Lecture - 10

Digital Signal Processing

ENGN 4537/ 6537

Semester 2, 2023

Transform Analysis of Linear Time-Invariant (LTI) Systems

How to analyze a system given the Z-Transform and/or Fourier Transform

- Analysis of LTI systems described by Difference Equations

- Analysis of LTI systems described by Fourier Transform
- All-Pass and Minimum-Phase Systems

Minimum-Phase Systems

- Definition: A minimum-phase system is a stable and causal system, which also has a stable and causal inverse system, i.e. all poles and zeros inside the unit circle.

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

$$H_i(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{N}(1 - d_k z^{-1})}{\prod_{k=1}^{M}(1 - c_k z^{-1})}$$

Minimum-Phase Systems

- Given a magnitude-squared function:

$$
\begin{aligned}
C(z) &= H(z)H^*(1/z^*) \\
&= \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}(1-c_k z^{-1})(1-c_k^* z)}{\prod_{k=1}^{N}(1-d_k z^{-1})(1-d_k^* z)}
\end{aligned}
$$

- We can uniquely determine $H(z)$ provided that it is minimum-phase since all the zeros and poles must lie inside the unit circle.

Minimum-Phase Systems

- We have seen that if the squared magnitude response ($C(z)$) is specified, and the corresponding M and N are fixed for a rational stable and causal system, there are finite choices of the zeros and phase responses.

- If we impose the additional restriction that the system is minimum-phase, then we can **uniquely** determine the zeros.

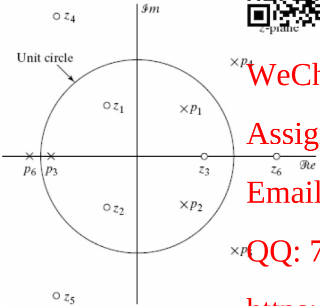
Minimum-Phase Systems

- Given the squared magnitude response of a system, and the corresponding M and N, there is a unique system whose zeros and poles are all inside the unit circle.

- Therefore, the relationship between magnitude and phase for a minimum-phase system is **unique**.

Revisit Example 5.10 (Lecture 9)

Given the C(z) zero-pole plot below, find the system function $H(z)$ that corresponds to a causal system with real-valued time domain impulse response.



Ans:

Poles and Zeros of $H(z)$:

- Poles: p_1, p_2, p_3
- Zeros:
  - *Option 1*: z_3 and (z_1, z_2).
  - *Option 2*: z_3 and (z_4, z_5).
  - *Option 3*: z_6 and (z_1, z_2).
  - *Option 4*: z_6 and (z_4, z_5).

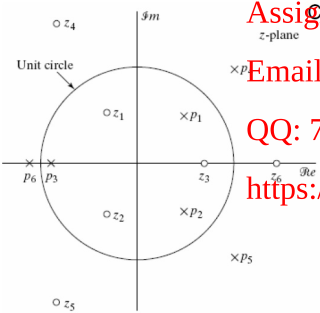
- There are 4 possible options for $H(z)$
- One of these options is a minimum-phase system.

Minimum-Phase Systems
Revisit Example 5.10

Given the magnitude-squared function:

$$C(z) = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})(1 - c_k^* z)}{\prod_{k=1}^{N}(1 - d_k z^{-1})(1 - d_k^* z)}$$

If $H(z)$ is minimum-phase, What are the poles and zeros of $H(z)$?



Ans:

Poles and Zeros of $H(z)$:

- Poles: p_1, p_2, p_3
- Zeros: z_3 and (z_1, z_2).

Properties of Minimum-Phase Systems

- Where does the [illegible]imum-phase come from?

Properties of Minimum-Phase Systems

- Where does the [illegible]imum-phase come from?
- Given the magnitude squared function

$$C(z) = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})(1 - c_k^* z)}{\prod_{k=1}^{N}(1 - d_k z^{-1})(1 - d_k^* z)}$$

there exists a limited number of possibilities for the

corresponding system function ($H_1(z), H_2 z \cdots$ etc.)

Properties of Minimum-Phase Systems

- Given the magnitude squared function

$$C(z) = \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})(1 - c_k^* z)}{\prod_{k=1}^{N}(1 - d_k z^{-1})(1 - d_k^* z)}$$

- All of these possibilities have the same magnitude response.

- $H_{min}(z)$ is one of them.

- All other possibilities for $H(z)$ can be decomposed in to a multiplication of $H_{min}(z)$ and a suitable $H_{ap}(z)$.

$$H_1(z) = H_{min}(z) H_{ap}^1(z)$$

$$H_2(z) = H_{min}(z) H_{ap}^2(z)$$

$$\vdots$$

$$H_k(z) = H_{min}(z)$$

Properties of Minimum-Phase Systems

- Among all these systems that have the same magnitude squared function, the minimum-phase system has the minimum phase delay, minimum group delay and minimum energy delay.

Properties of Minimum-Phase Systems

Minimum Phase Lag

- Phase lag is defined by $\tau_p(\omega) = -\frac{arg[H(e^{j\omega})]}{\omega}$

Properties of Minimum-Phase Systems

Minimum Phase Lag

- Phase lag is defined by $\tau_p(\omega) = -\frac{arg[H(e^{j\omega})]}{\omega}$

- Recall that

$$H(z) = H_{min}(z)H_{ap}(z)$$

- The unwrapped phased of any non minimum-phase system

$$arg[H(e^{j\omega})] = arg[H_{min}(e^{j\omega})] + arg[H_{ap}(e^{j\omega})]$$

Properties of Minimum-Phase Systems

Minimum Phase Lag Property

- The unwrapped phase of any non minimum-phase system

$$\arg[H(e^{j\omega})] = \arg[H_{min}(e^{j\omega})] + \arg[H_{ap}(e^{j\omega})]$$

- Fact: $arg[H_{ap}(e^{j\omega})] < 0$ for $0 \leqslant \omega \leqslant \pi$. (See Lecture 09)

Properties of Minimum-Phase Systems

Minimum Phase Lag P[illegible]

- The unwrapped [illegible] any non minimum-phase system

$$\arg[H(e^{j\omega})] = \arg[H_{min}(e^{j\omega})] + \arg[H_{ap}(e^{j\omega})]$$

- Fact: $arg[H_{ap}(e^{j\omega})] < 0$ for $0 \leqslant \omega \leqslant \pi$. (See Lecture 09)
- Hence, $\arg[H(e^{j\omega})] < \arg[H_{min}(e^{j\omega})]$ for $0 \leqslant \omega \leqslant \pi$
- Hence, $-\arg[H(e^{j\omega})] > -\arg[H_{min}(e^{j\omega})]$ for $0 \leqslant \omega \leqslant \pi$
- $\dfrac{-\arg[H(e^{j\omega})]}{\omega} > \dfrac{-\arg[H_{min}(e^{j\omega})]}{\omega}$ for $0 \leqslant \omega \leqslant \pi$
- Phase Lag $H(e^{j\omega}) >$ Phase Lag $H_{\text{min}}(z)$.
- Minimum-phase systems are minimum phase-lag systems.

Properties of Minimum-Phase Systems

$$H_{min}(z)H_{ap}(z)$$

Minimum Group Del[illegible]ty

- The group delay of any non minimum-phase system

$$grd[H(e^{j\omega})] = grd[H_{min}(e^{j\omega})] + grd[H_{ap}(e^{j\omega})]$$

- Fact: $grd[H_{ap}(e^{j\omega})] \geq 0$ for $0 \leq \omega \leq \pi$. (See Lecture 09)

Properties of Minimum-Phase Systems

$$H_{min}(z)H_{ap}(z)$$

Minimum Group Delay Property

- The group delay of any non minimum-phase system

$$grd[H(e^{j\omega})] = grd[H_{min}(e^{j\omega})] + grd[H_{ap}(e^{j\omega})]$$

- Fact: $grd[H_{ap}(e^{j\omega})] \geq 0$ for $0 \leq \omega \leq \pi$. (See Lecture 09)

- $grd[H_{min}(e^{j\omega})]$ is always less than $grd[H(e^{j\omega})]$.

- Minimum-phase systems are minimum group delay systems.
- Thus, introduces the least amount of delay to the input.

Properties of Minimum-Phase Systems

Minimum Energy De[illegible]ty

- We define the p[illegible]gy of impulse response (time domain) is:

$$E[n] = \sum_{m=0}^{n} |h(m)|^2.$$

Properties of Minimum-Phase Systems

Minimum Energy De[illegible]ty

- We define the p[illegible]gy of impulse response (time domain) is:

$$E[n] = \sum_{m=0}^{n} |h(m)|^2.$$

- What does $E[n]$ mean? At time n, the energy of n samples is added up.

Properties of Minimum-Phase Systems

Minimum Energy Delay Property

- We define the partial energy of impulse response (time domain) is:

$$E[n] = \sum_{m=0}^{n} |h(m)|^2.$$

- What does $E[n]$ mean? At time n, the energy of n samples is added up.
- If $h[n]$ has all of its energy concentrated around $n = 0$, is this a good/bad thing? A good thing, as this will result in low delay.

Properties of Minimum-Phase Systems

$$\sum_{m=0}^{n} |h(m)|^2.$$

- Given $C(z)$, the minimum-phase system has the most amount of partial energy concentrated around $n = 0$; i.e., the energy of the minimum-phase system is delayed the least of all systems having the same magnitude response.

- $\sum_{m=0}^{n} |h(m)|^2 \leqslant \sum_{m=0}^{n} |h_{min}(m)|^2$

Minimum-Phase & All-Pass Decomposition

- Any rational sys… …e decomposed as:

$$= H_{min}(z)H_{ap}(z)$$

where $H_{min}(z)$ is a minimum-phase system and $H_{ap}(z)$ is an all-pass system.

Minimum-Phase & All-Pass Decomposition

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- The magnitude response is equal to the minimum-phase response i.e. $|H(z)| = |H_{min}(z)|$ (why?)

Minimum-Phase & All-Pass Decomposition

- Any rational system can be decomposed as:

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$$|H(z)| = |H_{min}(z)||H_{ap}(z)|$$

given $|H_{ap}(z)| = 1$.

Minimum-Phase & All-Pass Decomposition

- Suppose $H(z)$ h[illegible]o outside the unit circle at $z = \frac{1}{c^*}$, $|c| < 1$ and the [illegible] poles/zeros are inside the unit circle.

$$H(z) = H_1(z)(z^{-1} - c^*)$$

where $H_1(z)$ is minimum-phase.

Minimum-Phase & All-Pass Decomposition

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- An equivalent expression for $H(z)$ is

$$H(z) = H_1(z)(1 - cz^{-1}) \frac{(z^{-1} - c^*)}{1 - cz^{-1}}.$$

Minimum-Phase & All-Pass Decomposition

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  (additional zero at $z = c$ which is at the conjugate reciprocal location of $1/c^*$)

Minimum-Phase & All-Pass Decomposition

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- $H_{min}(z) = H_1(z)(1 - cz^{-1})$ is minimum-phase

  (additional zero at $z = c$ which is at the conjugate reciprocal location of $1/c^*$)

- $H_{ap}(z) = \frac{(z^{-1} - c^*)}{1 - cz^{-1}}$ is an all-pass system.

- The added pole to the all-pass system (inside the unit circle) must be canceled by a zero, which we have added to the minimum-phase system.

Minimum-Phase & All-Pass Decomposition Example 5.12(a)

Find the minimum-phase/all-pass decomposition of the following system:

$$H_1(z) = \frac{1 + 3z^{-1}}{1 + \frac{1}{2} z^{-1}}$$

Minimum-Phase & All-Pass Decomposition Example 5.12(a)

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(1) What are the poles and zeros?

Minimum-Phase & All-Pass Decomposition Example 5.12(a)

Find the minimum-phase and all-pass decomposition of the following system:

$$H_1(z) = \frac{1 + 3z^{-1}}{1 + \frac{1}{2} z^{-1}}$$

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Pole: $z = -\frac{1}{2}$ and Zero: $z = -3$

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(2) How to create a minimum-phase and all-pass system from $H(z)$?

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(1) What are the poles and zeros?
Pole: $z = -\frac{1}{2}$ and Zero: $z = -3$

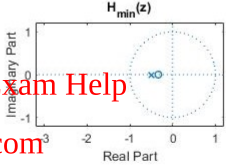
(2) How to create a minimum-phase and all-pass system from $H(z)$?

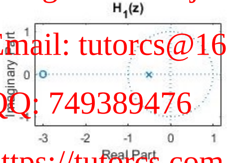
$$H_1(z) = 3\frac{z^{-1} + \frac{1}{3}}{1 + \frac{1}{2}z^{-1}} = 3\frac{1}{1 + \frac{1}{2}z^{-1}}\left(z^{-1} + \frac{1}{3}\right)$$

$$= 3\frac{1 - \left(-\frac{1}{3}z^{-1}\right)}{1 + \frac{1}{2}z^{-1}} \, \frac{z^{-1} + \frac{1}{3}}{1 - \left(-\frac{1}{3}z^{-1}\right)}$$

Minimum-Phase & All-Pass Decomposition Example 5.12(a)

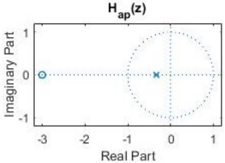
$$H_1(z) = \frac{1}{1 \cdots} \cdots 3\frac{1-(-\frac{1}{3}z^{-1})}{1+\frac{1}{2}z^{-1}}\frac{z^{-1}+\frac{1}{3}}{1-(-\frac{1}{3}z^{-1})}$$





$$H_{min}(z) = 3\frac{1+\frac{1}{3}z^{-1}}{1+\frac{1}{2}z^{-1}}$$

$$H_{ap}(z) = \frac{z^{-1}+\frac{1}{3}}{1+\frac{1}{3}z^{-1}}$$



Minimum-Phase & All-Pass Decomposition Example 5.12(b)

Find the minimum-phase/all-pass decomposition of the following system:

$$H_2(z) = \frac{(1 + \frac{3}{2}e^{j\pi/4}z^{-1})(1 + \frac{3}{2}e^{-j\pi/4}z^{-1})}{1 - \frac{1}{3}z^{-1}}$$

Minimum-Phase & All-Pass Decomposition Example 5.12(b)

Find the minimum-phase/all-pass decomposition of the following system:

$$H_2(z) = \frac{(1 + \frac{3}{2}e^{j\pi/4}z^{-1})(1 + \frac{3}{2}e^{-j\pi/4}z^{-1})}{1 - \frac{1}{3}z^{-1}}$$

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Minimum-Phase & All-Pass Decomposition Example 5.12(b)

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(1) What are the poles and zeros?

Poles: $z = 0$ and $z = \frac{1}{3}$

Zeros: $z = -\frac{3}{2}e^{j\pi/4}$, $z = -\frac{3}{2}e^{-j\pi/4}$

Minimum-Phase & All-Pass Decomposition Example 5.12(b)

Find the minimum-phase and all-pass decomposition of the following system:

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(1) What are the poles and zeros?

Poles: $z = 0$ and $z = \frac{1}{3}$

Zeros: $z = -\frac{3}{2}e^{j\pi/4}$, $z = -\frac{3}{2}e^{-j\pi/4}$

(2) How to create a minimum-phase and all-pass system from $H(z)$?

$$H_2(z) = \frac{\frac{3}{2}e^{j\pi/4}(\frac{2}{3}e^{-j\pi/4} + z^{-1})\frac{3}{2}e^{-j\pi/4}(\frac{2}{3}e^{j\pi/4} + z^{-1})}{1 - \frac{1}{3}z^{-1}}$$

$$= \frac{9}{4}\frac{(z^{-1} + \frac{2}{3}e^{-j\pi/4})(z^{-1} + \frac{2}{3}e^{j\pi/4})}{1 - \frac{1}{3}z^{-1}}$$

Minimum-Phase & All-Pass Decomposition Example 5.12(b)

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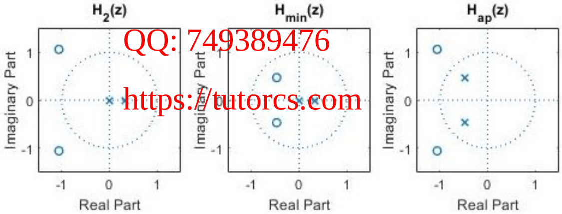
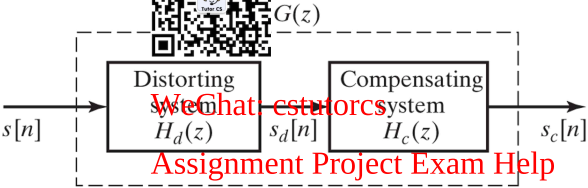
$$= \underbrace{\frac{9}{4}\frac{(1 + \frac{2}{3}e^{j\pi/4}z^{-1})(1 + \frac{2}{3}e^{-j\pi/4}z^{-1})}{1 - \frac{1}{3}z^{-1}}}_{H_{min}(z)}\underbrace{\frac{(z^{-1} + \frac{2}{3}e^{-j\pi/4})(z^{-1} + \frac{2}{3}e^{j\pi/4})}{(1 + \frac{2}{3}e^{j\pi/4}z^{-1})(1 + \frac{2}{3}e^{-j\pi/4}z^{-1})}}_{H_{ap}(z)}$$

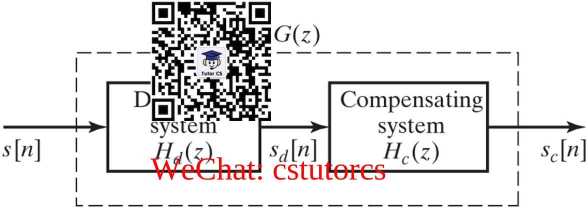


Compensating Non-Minimum-Phase Systems



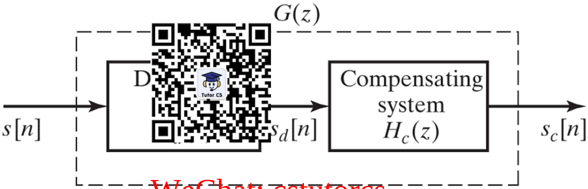
- Perfect compensation: $s_c[n] = s[n]$, i.e., $G(z) = 1$, where $H_c(z)$ is the inverse of $H_d(z)$.

Compensating Non-Minimum-Phase Systems



- If $H_d(z)$ and $H_c(z)$ are both causal and stable, then perfect compensation is possible only if $H_d(z)$ is minimum-phase.
- If $H_d(z)$ is non-minimum-phase, we only compensate its min phase component (why?).

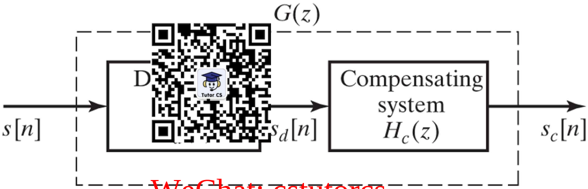
Compensating Non-Minimum-Phase Systems



- If $H_d(z)$ is non-minimum-phase, we only compensate its min phase component:

$$H_d(z) = H_{d\,min}(z) H_{ap}(z).$$

Compensating Non-Minimum-Phase Systems



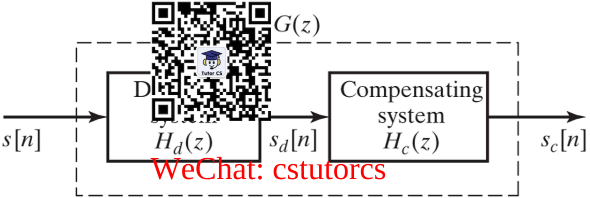
- If $H_d(z)$ is non-minimum-phase, we only compensate its min phase component:

$$H_d(z) = H_{d\ min}(z)H_{ap}(z).$$

- Choosing $H_c(z) = \frac{1}{H_{d\ min}(z)}$, then the overall system function:

$$G(z) = H_d(z)H_c(z) = H_{ap}(z).$$

Compensating Non-Minimum-Phase Systems



- Choosing $H_c(z) = \frac{1}{H_{d\,min}(z)}$, then the overall system function

$$G(z) = H_d(z) H_c(z) = H_{ap}(z).$$

- Magnitude is exactly compensated but phase distortions occur due to $\angle H_{ap}(z)$.

Homework

- Read and understand 5.0 up to 5.6 of the textbook.
- Problems related to this lecture: 5.12, 5.13, 5.14, 5.15, 5.17, 5.18, 5.19, 5.24 and 5.28.