Lecture - 7

Digital Signal Processing

ENGN 4537/ 6537

Semester 2, 2023

Announcements

- MatLab Project ... t released
  - Worth 30% ... se assessment
  - Minimum 50% score required to pass this course.
  - Due by 5:00 pm on Friday, 20 October 2023 (Week-11).
  - Late submissions will receive a score of 0.
  - Marking will be strict - plagiarism will be dealt with seriously
  - Drop-in sessions arranged (check Wattle)
- Teaching Feedback Survey-1 open in Wattle until Friday, 25th August.

Course Outline

- Discrete-time Signals and Systems
- The Z-Transform
- Transform Analysis of Linear Time-Invariant (LTI) Systems
- Sampling of Continuous-Time Signals
- The Discrete Fourier Transform (DFT)
- Computation of DFT using FFT algorithms
- Structure of Discrete-time Systems
- Filter Design Techniques
- Fourier Analysis of Signals using DFT

Transform Analysis of Linear Time-Invariant (LTI) Systems

We will study how to analyze a system given the Z-Transform and/or Fourier Transform

- Analysis of LTI systems described by Difference Equations

- Analysis of LTI systems described by Fourier Transform
- All Pass and Minimum Phase Systems

Analysis of LTI systems described by Difference Equations

An important subclass of LTI systems can be described by an N^{th} order linear constant-coefficient difference equation. The general definition of a difference equation is given by:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Analysis of LTI systems described by Difference Equations

- They are import[illegible]se discrete time filters are mostly realized (designed and implemented) through difference equations.
  - Infinite impulse response (IIR) filters
  - Finite impulse response (FIR) filter
- They are also a useful tool for modeling a given system
  - for a system introducing distortion
  - for characterizing a random process

Revisited Example: Accumulator

$$h[n] = \sum_{k} \; = \begin{cases} 1, & n \geqslant 0 \\ 0, & n < 0 \end{cases} = u[n]$$

Revisited Example: Accumulator

$$h[n] = \quad {}_{k} \quad = \begin{cases} 1, & n \geqslant 0 \\ 0, & n < 0 \end{cases} = u[n]$$

- Lets compute the convolution of $x[n]$ and $h[n]$, as follows:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x[k]u[n-k] = \sum_{k=-\infty}^{n} x[k]$$

Revisited Example: Accumulator

- We can rewrite ulator as a follows:

$$y[n] = x[n \quad \sum_{k=-\infty}^{n} x[k] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

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- With a time delay, we can write:

$$y[n-1] = \sum_{k=-\infty}^{n-1} x[k]$$

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- With a time delay, we can write:

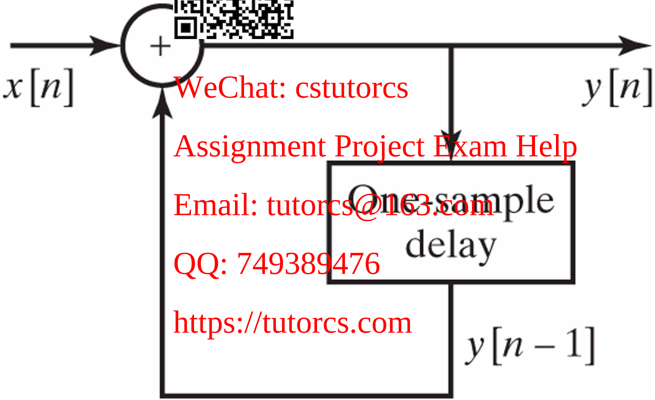
$$y[n-1] = \sum_{k=-\infty}^{n-1} x[k]$$

- Therefore, we have:

$$y[n] = x[n] + y[n-1] \quad \Rightarrow \quad y[n] - y[n-1] = x[n]$$

Revisited Example: Accumulator

$$y[n] = x[n] + \quad \Rightarrow \quad y[n] - y[n-1] = x[n]$$



Revisited Example: Accumulator

$$[n-1] = x[n]$$

Compare the accumulator difference equation with the general definition for difference equations:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$N = \ldots, \quad a_0 = \ldots, \quad a_1 = \ldots$$

$$M = \ldots, \quad b_0 = \ldots$$

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$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$N = 1, \quad a_0 = 1, \quad a_1 = -1$$

$$M = 0, \quad b_0 = 1$$

Important Example: Moving Average (Chapter 2)

$$h[n] = \quad u[n] - u[n - M_2 - 1]\big)$$

- Does this impulse response have a finite or infinite duration?

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- Does this impulse response have a finite or infinite duration?

- Lets compute the convolution of $x[n]$ and $h[n]$ as follows:

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$\therefore y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n-k]$$

Important Example: Moving Average (Chapter 2)

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n-k]$$

Compare the moving average equation with the general definition for difference equations:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$N = \ldots, \quad a_0 = \ldots$$

$$M = \ldots, \quad b_k = \ldots$$

Important Example: Moving Average (Chapter 2)

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n-k]$$

Compare the moving average equation with the general definition for difference equations:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$N = 0, \quad a_0 = 1$$

$$M = M_2, \quad b_k = \frac{1}{M_2 + 1}, 0 \leqslant k \leqslant M_2$$

However, there are other ways to write the relationship between $x[n]$ and $y[n]$.

Important Example: Moving Average (Chapter 2)

We can rewrite $h[n]$ as follows:

$$h[n] = \frac{1}{M_2+1}\left(u[n] - u[n-M_2-1]\right)$$
$$= \frac{1}{M_2+1}\Big(\left(\delta[n] - \delta[n-M_2-1]\right) * u[n]\Big)$$

What is happening on the RHS? What is the effect of $\delta[n]$?

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What is happening on the RHS? What is the effect of $\delta[n]$?

- $\delta[n]$ convolved with any signal $x[n]$ will give $x[n]$, i.e., $\delta[n] * u[n] = u[n]$.

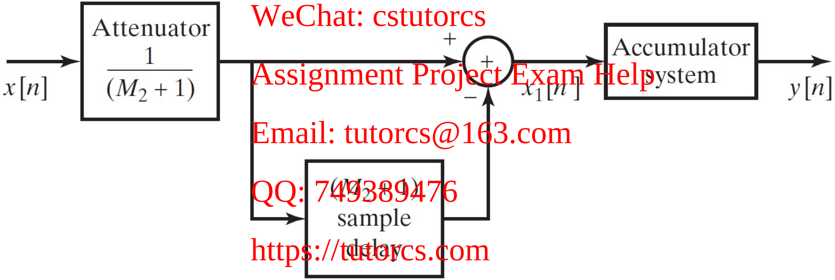
- We write $h[n]$ this way so that we get a different difference equation.

- $u[n]$ is just the accumulator from the previous example.

Therefore, we will use the accumulator system to get the moving average system.

Important Example: Moving Average (Chapter 2)

$$h[n] = \frac{\quad}{M \quad} \;\;] - \delta[n - M_2 - 1]) * u[n]$$



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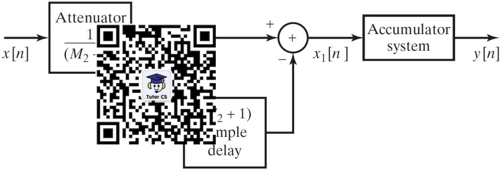

In the first system:

$$x_1[n] = \frac{1}{M_2+1}\left(x[n] - x[n-M_2-1]\right)$$

In the second system (from Accumulator example):

$$y[n] - y[n-1] = x_1[n]$$

Therefore:

$$y[n] - y[n-1] = \frac{1}{M_2+1}\left(x[n] - x[n-M_2-1]\right)$$

Important Example: Moving Average (Chapter 2)

$$y[n] - y[n-1] = \frac{1}{M_2+1}\left(x[n] - x[n-M_2-1]\right)$$

Compare the moving average equation with the general definition for difference equations:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$N = \ldots, \quad a_0 = \ldots, \quad a_1 = \ldots$$

$$M = \ldots, \quad b_0 = \ldots, \quad b_k = \ldots, \quad b_{M_2+1} = \ldots$$

Important Example: Moving Average (Chapter 2)

$$y[n] - y[n-1] = \frac{1}{M_2+1}\left(x[n] - x[n-M_2-1]\right)$$

Compare the moving average equation with the general definition for difference equations:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$N = 1, \quad a_0 = 1, \quad a_1 = -1$$

$$M = M_2 + 1, \quad b_0 = \frac{1}{M_2+1},$$

$$b_k = 0, 1 \leqslant k \leqslant M_2, \quad b_{M_2+1} = \frac{-1}{M_2+1}$$

The difference equation is not unique in general.

Exercise

See also a related Example 5.5 in Chapter 5 (Refer slide 50).

Finding the Output of a Linear Constant-Coefficient Difference Equation

Given a linear constant-coefficient difference equation with $N \geqslant 1$:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- The output is not uniquely defined. It depends on auxiliary conditions.
- Linearity, time invariance and causality of the system will depend on the auxiliary conditions.

Finding the Output of a Linear Constant-Coefficient Difference Equation

Given a linear constant-coefficient difference equation with $N \geqslant 1$:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- If the system is linear, time invariant and causal, then the solution is unique.
- This means: if $x[n] = 0, n < n_0$ then we have to have $y[n] = 0, n < n_0$.
- Read Section 2.5 Chapter 2 of the textbook for more details.

Finding the Output of a Linear Constant-Coefficient Difference Equation

Given a linear constant-coefficient difference equation with $\mathrm{N} = 0$:

$$a_0 y[n] = \sum_{k=0}^{M} b_k x[n-k] \quad \Rightarrow \quad y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k]$$

- $y[n]$ can be **uniquely** determined by the input.

- Recursion is not needed to compute output $y[n]$ and we have a finite impulse response (FIR) $h[n]$ of the form:

$$h[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} \delta[n-k] = \begin{cases} \frac{b_k}{a_0}, & 0 \leqslant n \leqslant M \\ 0, & \text{otherwise} \end{cases}$$

System function of LTI systems described by Difference Equations

$$\sum_{k=0}^{N} \ldots k] = \sum_{k=0}^{M} b_k x[n-k]$$

From the general definition of the difference equation, what is the system function, $H(z)$?

System function of LTI systems described by Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

From the general definition of the difference equation, what is the system function, $H(z)$?

By taking the z-transform of the above equation and using linearity and time shifting property, we get

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

System function of LTI systems described by Difference Equations

Now we can solve fo[illegible] rearranging:

$$Y(z)\sum_{k=0}^{N} a_k z^{-k} = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$Y(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} X(z)$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Note: Use this method if you cannot directly determine the impulse response from the difference equation.

System function of LTI systems described by Difference Equations

Therefore, the system function becomes:

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Recall: $H(z)$ has

- M zeros and N poles at non-zero locations, assuming $a_k > 0$ and $b_k > 0$.
- If $M > N$, then there are $(M - N)^{\text{th}}$ order poles at $z = 0$.
- If $M < N$, then there are $(N - M)^{\text{th}}$ order zeros at $z = 0$.

System function of LTI systems described by Difference Equations

$$\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Note that $H(z)$ can be converted into a product form (by factorising):

$$H(z) = \frac{b_0 \prod_{k=1}^{M} (1 - c_k z^{-1})}{a_0 \prod_{k=1}^{N} (1 - d_k z^{-1})}$$

- c_k are the non-zero zeros of $H(z)$ and d_k are the non-zero poles of $H(z)$.

- This product form allows us to use partial fraction expansion to find the impulse response $h[n]$ in time domain.

From System function to Difference Equation

If we are given

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

How do we obtain the Difference Equation?

From System Function to Difference Equation

Note that

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{Y(z)}{X(z)}$$

Hence:

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z) \Rightarrow$$

Taking inverse z-transform,

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

From System Function to Difference Equation
Example 5.1:

$$H(z) \quad \frac{(1+z^{-1})^2}{.5z^{-1})(1+0.75z^{-1})}$$

What is the difference equation of the system function?

From System Function to Difference Equation

Example 5.1:

$$H(z) = \frac{(1 + z^{-1})^2}{(1 - 0.5z^{-1})(1 + 0.75z^{-1})}$$

What is the difference equation of the system function?

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}} = \frac{Y(z)}{X(z)}$$

From System Function to Difference Equation

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$$\left(1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}\right)Y(Z) = \left(1+2z^{-1}+z^{-2}\right)X(Z)$$

From System Function to Difference Equation
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Ans.

$$y[n]+\frac{1}{4}y[n-1]-\frac{3}{8}y[n-2] = x[n]+2x[n-1]+x[n-2]$$

Key questions to answer given a system in terms of difference equation

- What are the conditions for stability and causality?

- If $H(z)$ models a system that happens to distort the input signal, can it be reversed using an inverse system?

- How does the impulse response $h[n]$ relate to parameters of difference equation?

Stability, Causality and ROC

- When deriving the difference equation from $H(z)$, we did not consider the ROC.

- The difference equation does not uniquely specify the impulse response $h[n]$. There is freedom to choose the ROC.

Stability, Causality and ROC

If the ROC is not giv

- Choose ROC to usality - ROC must be outside the outermost pole.

- Choose ROC to ensure stability - ROC must include the unit circle.

Stability, Causality and ROC

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- Normally we try to choose ROC for both causality and stability (if possible at all).

Stability, Causality and ROC

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- Choose ROC to usality - ROC must be outside the outermost pole.

- Choose ROC to ensure stability - ROC must include the unit circle.

- Normally we try to choose ROC for both causality and stability (if possible at all).

- **For a system to be both causal and stable: All poles must be inside the unit circle.**

Stability, Causality and ROC
Example 5.2

Can the system given by the difference equation be both causal and stable?

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = x[n]$$

By taking the z-transform:

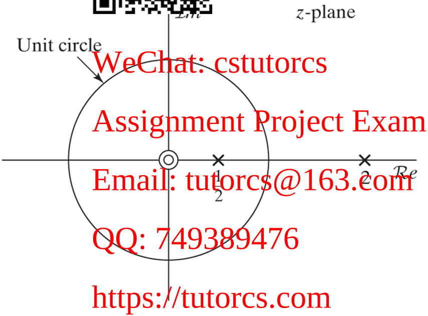
$$Y(z)\left(1 - \frac{5}{2}z^{-1} + z^{-2}\right) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{\left(1 - \frac{5}{2}z^{-1} + z^{-2}\right)} = \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - 2z^{-1}\right)}$$

$$= \frac{z^2}{\left(z - \frac{1}{2}\right)(z - 2)}$$

Stability, Causality and ROC
Example 5.2

Corresponding pole-z[illegible]f $H(z) = \frac{z^2}{\left(z-\frac{1}{2}\right)(z-2)}$ is



Can the given system be both causal and stable?

Stability, Causality and ROC
Example 5.2

Corresponding pole-z… f $H(z) = \frac{z^2}{\left(z-\frac{1}{2}\right)(z-2)}$ is

Can the given system be both causal and stable?
Ans. No

Key questions to answer given a system in terms of difference equation

- What are the conditions for stability and causality

- If $H(z)$ models a system that happens to distort the input signal, can it be reversed using an inverse system?

- How does the impulse response $h[n]$ relate to parameters of difference equation?

Inverse Systems

- Given an LTI system $H(z)$, the corresponding inverse system $H_i(z)$ leads to the overall system function of unity:

$$G(z) = H(z)H_i(z) = 1,$$

which implies

$$H_i(z) = \frac{1}{H(z)}.$$

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What does this mean in the time domain?

- In the time domain, $g[n] = h[n] * h_i[n] = \delta[n]$

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What does this mean in the time domain?

- In the time domain, $g[n] = h[n] * h_i[n] = \delta[n]$
- The inverse system undoes what the first filter has done.

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What does this mean in the frequency domain?

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- In the time domain, $g[n] = h[n] * h_i[n] = \delta[n]$
- The inverse system undoes what the first filter has done.

What does this mean in the frequency domain?

- In the frequency domain $H_i(e^{j\omega}) = \frac{1}{H(e^{j\omega})}$

Inverse Systems

- Given an LTI sy[illegible]), the corresponding inverse system $H_i(z)$ leads to t[illegible] system function of unity:

$$[illegible] = H(z)H_i(z) = 1,$$

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Do all systems have an inverse?

Inverse Systems

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which implies

$$H_i(z) = \frac{1}{H(z)}.$$

Do all systems have an inverse?

- Not all systems have an inverse, i.e., the ideal low pass filter does not (no way to recover high-frequency components above the cutoff frequency that are already set to zero).

Inverse Systems - Rational System functions

- Given a rational system function corresponding to a difference equation:

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

- Its inverse $H_i(z) = 1/H(z)$ is

$$H_i(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{N}(1 - d_k z^{-1})}{\prod_{k=1}^{M}(1 - c_k z^{-1})}$$

Inverse Systems - Rational System functions

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

$$H_i(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{N}(1 - d_k z^{-1})}{\prod_{k=1}^{M}(1 - c_k z^{-1})}$$

- How are the poles and zeros of $H(z)$ and $H_i(z)$ related?

- If $H_i(z)$ exists, then the ROC of $H(z)$ and $H_i(z)$ must overlap (why? Refer to Table 3.2 Property 9).

Sequence	Transform	ROC
$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Inverse Systems - Rational System functions

- Causal $H(z)$ im is $|z| > \max_k |d_k|$.
- Stable and causal $H(z)$ implies $\max_k |d_k| < 1$

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- Causal $H_i(z)$, ROC is $|z| > \max_k |c_k|$

- Stable and causal $H_i(z)$ implies $\max_k |c_k| < 1$

- If the system function and its inverse are both stable and causal, what can you conclude?

Inverse Systems - Rational System functions

- The **poles and zeros** of $H(z)$ must all be in the unit circle in order to have **bo[illegible]le and causal system AND inverse system**

- Such systems are called **minimum phase systems** (will be studied in the next lectures).

Inverse Systems - Rational System functions

- The **poles and zeros** of $H(z)$ must all be in the unit circle in order to have **bo[illegible]ble and causal system AND inverse system**.
- Such systems are called **minimum phase systems** (will be studied in the next lectures).

- Example 5.3

$$H(z) = \frac{1 - 0.5z^{-1}}{1 - 0.9z^{-1}} \text{ with ROC } |z| > 0.9$$

The inverse is:

$$H_i(z) = \frac{1 - 0.9z^{-1}}{1 - 0.5z^{-1}},$$

with ROC $|z| > 0.5$.

Key questions to answer given a system in terms of difference equation

- What are the conditions for stability and causality

- If $H(z)$ models a system that happens to distort the input signal, can it be reversed using an inverse system?

- How does the impulse response $h[n]$ relate to parameters of difference equation?

Impulse Response of Rational System Functions

Given rational system $H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$ find its impulse response $h[n]$.

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- Assuming only first-order poles and $M \geqslant N$,

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

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- What is $h[n]$, if the system is causal?

$$h[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N} A_k d_k^n u[n]$$

Impulse Response of Rational System Functions

- If the system is causal, then $h[n]$ is of the form

$$h[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N} A_k d_k^n u[n]$$

- What parts of $h[n]$ is infinite or finite?

Impulse Response of Rational System Functions

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- What parts of $h[n]$ is infinite or finite?

- The first part $h[n] = \sum_{r=0}^{M-N} B_r \delta[n-r]$ is a finite impulse response (FIR) system.

- The second part with $u[n]$ included leads to an infinite impulse response (IIR) system.

Impulse Response of Rational System Functions

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- The second part with $u[n]$ included leads to an infinite impulse response (IIR) system.

- If $M < N$, then $h[n]$ corresponds to IIR system.
- If $N = 0$, then $h[n]$ corresponds to FIR system.

FIR System Poles and Zeros

- FIR system:

$$= \sum_{r=0}^{M} B_r \delta[n-r]$$

$$H(z) = \sum_{r=0}^{M} B_r z^{-r}$$

- What are the poles and zeros of this FIR system?

FIR System Poles and Zeros

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$$= \sum_{r=0}^{M} B_r \delta[n-r]$$

$$H(z) = \sum_{r=0}^{M} B_r z^{-r}$$

- What are the poles and zeros of this FIR system?

Ans. Refer slide 26: For FIR systems, $N = 0$.

  - M zeros at non-zero locations
  - M^{th} order pole at $z = 0$.

FIR systems have poles only at $z = 0$.

FIR System Poles and Zeros

Proof:

$$H(z) = \sum_{r=0}^{M} B_r z^{-r} = \frac{1-(B_r)z^{-(M+1)}}{1-(B_r)z^{-1}} = \frac{1}{z^M}\left(\frac{z^{M+1}-B_r}{z-B_r}\right)$$

Therefore, there are $M+1$ roots of the numerator polynomial (zeros) at the following z-plane locations:

$$z_k = B_r e^{j(2\pi k/M+1)}, \qquad k = 0, 1, \ldots, M$$

Plus, there is an M-th order pole at $z = 0$ and a pole at $z = B_r$. Hence, there is a pole-zero cancellation at $z = B_r$, when $k = 0$.

- M zeros at $z_k = B_r e^{j(2\pi k/M+1)}$, $k = 1, \ldots, M$
- M^{th} order pole at $z = 0$.

i.e., Poles only at $z = 0$. True for all FIR systems.

FIR System Poles and Zeros Example 5.5

Consider the impulse response of FIR system (a is real and positive)

$$h[n] = \begin{cases} a^n, & 0 \leqslant n \leqslant M, \\ 0, & \text{otherwise} \end{cases}$$

Then the system function is

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = \sum_{n=0}^{M} a^n z^{-n} = \frac{1 - a^{(M+1)} z^{-(M+1)}}{1 - a z^{-1}} = \frac{1}{z^M} \left(\frac{z^{M+1} - a^{(M+1)}}{z - a} \right)$$

From $H(z)$,

- Roots of numerator (zeros): $z_k = a e^{j(2\pi k/(M+1))}, \quad k = 0, 1, \ldots, M$
- Roots of denominator (poles): $z = a$, M^{th} order pole at $z = 0$

Pole at $z = a$ is cancelled by zero at $k = 0$. Hence, the given FIR system has

- Zeros at $z_k = a e^{j(2\pi k/(M+1))}, \quad k = 1, \ldots, M$
- M^{th} order pole at $z = 0$

FIR System Poles and Zeros
Example 5.5

$$, \quad 0 \leqslant n \leqslant M,$$

otherwise

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=0}^{M} a^k x[n-k]$$

$$y[n] = \sum_{k=0}^{M} a^k x[n-k] \quad \left(\text{can also be obtained from } H(z) = \sum_{n=0}^{M} a^n z^{-n}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - a^{(M+1)} z^{-(M+1)}}{1 - az^{-1}}$$

$$Y(z)\left(1 - az^{-1}\right) = X(z)\left(1 - a^{(M+1)} z^{-(M+1)}\right)$$

Taking inverse z-transform,

$$y[n] - ay[n-1] = x[n] - a^{(M+1)} x[n-M-1]$$

The two difference equations represent the same system.

Homework

- Read and understand section 5.2 of the book.
- Related Problems: 5.2, 5.3, 5.4, 5.5, 5.6 and 5.7