ECON6300/7320/8300
Advanced Microeconometrics
Simulation based methods

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Lecture 7

Introduction

- This lecture considers simulation-based methods in estimation and inference
- Justifies why these methods work and are popular
- Introduces resampling methods for routine and refined inference
- Explains the canonical resampling method: the bootstrap

Simulation-based methods

- "Simulation is the **imitation** of the operation of a **real-world process or system** over time." (ref. *Wikipedia*)

Example: A fire drill; mimicking an emergency evacuation; properties of a roulette wheel;

- Simulation in econometrics is the imitation of some stochastic process
- Simulation traditionally used to study the properties of statistics, statistical procedures such as methods of inference.

Simulation-based methods

Example: Central limit theorem for the sample mean of an i.i.d. random variable

$$\frac{\bar{y} - \mu}{s/\sqrt{N}} \sim N[0, 1]$$

$$\bar{y} \sim N[\mu, \sigma^2/N]$$

- Simulation can help us compare the properties and performance of competing estimation and test procedures in an ideal setting that conforms to the underlying assumptions of the statistical model.
- Compare $\frac{\bar{y}-\mu}{s/\sqrt{N}}$ with $\frac{\bar{y}-\mu}{s/\sqrt{N-50}}$ in a simulation set up

Monte-Carlo Experiments

1. Write down the data generating process ($y_i \sim N[\mu, \sigma^2]$, $i = 1, ..., N$)
2. Choose values for the parameters ($N = 100$, $\mu = 2$, $\sigma^2 = 1$)

3. Set the number of simulations ($S = 10,000$)
4. Simulate the data generating process and compute the statistics of interest S indepdenent times. (For $s = 1, ..., S$ draw $y_i^{(s)} \sim N[\mu, \sigma^2]$ and compute $\frac{\overline{y}^{(s)} - \mu}{s^{(s)}/\sqrt{N}}$, $\frac{\overline{y}^{(s)} - \mu}{s^{(s)}/\sqrt{N-50}}$)

5. Compare the distributions of the statistics of interest

Monte-Carlo Experiments

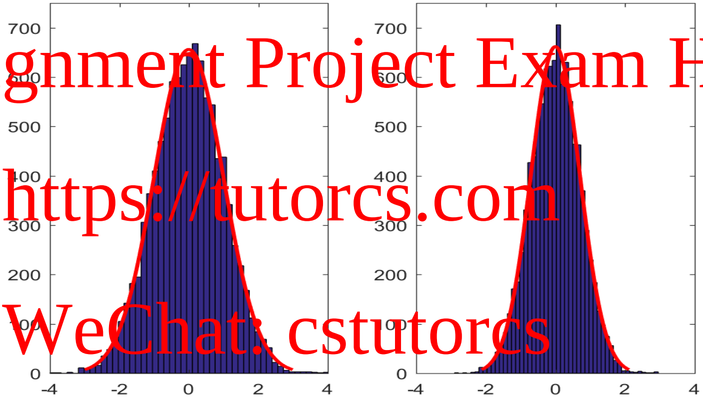

Figure: Histograms of 10,000 values of $\frac{\overline{y}^{(s)}-\mu}{s^{(s)}/\sqrt{N}}$ and $\frac{\overline{y}^{(s)}-\mu}{s^{(s)}/\sqrt{N-50}}$ each from a sample of size 100

Monte-Carlo in STATA

```
. program onesample,rclass
        drop _all
        quietly set obs 100
        generate x=rnormal(2,1)
        summarize x
        return scalar s1=(r(mean)-2)/(r(sd)/sqrt(r(N)))
        return scalar s2=(r(mean)-2)/(r(sd)/sqrt(r(N)-50))
  end
```


Figure: Code to generate data and compute statistics

Monte-Carlo in STATA

```
. onesample

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
           x |       100    1.949839    .9912055   -.063986   4.881399

. return list

scalars:
                  r(s2) =  -.3578420246746993
                  r(s1) =  -.5060650444820074
```


Figure: Run for one dataset

Monte-Carlo in STATA

```
. simulate stat1=r(s1) stat2=r(s2), seed(10101) reps(10000) nodots: onesample

      command:  onesample
        stat1:  r(s1)
        stat2:  r(s2)

. sum stat1 stat2

    Variable |        Obs        Mean    Std. Dev.       Min        Max
-------------+---------------------------------------------------------
       stat1 |     10,000    .0086302    1.002769  -3.996454   3.955304
       stat2 |     10,000    .0061025    .7090[illegible]44  -2.82592   2.796822

. histogram stat1, normal xtitle("s1 from many samples")
(bin=40, start=-3.9964538, width=.19879394)

. graph save g1
(file g1.gph saved)

. histogram stat2, normal xtitle("s2 from many samples")
(bin=40, start=-2.8259196, width=.14056854)

. graph save g2
(file g2.gph saved)

. graph combine g1.gph g2.gph
```

Figure: Simulate 10,000 datasets and draw histograms

Summary

- Simulation by Monte Carlo experimentation is a useful and powerful methodology for investigating the properties of econometric estimators and tests.
- Power of the method derives from being able to define and control the statistical environment in which the investigator specifies the data generating process and generates data used in subsequent experiments.
- Monte Carlo experiments can be used to verify that valid methods of statistical inference are being used.
- Many methods rely on asymptotic results as guiding approximations. Need to check whether these provide a good approximation in samples of the size typically available to the investigators.

Newer applications of simulations

- Many econometric models involve latent variables (e.g. probit and logit, unobserved heterogeneity, random parameter variation).
- Many models involve measures that cannot be analytically or exactly evaluated. In many cases these expressions are integrals that lack an exact solution. In many cases such integrals can be evaluated by numerical integration or by Monte Carlo simulation.
- Both methods provide approximate solutions in situations where exact answers may not exist.
- Refinements are often possible that improve the accuracy of the approximate methods.

Simulation-based Estimation: An Example

- Consider a RE linear panel regression model with a single regressor x, for simplicity:

$$y_{it} = \beta_0 + \alpha_i + \beta_1 x_{it} + \varepsilon_{it} \tag{1}$$

  where α_i is individual-specific **unobserved** heterogeneity term, and ε_{it} is i.i.d. $N[0, \sigma_\varepsilon^2]$ error uncorrelated with α_i and x_{it}.

- The joint **likelihood** conditional on x, α is given by:

$$\begin{aligned} L(y_{it}|x_{it}, \alpha_i) &= \prod_{i=1}^{N}\prod_{t=1}^{T}(2\pi\sigma_\varepsilon^2)^{-1/2} \\ &\quad \times \exp\left[-\frac{(y_{it} - \beta_0 - \alpha_i - \beta_1 x_{it})^2}{2\sigma_\varepsilon^2}\right] \end{aligned}$$

- Assume that $\alpha_i \sim \mathcal{N}[\alpha, \sigma_\alpha^2]$; i.e. $\alpha_i = \alpha + \sigma_\alpha \eta_i$ where $\eta_i \sim \mathcal{N}[0, 1]$. Then rewrite the likelihood equation as follows:

$$L(y_{it}|x_{it}, \eta_i) = \prod_{i=1}^{N} \prod_{t=1}^{T} (2\pi\sigma_\varepsilon^2)^{-1/2} \times \exp\left[-\frac{(y_{it} - (\beta_0 + \alpha) - \beta_1 x_{it} - \sigma_\alpha \eta_i)^2}{2\sigma_\varepsilon^2} \right]$$

- The above likelihood cannot be maximized directly because η_i are unobserved.
- Since we have specified the distribution of η we can replace the unobserved values by a draw from the $\mathcal{N}[0, 1]$ distribution.
- Unobservables become observables.
- Let $\eta_i^{(s)}$ be a draw from $\mathcal{N}[0, 1]$. Then

$$
\begin{aligned}
L(y_{it}|x_{it}, \eta_i &= \eta_i^{(s)}) = \prod_{i=1}^{N}\prod_{t=1}^{T}(2\pi\sigma_\varepsilon^2)^{-1/2} \\
&\times \exp\left[-\frac{(y_{it} - (\beta_0 + \alpha) - \beta_1 x_{it} - \sigma_\alpha \eta_i^{(s)})^2}{2\sigma_\varepsilon^2}\right]
\end{aligned}
$$

- To convert $L(y_{it}|x_{it,}\eta_i = \eta_i^{(s)})$ to $L(y_{it}|x_{it})$ we need to average out the effect of $\eta_i^{(s)}$. This operation is called "integrating out" η_i

$$
\begin{aligned}
L(y_{it}|x_{it}) &= E_{\eta}\left[L(y_{it}|x_{it,}\eta_i)\right] = \int L(y_{it}|x_{it,}\eta_i) f(\eta_i) d\eta \\
&\simeq \frac{1}{S}\sum_{s=1}^{S} L(y_{it}|x_{it,}\eta_i^{(s)})
\end{aligned}
$$

where S is a suitable large integer to ensure that the expression on the RHS converges to its expected value.

Simulation-based Estimation

- **Maximum simulated likelihood** estimator is defined as the estimator which maximizes $\ln\left(\frac{1}{S}\sum_{s=1}^{S} L(y_{it}|x_i, \eta_i^{(s)})\right)$
- The maximized likelihood will deliver the MLE of $((\beta_0 + \alpha), \beta_1, \sigma_\alpha^2, \sigma_\varepsilon^2)$ where σ_α^2 is a measure of the dispersion of unobserved heterogeneity
- The same approach can be used for dealing with latent variables in higher dimensions, e,g, heterogeneity in β_1 and β_0.
- This approach has been used in STATA to handle RE model in a maximum likelihood setting.

Distribution of the MSL Estimator

- MSL estimator $\rightarrow$ same probability limit as the ML estimator if plim $N^{-1}L_N^{(s)}(\theta) = N^{-1}L_N(\theta)$;
  - i.e., if $\ln f_i^{(s)} - \ln f_i \xrightarrow{p} 0$, which in turn happens if $f_i^{(s)} - f_i \xrightarrow{p} 0$ as $S \rightarrow \infty$.
- Even if the MSL estimator is consistent, it is possible that simulation error will inflate the variance of the MSL estimator compared to the ML estimator.
- A formal statement of conditions under which the MSL estimator is consistent and efficient is given in the references.

Introduction to Bootstrap Methods

- Exact finite sample results usually unavailable. Instead use asymptotic theory. Can we do better?
- **Bootstrap** due to Efron (1979) is an alternative method. Approximate distribution of statistic by Monte Carlo simulation, with sampling from the empirical distribution or the fitted distribution of the observed data.
- Like conventional methods relies on asymptotic theory so only exact in infinitely large samples.
- **Simplest bootstraps** no better than usual asymptotics but may be easier to implement.
- **More complicated bootstraps provide asymptotic refinement** so perhaps better finite sample approximation.

Bootstrap Summary

- Sample: $\{\mathbf{w}_1, \ldots, \mathbf{w}_N\}$ iid over i. Usually: $\mathbf{w}_i = (y_i, \mathbf{x}_i)$
- Estimator: $\widehat{\theta}$ is smooth root-N consistent and is asymptotically normal. For simplicity we consider scalar θ.
- Statistics of interest:
  - estimate $\widehat{\theta}$
  - standard errors $s_{\widehat{\theta}}$
  - $t-$statistic $t = (\widehat{\theta} - \theta_0)/s_{\widehat{\theta}}$ where θ_0 is H_0 value
  - associated critical value or $p-$value for the test
  - confidence interval.

Bootstrap Without Refinement

- Consider estimating the variance of the sample mean

$$\widehat{\mu} = N^{-1} \sum_{i=1}^{N} y_i$$

where $y_i \sim iid[\mu, \sigma^2]$

- If we obtain S samples of size N from the population to obtain S estimates $\widehat{\mu}_s$ for $s = 1, ..., S$.

$$\widehat{V}[\widehat{\mu}] = (S-1)^{-1} \sum_{s=1}^{S} (\widehat{\mu}_s - \overline{\widehat{\mu}})^2$$

, where $\overline{\widehat{\mu}} = S^{-1} \sum_{s=1}^{S} \widehat{\mu}_s$.

Multiple samples

- This approach not possible in practice as we only have one sample!
- **Bootstrap idea:** Treat the sample as the population.
- Draw B **bootstrap samples** of size N from the sample $y_1, ..., y_N$ **with replacement.**
- In each bootstrap sample some original data points appear more than once while others do not appear at all.
- Gives B estimates $\widehat{\mu}_b$, $b = 1, ..., B$.
- Then

$$\widehat{V}[\widehat{\mu}] = (B-1)^{-1} \sum_{b=1}^{B} (\widehat{\mu}_b - \overline{\widehat{\mu}})^2$$

, where $\overline{\widehat{\mu}} = B^{-1} \sum_{b=1}^{B} \widehat{\mu}_b$.

Bootstrap Example

- Data generated from an exponential distribution with an exponential mean with two regressors

$$y_i | x_{2i}, x_{3i} \sim \text{exponential}(\lambda_i), \; i = 1, ..., 50$$

$$\lambda_i = \exp(\beta_1 + \beta_2 x_{2i} + \beta_3 x_{3i})$$

$$(x_{2i}, x_{3i}) \sim \mathcal{N}\left[\begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}; \begin{pmatrix} 0.1^2 & 0.005 \\ 0.005 & 0.1^2 \end{pmatrix}\right]$$

$$(\beta_1, \beta_2, \beta_3) = (-2, 2, 2)'.$$

- MLE for 50 observations gives $\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\beta}_3$
- Focus on statistical inference for β_3 :

$$\widehat{\beta}_3 = 4.664 \qquad s_3 = 1.741 \qquad t_3 = 2.679$$

- Use bootstrap with (y_i, x_{2i}, x_{3i}) jointly resampled with replacement $B = 999$ times.
- The bootstrapped $\widehat{\beta}^*_{3,1}, \ldots, \widehat{\beta}^*_{3,999}$ had mean 4.716 and standard deviation of 1.939

- **Bootstrap standard error estimate** $= 1.939$

Asymptotic Refinements

- Example: Nonlinear estimators asymptotically unbiased but biased in finite samples
- Often can show that

$$E[\widehat{\theta} - \theta_0] = \frac{a_N}{N} + \frac{b_N}{N^2} + \frac{c_N}{N^3} + \dots$$

where a_N, b_N and c_N are bounded constants that vary with the data and estimator.

- Consider alternative estimator $\widetilde{\theta}$ with

$$E[\widetilde{\theta} - \theta_0] = \frac{B_N}{N^2} + \frac{C_N}{N^3} + \dots$$

where B_N and C_N are bounded constants.

- Both estimators are unbiased as $N \to \infty$. The latter is an **asymptotic refinement.**

Asymptotically Pivotal Statistic

- $\alpha =$ nominal size for a test, (e.g. $\alpha = 0.05$).
- Actual size $= \alpha + O(N^{-1/2})$ for usual one-sided tests.
- Asymptotic refinement requires statistic to be an **asymptotically pivotal statistic**, meaning limit distribution does not depend on unknown parameters.
- Example: Sampling from $y_i \sim [\mu, \sigma^2]$.
  - $\widehat{\mu} = \bar{y} \overset{a}{\sim} \mathcal{N}[\mu, \sigma^2/N]$ is not asymptotically pivotal since distribution depends on unknown σ^2.
  - $t = (\widehat{\mu} - \mu_0)/s_{\widehat{\mu}} \overset{a}{\sim} \mathcal{N}[0, 1]$, the **studentized statistic**, is asymptotically pivotal.
- Estimators are usually not asymptotically pivotal. Conventional test statistics usually are.

A general bootstrap algorithm

1. Given data $\mathbf{w}_1, ..., \mathbf{w}_N$ draw a bootstrap sample of size N and denote this new sample $\mathbf{w}_1^*, ..., \mathbf{w}_N^*$.
2. Calculate an appropriate statistic using the bootstrap sample. Examples include:
   (a) estimate $\widehat{\theta}^*$ of θ; (b) standard error $s_{\widehat{\theta}^*}$ of estimate $\widehat{\theta}^*$;
   (c) $t-$statistic $t^* = (\widehat{\theta}^* - \widehat{\theta})/s_{\widehat{\theta}^*}$ centered at $\widehat{\theta}$.
3. After repeating steps 1-2 B independent times we have B bootstrap replications of $\widehat{\theta}_1^*, ..., \widehat{\theta}_B^*$ or $t_1^*, ..., t_B^*$ or .....
4. Use these B bootstrap replications to obtain a bootstrapped version of the statistic.

Bootstrap Sampling Methods (step 1)

- Empirical distribution function (EDF) bootstrap or nonparametric bootstrap: $\mathbf{w}_1^*, ..., \mathbf{w}_N^*$ obtained by sampling with replacement from $\mathbf{w}_1, ..., \mathbf{w}_N$.
  - Also called **paired bootstrap** as for single equation models the pair $(y_i, \mathbf{x}_i)$ is being resampled.
- **Parametric bootstrap** for fully parametric models
  Suppose $y|\mathbf{x} \sim F(\mathbf{x}, \theta_0)$ and $\widehat{\theta} \xrightarrow{p} \theta_0$. Bootstrap generate y_i^* by random draws from $F(\mathbf{x}_i, \widehat{\theta})$ where $\mathbf{x}_i$ may be the original sample (or the first resample $\mathbf{x}_i^*$ from $\mathbf{x}_1, ..., \mathbf{x}_N$)
- **Residual bootstrap** for additive error regression
  Suppose $y_i = g(\mathbf{x}_i, \beta) + u_i$.
  Form fitted residuals $\widehat{u}_1, ..., \widehat{u}_N$.
  Bootstrap residuals $(\widehat{u}_1^*, ..., \widehat{u}_N^*)$ yield bootstrap sample $(y_1^*, \mathbf{x}_1), ..., (y_N^*, \mathbf{x}_N)$, for $y_i^* = g(\mathbf{x}_i, \widehat{\beta}) + \widehat{u}_i^*$.

Number of Bootstraps (step 3)

- Asymptotic refinement can occur even for low B but higher B is even better.
- For standard error computation Efron and Tibsharani (1993, p.52) say $B = 50$ is often enough and $B = 200$ is almost always enough.
- Hypothesis tests and confidence intervals at standard levels of statistical significance involve the tails of the distribution, so more replications are needed.
  - For hypothesis testing at level α choose B so that $\alpha(B + 1)$ is an integer. e.g. at $\alpha = .05$ let $B = 399$ rather than 400.
  - Use $B \geq 399$ if $\alpha = 0.05$ and $B \geq 1499$ if $\alpha = 0.01$.

Standard error estimation

- The **bootstrap estimate of variance** of an estimator is the usual formula for estimating a variance, applied to the B bootstrap replications $\widehat{\theta}_1^*, \ldots, \widehat{\theta}_B^*$:

$$s^2_{\widehat{\theta},\mathrm{Boot}} = \frac{1}{B-1}\sum_{b=1}^{B}\left(\widehat{\theta}_b^* - \overline{\widehat{\theta}}^*\right)^2,$$

  where $\overline{\widehat{\theta}}^* = B^{-1}\sum_{b=1}^{B}\widehat{\theta}_b^*$.
- The **bootstrap estimate of the standard error**, $s_{\widehat{\theta},\mathrm{Boot}}$, is obtained by taking the square root.

- This bootstrap provides no asymptotic refinement.
- But it is extraordinarily useful when it is difficult to obtain standard errors using conventional methods:
  - Sequential two-step m-estimator
  - 2SLS estimator with heteroskedastic errors (if no White option).
  - Functions of other estimates e.g. $\widehat{\theta} = \widehat{\alpha}/\widehat{\beta}$
  - Clustered data with many small clusters, such as short panels. (Resample the clusters.)

Tests with Asymptotic Refinement

- $T_N = (\widehat{\theta} - \theta_0)/s_{\widehat{\theta}} \overset{a}{\sim} \mathcal{N}[0,1]$ is asymptotically pivotal.
- Bootstrap gives B test statistics $t_1^*, \ldots, t_B^*$, where

$$t_b^* = (\widehat{\theta}_b^* - \widehat{\theta})/s_{\widehat{\theta}_b^*},$$

  is **centered around the original estimate** $\widehat{\theta}$ since resampling is from a distribution centered around $\widehat{\theta}$.
- Let $t_1^*, \ldots, t_B^*$ denote statistics ordered from smallest to largest.

- Upper alternative test: $H_0 : \theta \leq \theta_0$ vs $H_a : \theta > \theta_0$. The **bootstrap critical value** (at level α) is the upper α quantile of the B ordered test statistics.
  - e.g. if $B = 999$ and $\alpha = 0.05$ then the critical value is the 950^{th} highest value of t^*, since then $(B+1)(1-\alpha) = 950$.
- Can also compute a **bootstrap** $p-$**value**.
  - e.g. if original statistic t lies between the 914^{th} and 915^{th} largest values and $B = 999$ then

$$p_{\text{boot}} = 1 - 914/(B+1) = 0.086$$

- For a **non-symmetrical test** the bootstrap **critical values** (at level α) are the lower $\alpha/2$ and upper $\alpha/2$ quantiles of the ordered test statistics t^*.
- For a **symmetrical test** we instead order $|t^*|$ and the bootstrap **critical value** (at level α) is the upper α quantile of the ordered $|t^*|$.
- These tests, using the **percentile-t method**, provide asymptotic refinements.
  - One-sided t test and a non-symmetrical two-sided t test the true size $\alpha + O(N^{-1/2}) \to \alpha + O(N^{-1})$.
  - Two-sided symmetrical t test or an asymptotic chi-square test true size $\alpha + O(N^{-1}) \to \alpha + O(N^{-2})$.

Tests without Asymptotic Refinement

- Alternative bootstrap methods can be used. While asymptotically valid there is no asymptotic refinement.

1. Compute $t = (\widehat{\theta} - \theta_0)/s_{\widehat{\theta},\text{boot}}$ where $s_{\widehat{\theta},\text{boot}}$ replaces the usual estimate $s_{\widehat{\theta}}$. Compare this test statistic to standard normal critical values.
2. For two-sided test of $H_0 : \theta = \theta_0$ against $H_a : \theta \neq \theta_0$ find the lower $\alpha/2$ and upper $\alpha/2$ quantiles of the bootstrap estimates $\widehat{\theta}_1^*, ..., \widehat{\theta}_B^*$. Reject H_0 if θ_0 falls outside this region. This is called the **percentile method**.

- These two bootstraps do not require computation of $s_{\widehat{\theta}}$, the usual standard error estimate based on asymptotic theory.

Bootstrap Example (continued)

- Summary statistics and percentiles based on 999 paired bootstrap resamples for:
  1. estimate $\widehat{\beta}_3^*$;
  2. the associated statistics $t_3^* = (\widehat{\beta}_3^* - 4.664)/s_{\widehat{\beta}_3^*}$
  3. $t(47)$ quantiles;
  4. standard normal quantiles.

	$\widehat{\beta}_3^*$	t_3^*	z=t(∞)	t(47)
Mean	4.716	0.026	1.021	1.000
St.Dev.	1.939	1.047	1.000	1.021
1%	0.336	-2.664	-2.326	-2.408
2.5%	0.501	-2.183	-1.960	-2.012
5%	1.545	-1.728	-1.645	-1.678
25%	3.570	-0.621	-0.675	-0.680
50%	4.772	0.062	0.000	0.000
75%	5.971	0.703	0.675	0.680
95%	7.811	1.706	1.645	1.678
97.5%	8.484	2.060	1.960	2.012
99.0%	9.427	2.529	2.326	2.408

Figure: Distribution of bootstrapped estimator, bootstrapped t-statistic, standard normal and t

Hypothesis Testing with Asymptotic Refinement:

Test $H_0 : \beta_3 = 0$ vs $H_a : \beta_3 \neq 0$ at level .05:

1. Compute $t_3^* = (\widehat{\beta}_3^* - 4.664)/s_{\widehat{\beta}_3^*}$.
2. Bootstrap critical values are -2.183 and 2.066.
3. Reject H_0 since original $t_3 = 2.679 > 2.066$,

Hypothesis Testing without Asymptotic Refinement:

- **Hypothesis Testing without Asymptotic Refinement 1:**
  Use the bootstrap standard error estimate of 1.939, rather than the asymptotic standard error estimate of 1.741.
  Then $t_3 = (4.664 - 0)/1.939 = 2.405$.
  Reject H_0 at level .05 as $2.405 > 1.960$ using standard normal critical values.
- **Hypothesis Testing without Asymptotic Refinement 2:**
  From Table, 95 percent of the bootstrap estimates $\widehat{\beta}_3^*$ lie in the range (0.501, 8.484).
  Reject $H_0 : \beta_3 = 0$ as this range does not include the hypothesized value of 0.

Confidence Intervals

- **Asymptotic refinement using the percentile-t method:** The $100(1-\alpha)$ percent confidence interval is

$$(\widehat{\theta} - t^*_{[1-\alpha/2]} \times s_{\widehat{\theta}}, \widehat{\theta} - t^*_{[\alpha/2]} \times s_{\widehat{\theta}}),$$

  where $\widehat{\theta}$ and $s_{\widehat{\theta}}$ are the estimate and standard error from the original sample and $t^*_{[1-\alpha/2]}$ and $t^*_{[\alpha/2]}$ denote the lower and upper $\alpha/2$ quantiles $\alpha/2$ of the $t-$statistics $t^*_1, \dots, t^*_B$.

- **No asymptotic refinement** (though valid).
  1. $(\widehat{\theta} - z_{[1-\alpha/2]} \times s_{\widehat{\theta},\text{boot}}, \widehat{\theta} - z_{[\alpha/2]} \times s_{\widehat{\theta},\text{boot}})$
  2. $(\widehat{\theta}^*_{[1-\alpha/2]}, \widehat{\theta}^*_{[\alpha/2]})$ the percentile method.

Bootstrap Applications

1. **Heteroskedastic Errors:** For asymptotic refinement use the wild bootstrap.
2. **Panel Data and Clustered Data:** Resample all observation in the cluster .e.g. For panel data resample over i only. No refinement but gives heteroscedasticity and panel robust standard errors

3. **Hypothesis and Specification Tests:** consider more complicated tests.

4. **Time Series:** bootstrap residuals if possible or do moving blocks.