremines the ex[illegible]rn of an asset?

Example: Mobil (a US pet[illegible] 1978:01-1987:12 with $T = 120$.



Dependent Variable: E_MOBIL				
Method: Least Squares				
Sample: 1978M01 1987M12				
Included observations: 120				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.004241	0.005881	0.721087	0.4723
E_MARKET	0.714695	0.085615	8.347761	0.0000
R-squared	0.371287	Mean dependent var		0.009353
Adjusted R-squared	0.365959	S.D. dependent var		0.080468
S.E. of regression	0.064074	Akaike info criterion		-2.641019
Sum squared resid	0.484452	Schwarz criterion		-2.594561
Log likelihood	160.4612	F-statistic		69.68511
Durbin-Watson stat	2.087124	Prob(F-statistic)		0.000000



Application 2: The term structure of interest rates

- Interest rates are the price of money and, in equilibrium, interest rates equate the amount of borrowing to the amount of saving.
- The Term Structure of Interest Rates shows the relation between interest rates for different term-to-maturity loans.
- In the most basic sense, theories to explain the term structure are still based on interest rates equating the supply and demand for loanable funds.
- Different rates may exist over different terms because of expectations of changing inflation and differing preferences regarding longer-term vs. shorter-term saving.

The term structure of interest rate

The relation of long and [illegible]s?

- Monthly return on t[illegible]h bond at time n:$R_{n,t}$, $n = 1, 3$

  Rule of one price (E[illegible] Hypothesis)

$$(1 + R_{3,t})^3 = (1 + R_{1,t})(1 + E_t R_{1,t+1})(1 + E_t R_{1,t+2})$$

$$R_{3,t} \approx [R_{3,t} + E_t R_{1,t+1} + E_t R_{1,t+2}]/3$$

  where E_t is expectation formed at time t

- If $R_{1,t}$ follows a random walk: $R_{1,t+1} = R_{1,t} + v_{t+1}$ with $E_t v_{t+1} = 0$ then $E_t R_{1,t+1} = E_t R_{1,t+2} = R_{1,t}$ and

$$R_{3,t} \approx R_{1,t}$$

- Test the null $H_0 : \beta_0 = 0, \beta_1 = 1$ in $R_{3,t} = \beta_0 + \beta_1 R_{1,t} + u_t$.

Term structure of interest rate: example

- Consider that given [illegible]s for inflation over the next year, investors require 4% for a one[illegible]
- Suppose investors cu[illegible]xpect inflation for the next year (the second year) to be higher so that they expect to require 6% for a one year loan (starting one year from now).
- Then, the Pure-Expectations Hypothesis, is consistent with the current 2-year spot rate defined as follows:

$$(1+R_{2,t})^2 = (1+R_{1,t})(1+E_t[R_{1,t+1}]) = (1.04)x(1.06)$$

so $R_{2,t} = 4.995238\%$

- Restated, if we observe $R_{1,t} = 4\%$ and $R_{2,t} = 4.995238\%$, then, under the Pure-Expectations Hypothesis, we would have $E_t[R_{1,t+1}]$ to be 6%.

Term Structure

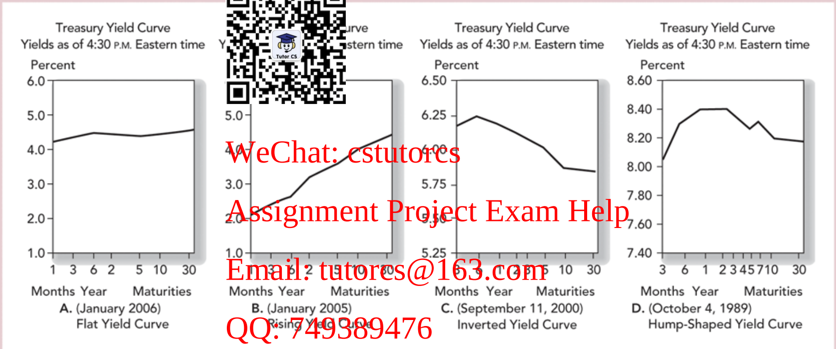


FIGURE 15.1 Treasury yield curves

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Application 3: Present Value (Gordon) Model

- The price of the asset today is a discounted sum of all possible future cash flows (or dividends D)

$$P_t = \sum_{j}^{\infty} \frac{E_t\left(D_{t+j}\right)}{(1+R)^j}$$

$$E_t\left(D_{t+j}\right) = D_t$$

Present Value (Gordon) Model

$$
\begin{aligned}
P_t &= \text{[illegible]} + \frac{1}{(1+R)^2} + K \Big] \\
&= \frac{D_t}{1+R}\left[1 + \frac{1}{1+R} + \frac{1}{(1+R)^2} + K\right] \\
&= \frac{D_t}{1+R}\left[\frac{1}{1-1/(1+R)}\right] \\
&= \frac{D_t}{R}
\end{aligned}
$$

- we used the property of the infinite converging geometric progression series: $\sum_{k=0}^{\infty} a^k = 1/(1-a)$

Present Value (Gordon) Model

- the model is still nonlinear, but we may take the logs: $\log(P_t) = -\log(R_t) + \log(D_t)$
- And once again use OLS to test whether the model is correct

$$\log(P_t) = \alpha + \beta \log(D_t) + u_t$$

Test the null hypothesis $H_0 : \beta = 1$

Ordinary Least Squares (OLS) Estimation

Consider a K-**variable regression model**

$$Y_i = \beta_1 + \beta_2 X_{i2} + \beta_3 X_{i3} + \ldots + \beta_K X_{iK} + \mu_i, \quad i = 1, \ldots, N$$

The OLS estimator minimises the sum of squared residuals of the sample regression function, i.e.

$$\min_{\hat{\beta}_1, \ldots, \hat{\beta}_K} \sum \hat{\mu}_i^2 = \sum \left(Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_{i2} - \hat{\beta}_3 X_{i3} - \ldots - \hat{\beta}_K X_{iK} \right)^2$$

This yields a system of K equations (first order conditions) in K unknowns which can be solved for the OLS estimator.

Model in matrix Form

The equation above is ... expression for the system

$$Y_1 = \beta_1 + \beta_2 X_{12} + \beta_3 X_{13} + \ldots + \beta_K X_{1K} + \mu_1$$
$$Y_2 = \beta_1 + \beta_2 X_{22} + \beta_3 X_{23} + \ldots + \beta_K X_{2K} + \mu_2$$
$$\vdots$$
$$Y_N = \beta_1 + \beta_2 X_{N2} + \beta_3 X_{N3} + \ldots + \beta_K X_{NK} + \mu_N$$

Alternatively, this system can be written in matrix notation

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix} = \begin{bmatrix} 1 & X_{12} & X_{13} & \cdots & X_{1K} \\ 1 & X_{22} & X_{23} & \cdots & X_{2K} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & X_{N2} & X_{N3} & \cdots & X_{NK} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_K \end{bmatrix} + \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{bmatrix}$$

Model in matrix Form

This matrix notation can ... d to

$$\ldots + \mu$$

by defining

- y is a $N \times 1$ column vector holding N observations on Y
- X is a $N \times K$ matrix holding N observations on the K explanatory variables
- β is a $K \times 1$ vector holding K unknown parameters
- μ is a $N \times 1$ column vector holding N disturbances μ

Deriving OLS Estimator

The OLS estimator can be obtained by minimising the sum of squared residuals from the sample regression function. In matrix notation, this amounts to minimising $\hat{\mu}'\hat{\mu}$ since:

$$\hat{\mu}'\hat{\mu} = \begin{bmatrix} \hat{\mu}_1 & \hat{\mu}_2 & \cdots & \hat{\mu}_N \end{bmatrix} \begin{bmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \\ \vdots \\ \hat{\mu}_N \end{bmatrix} = \hat{\mu}_1^2 + \hat{\mu}_2^2 + \cdots + \hat{\mu}_N^2 = \sum \hat{\mu}_i^2$$

with $\hat{\mu} = y - X\hat{\beta}$.
Therefore

$$\begin{aligned} \hat{\mu}'\hat{\mu} &= \left(y - X\hat{\beta}\right)'\left(y - X\hat{\beta}\right) \\ &= y'y - \hat{\beta}'X'y - y'X\hat{\beta} + \hat{\beta}'X'X\hat{\beta} \\ &= y'y - 2y'X\hat{\beta} + \hat{\beta}'X'X\hat{\beta} \end{aligned}$$

Deriving OLS Estimator

The OLS estimator for [illegible]rived from the first order conditions

$$\frac{\partial \hat{u}'\hat{u}}{\partial \hat{\beta}} \ [illegible] + 2X'X\hat{\beta} = 0$$

as

$$X'X\hat{\beta} = X'y$$

$$\hat{\beta} = (X'X)^{-1} X'y$$

Features of the $(X'X)$ matrix:

- Symmetrical (see example below)
- Invertible if there is no exact multicollinearity

Linear Regression: Basic Assumptions

1. No collinearity in X (for the inverse of $X'X$ to exist

   Intuitively: all regressors non-redundant

2. Zero conditional mean $E(\mu|X) = 0$ (μ independent of X)

   Violation means endogeneity, a serious problem

$$
\begin{aligned}
\hat{\beta} &= (X'X)^{-1}X'y = (X'X)^{-1}X'(X\beta+\mu) \\
&= (X'X)^{-1}X'X\beta + (X'X)^{-1}X'\mu = \beta + (X'X)^{-1}X'\mu \\
E(\hat{\beta}) &= \beta + E\left[(X'X)^{-1}X'\mu\right] \\
&= \beta + (X'X)^{-1}X'E\left[\mu|X\right]
\end{aligned}
$$

The latter assuming X deterministic. Otherwise, we can use the law of iterated expectations.

3. Homoskedasticity: $Var(\mu_i|X) = \sigma^2$ for all i

Linear regression: variance-covariance

$$
\begin{aligned}
var\left(\hat{\beta}\right) &= E\left[\left(\hat{\beta}-\beta\right)\left(\hat{\beta}-\beta\right)'\right] \\
&= E\left[\left((X'X)^{-1}X'\mu\right)\left((X'X)^{-1}X'\mu\right)'\right] \\
&= E\left[(X'X)^{-1}X'\mu\mu'X(X'X)^{-1}\right] \\
&= (X'X)^{-1}X'E\left[\mu\mu'\right]X(X'X)^{-1} \\
&= (X'X)^{-1}X'\sigma^2 I_N X(X'X)^{-1} \\
&= \sigma^2(X'X)^{-1}
\end{aligned}
$$

An unbiased estimator of σ^2 is given by

$$
\hat{\sigma}^2 = \frac{\sum \hat{\mu}_i^2}{N-K} = \frac{\hat{\mu}'\hat{\mu}}{N-K} = \frac{y'y-\hat{\beta}'X'y}{N-K}
$$

Linear regression: Properties (BLUE)

1. **Unbiased**: $E(\hat{\beta}) =$
2. **Consistent**: plim$\hat{\beta}$ = ... es to infinity (∞).

   $P\left(\hat{\beta} - \beta > \epsilon\right) \rightarrow 0$ as T goes to ∞.

   Intuitively $\hat{\beta}$ gets closer and closer to β as $T \rightarrow \infty$.

   (does not imply unbiaseness, may sill be that $E(\hat{\beta}) \neq \beta$)
3. **Asymptotically** normal (Why?): $\hat{\beta} = \beta + (X'X)^{-1}X'\mu$

   (or **Exat** normality if μ are normally distributed)

   $$\hat{\beta} \sim N\left(\beta, \sigma^2 (X'X)^{-1}\right)$$
4. **Efficient among linear estimators:**

   OLS has smallest variance among linear estimators