Lecture - 6

Digital Signal Processing

ENGN 4537/ 6537

Semester 2, 2023

Inverse z-Transform

The z-transform

$$\sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Question: If we are given $X(z)$, how can we infer $x[n]$?

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The inverse of z-transform is formally defined as

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

where C represents a closed contour within ROC of the z-transform.

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$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

where C represents a closed contour within ROC of the z-transform.

For the purpose of calculating inverse transforms for most practical purposes, we use **simpler techniques** rather than using this integral. **(We are not going to use this formula directly)**

Method 1: Inspection Method

Simply by inspection and familiarity with Table 3.1 and Table 3.2 of the z-transforms.

TABLE 3.1 SOME COM[illegible]S

Sequence	Transform	ROC
1. $\delta[n]$	[illegible]	All z
2. $u[n]$	$\frac{[illegible]}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{[illegible]}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Method 1: Inspection Method

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\| R_x$
4	3.4.4	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Method 1: Inspection Method

Good idea to apply t[illegible]our memory:

$$z^{-m} \rightarrow \delta[n-m], \quad m \text{ is a constant}$$

$$\frac{1}{1-az^{-1}} \rightarrow \begin{cases} a^n u[n], & \text{ROC:} |z| > a \\ -a^n u[-n-1], & \text{ROC:} |z| < a \end{cases}$$

Method 1: Inspection Method

Example

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, \quad |z| > |a| \quad \text{(Table 3.1 Transform Pair 5)}$$

Find the inverse z-transform of:

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

Method 1: Inspection Method

Example

$$a^n u[n] \quad \overset{\mathscr{Z}}{\longleftrightarrow} \quad \frac{1}{1 - az^{-1}}, \quad |z| > |a| \quad \text{(Table 3.1 Transform Pair 5)}$$

Find the inverse z-transform of:

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

Ans. According to Table 3.1 Transform Pair 5, if we set $a = \frac{1}{2}$, then:

$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$

Method 2: Partial Fraction Expansion

- Break down a rational function into a summation of smaller rational functions.
- Generic Rational z-Transforms :

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \quad (X(z) \text{ as ratio of polynomials in } z^{-1})$$

An equivalent expression is:

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

Factoring out z^{-M} from numerator and z^{-N} from denominator:

$$X(z) = \frac{z^{-M}(b_0 z^{M} + b_1 z^{M-1} + b_2 z^{M-2} + \cdots + b_M)}{z^{-N}(a_0 z^{N} + a_1 z^{N-1} + a_2 z^{N-2} + \cdots + a_N)}$$

$$= \frac{z^{N} \sum_{k=0}^{M} b_k z^{M-k}}{z^{M} \sum_{k=0}^{N} a_k z^{N-k}} \quad (X(z) \text{ as ratio of polynomials in } z)$$

Method 2: Partial Fraction Expansion

$$\frac{z^{N}\sum_{k=0}^{M} b_k z^{M-k}}{z^{M}\sum_{k=0}^{N} a_k z^{N-k}}$$

How many poles and zeros are there?

Method 2: Partial Fraction Expansion

$$\frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}}$$

How many poles and zeros are there?

- There are M zeros and N poles at non-zero locations, assuming $a_k > 0$ and $b_k > 0$.

Method 2: Partial Fraction Expansion

$$\frac{z^{N}\sum_{k=0}^{M} b_k z^{M-k}}{z^{M}\sum_{k=0}^{N} a_k z^{N-k}}$$

How many poles and zeros are there?

- There are M zeros and N poles at non-zero locations, assuming $a_k > 0$ and $b_k > 0$.

- If $M > N$, then there are $(M - N)^{\text{th}}$ order poles at $z = 0$.

- If $M < N$, then there are $(N - M)^{\text{th}}$ order zeros at $z = 0$.

Method 2: Partial Fraction Expansion

To obtain the partial fraction expansion, it is most convenient to express $X(z)$ in the following form:

$$X(z) = \frac{b_0 \prod_{k=1}^{M} (1 - c_k z^{-1})}{a_0 \prod_{k=1}^{N} (1 - d_k z^{-1})}$$

where c_k are the zeros of $X(z)$ and d_k are the poles of $X(z)$.

Method 2: Partial Fraction Expansion Case 1

Given

$$X(z) = \frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

Case 1: If $M < N$ and the poles are all 1st order, we can decompose $X(z)$ as:

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}}$$

$$= \frac{A_1}{1-d_1 z^{-1}} + \frac{A_2}{1-d_2 z^{-1}} + \cdots + \frac{A_m}{1-d_m z^{-1}} + \cdots + \frac{A_N}{1-d_N z^{-1}}$$

Then $A_m = (1-d_m z^{-1})X(z)|_{z=d_m}$ or $A_m = (1-d_m z^{-1})X(z)|_{z^{-1}=d_m^{-1}}$

- Note: $\frac{1}{(1-d_k z^{-1})^2}$ will result in second-order poles at $z = d_k$.

Method 2: Partial Fraction Expansion Case 1

Proof :

Multiply all sides by $(1 - d_m z^{-1})$

$$
(1 - d_m z^{-1})X(z)\Big|_{z=d_m} = \frac{b_0 \prod_{k=1}^{M}(1 - c_k z^{-1})}{a_0 \prod_{k=1}^{m-1}(1 - d_k z^{-1}) \prod_{k=m+1}^{N}(1 - d_k z^{-1})}\Bigg|_{z=d_m}
$$

$$
= \frac{A_1(1 - d_m z^{-1})}{1 - d_1 z^{-1}}\Bigg|_{z=d_m} + \frac{A_2(1 - d_m z^{-1})}{1 - d_2 z^{-1}}\Bigg|_{z=d_m} + \cdots
$$

$$
\cdots + \frac{A_m(1 - d_m z^{-1})}{1 - d_m z^{-1}}\Bigg|_{z=d_m} + \cdots + \frac{A_N(1 - d_m z^{-1})}{1 - d_N z^{-1}}\Bigg|_{z=d_m}
$$

$$
= 0 + 0 + \cdots + A_m + \cdots + 0
$$

Method 2: Partial Fraction Expansion
Case 1

S.No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}$, a	$\frac{A}{x-a}+\frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ where x^2+bx+c cannot be factorised further	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$,

Method 2: Partial Fraction Expansion
Case 1: Example 3.9

Consider the z-transfor

$$\frac{1}{\cdots - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

Method 2: Partial Fraction Expansion
Case 1: Example 3.9

Consider the z-transform

$$X(z) = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

$$X(z) = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \underbrace{\frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}}_{\text{(Factorise denominator)}} = \underbrace{\frac{A_1}{1 - \frac{1}{4}z^{-1}} + \frac{A_2}{1 - \frac{1}{2}z^{-1}}}_{\text{(Partial fraction expansion)}}$$

Clearing denominators: $1 = A_1\left(1 - \frac{1}{2}z^{-1}\right) + A_2\left(1 - \frac{1}{4}z^{-1}\right)$

Substituting $z^{-1} = 4$ gives $A_1 = -1$, and substituting $z^{-1} = 2$ gives $A_2 = 2$.

Another method: Solve equations $A_1 + A_2 = 1$ and $\frac{A_1}{2} + \frac{A_2}{4} = 0$

$$X(z) = \frac{-1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{2}z^{-1}}$$

How do we find A_k in general? Use $A_m = (1 - d_m z^{-1})X(z)\big|_{z^{-1}=d_m^{-1}}$

Method 2: Partial Fraction Expansion
Case 1: Example 3.9

$$X(z) = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{1}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})}$$

$$= \frac{A_1}{1 - \frac{1}{4}z^{-1}} + \frac{A_2}{1 - \frac{1}{2}z^{-1}}$$

Method 2: Partial Fraction Expansion
Case 1: Example 3.9

$$X(z) = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{1}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})}$$

$$= \frac{A_1}{1 - \frac{1}{4}z^{-1}} + \frac{A_2}{1 - \frac{1}{2}z^{-1}}$$

To find A_1 corresponding to the pole at $z = \frac{1}{4}$ (or $z^{-1} = 4$),

$$\left(1 - \frac{1}{4}z^{-1}\right) X(z) = \left(\frac{A_1}{1 - \frac{1}{4}z^{-1}} + \frac{A_2}{1 - \frac{1}{2}z^{-1}}\right)\left(1 - \frac{1}{4}z^{-1}\right)$$

$$\left.\frac{1}{1 - \frac{1}{2}z^{-1}}\right|_{z^{-1}=4} = A_1 + \left.\frac{A_2(1 - \frac{1}{4}z^{-1})}{1 - \frac{1}{2}z^{-1}}\right|_{z^{-1}=4}$$

$$\therefore A_1 = -1$$

Method 2: Partial Fraction Expansion
Case 1: Example 3.9

$$X(z) = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{1}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})}$$

$$= \frac{A_1}{1 - \frac{1}{4}z^{-1}} + \frac{A_2}{1 - \frac{1}{2}z^{-1}}$$

Method 2: Partial Fraction Expansion Case 1: Example 3.9

$$X(z) = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{1}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})}$$

$$= \frac{A_1}{1 - \frac{1}{4}z^{-1}} + \frac{A_2}{1 - \frac{1}{2}z^{-1}}$$

To find A_2 corresponding to the pole at $z = \frac{1}{2}$ (or $z^{-1} = 2$)

$$\left(1 - \frac{1}{2}z^{-1}\right) X(z) = \left(\frac{A_1}{1 - \frac{1}{4}z^{-1}} + \frac{A_2}{1 - \frac{1}{2}z^{-1}}\right)\left(1 - \frac{1}{2}z^{-1}\right)$$

$$\left.\frac{1}{1 - \frac{1}{4}z^{-1}}\right|_{z^{-1}=2} = \left.\frac{A_1(1 - \frac{1}{2}z^{-1})}{1 - \frac{1}{4}z^{-1}}\right|_{z^{-1}=2} + A_2$$

$$\therefore A_2 = 2$$

Method 2: Partial Fraction Expansion
Case 1: Example 3.9

$$X(z) = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{1}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})}$$

$$= \frac{A_1}{1 - \frac{1}{4}z^{-1}} + \frac{A_2}{1 - \frac{1}{2}z^{-1}}$$

$$\therefore X(z) = \frac{-1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{2}z^{-1}}$$

Method 2: Partial Fraction Expansion
Case 1

Note the z-transform

$$\frac{A_k}{1 - d_k z^{-1}} \rightarrow \begin{cases} A_k (d_k)^n u[n] & \text{ROC:} |z| > d_k \\ -A_k (d_k)^n u[-n-1] & \text{ROC:} |z| < d_k \end{cases}$$

ROC needs to be used to determine the corresponding sequence.

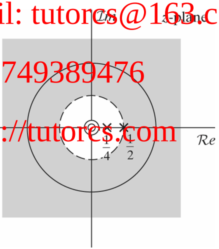
Method 2: Partial Fraction Expansion

Case 1: Example 3.9 (Completed)

$$X(z) = \frac{1}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}} = \left(\frac{-1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{2}z^{-1}}\right) = \frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{4})}$$

If further know that ROC is $|z| > \frac{1}{2}$, then we conclude that:

$$x[n] = -\left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{2}\right)^n u[n]$$



Method 2: Partial Fraction Expansion
Exercise

$$X(z) = \frac{\text{[illegible]}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{-1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{2}z^{-1}}$$

If the ROC is $\frac{1}{4} < |z| < \frac{1}{2}$, what would be the sequence?

Method 2: Partial Fraction Expansion
Exercise

$$X(z) = \frac{\quad}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{-1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{2}z^{-1}}$$

If the ROC is $\frac{1}{4} < |z| < \frac{1}{2}$, what would be the sequence?

Ans.

$$x[n] = -\left(\frac{1}{4}\right)^n u[n] - 2\left(\frac{1}{2}\right)^n u[-n-1]$$

As an exercise if ROC is $|z| < \frac{1}{4}$, determine the corresponding sequence.

Method 2: Partial Fraction Expansion
Case 2 - First-order poles

$$X(z) = \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{a_0 \prod_{k=1}^{N}(1 - d_k z^{-1})}$$

Case 2: If $M \geqslant N$, a polynomial of the order $M - N$ has to be added. If all poles are first order, then we will have:

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}.$$

Method 2: Partial Fraction Expansion
Case 2 - First-order poles

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}.$$

- Coefficients B_r can be obtained by long division of the numerator of $X(z)$ by its denominator.
- Coefficients A_k are obtained as in the case of $M < N$ (by multiplying $X(z)$ by $1 - d_k z^{-1}$ and evaluating at $z = d_k$ or $z^{-1} = d_k^{-1}$).

Method 2: Partial Fraction Expansion
Case 2 - First-order poles

Note the Z transform

$$B_r z^{-r} \rightarrow B_r \delta[n-r]$$

and

$$\frac{A_k}{1-d_k z^{-1}} \rightarrow \begin{cases} A_k (d_k)^n u[n] & \text{ROC:}|z| > d_k \\ -A_k (d_k)^n u[-n-1] & \text{ROC:}|z| < d_k \end{cases}$$

ROC needs to be used to determine the corresponding sequence.

Method 2: Partial Fraction Expansion
Case 2: Example

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$$

- From $X(z)$, $M = N$.
- We need to rearrange X(z) to have the following form:

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}.$$

- To find B_r we need to calculate the long division of $X(z)$, which gives quotient 2 with a remainder $(-1 + 5z^{-1})$.

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Method 2: Partial Fraction Expansion

Case 2: Example

$$X(z) = 2 + \frac{\cdots}{(1-\frac{1}{2}z^{-1})(1-z^{-1})} = 2 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}$$

Method 2: Partial Fraction Expansion

Case 2: Example

$$X(z) = 2 + \frac{\text{[illegible]}}{(1-\frac{1}{2}\text{[illegible]}\; z^{-1})} = 2 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}$$

To find A_1 corresponding to the pole at $z = \frac{1}{2}$ (or $z^{-1} = 2$)

$$\left(1-\frac{1}{2}z^{-1}\right)X(z) = \left(2 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}\right)\left(1-\frac{1}{2}z^{-1}\right)$$

$$\left[2\left(1-\frac{1}{2}z^{-1}\right) + \frac{-1+5z^{-1}}{1-z^{-1}}\right|_{z^{-1}=2} = 2\left(1-\frac{1}{2}z^{-1}\right)\Big|_{z^{-1}=2} + A_1 + \frac{A_2(1-\frac{1}{2}z^{-1})}{1-z^{-1}}\Big|_{z^{-1}=2}$$

$$\therefore A_1 = -9$$

Method 2: Partial Fraction Expansion

Case 2: Example

$$X(z) = 2 + \frac{\text{[illegible]}}{(1 - \text{[illegible]}\, z^{-1})} = 2 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

To find A_2 corresponding to the pole at $z = 1$ (or $z^{-1} = 1$)

$$\left(1 - z^{-1}\right) X(z) = \left(2 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}\right)\left(1 - z^{-1}\right)$$

$$\left[2\left(1 - z^{-1}\right) + \frac{-1 + 5z^{-1}}{1 - \frac{1}{2}z^{-1}}\right]_{z^{-1}=1} = 2\left(1 - z^{-1}\right)\Big|_{z^{-1}=1} + \frac{A_1\left(1 - z^{-1}\right)}{1 - \frac{1}{2}z^{-1}}\Bigg|_{z^{-1}=1} + A_2$$

$$\therefore A_2 = 8$$

Method 2: Partial Fraction Expansion

Case 2: Example

$$X(z) = 2 + \frac{-9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}$$

- If ROC $|z| > 1$:

Method 2: Partial Fraction Expansion

Case 2: Example

$$X(z) = 2 + \frac{-9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}$$

- If ROC $|z| > 1$: $x[n] = 2\delta[n] - 9(\frac{1}{2})^n u[n] + 8u[n]$

Method 2: Partial Fraction Expansion

Case 2: Example

$$X(z) = 2 + \frac{-9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}$$

- If ROC $|z| > 1$: $x[n] = 2\delta[n] - 9(\frac{1}{2})^n u[n] + 8u[n]$

- If ROC $|z| < \frac{1}{2}$:

Method 2: Partial Fraction Expansion

Case 2: Example

$$X(z) = 2 + \frac{-9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}$$

- If ROC $|z| > 1$: $x[n] = 2\delta[n] - 9(\frac{1}{2})^n u[n] + 8u[n]$

- If ROC $|z| < \frac{1}{2}$: $x[n] = 2\delta[n] + 9(\frac{1}{2})^n u[-n-1] - 8u[-n-1]$

Method 2: Partial Fraction Expansion

Case 2: Example

$$X(z) = 2 + \frac{-9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}$$

- If ROC $|z| > 1$: $x[n] = 2\delta[n] - 9(\frac{1}{2})^n u[n] + 8u[n]$

- If ROC $|z| < \frac{1}{2}$: $x[n] = 2\delta[n] + 9(\frac{1}{2})^n u[-n-1] - 8u[-n-1]$

- If ROC $\frac{1}{2} < |z| < 1$:

Method 2: Partial Fraction Expansion

Case 2: Example

$$X(z) = 2 + \frac{-9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}$$

- If ROC $|z| > 1$: $x[n] = 2\delta[n] - 9(\frac{1}{2})^n u[n] + 8u[n]$

- If ROC $|z| < \frac{1}{2}$: $x[n] = 2\delta[n] + 9(\frac{1}{2})^n u[-n-1] - 8u[-n-1]$

- If ROC $\frac{1}{2} < |z| < 1$: $x[n] = 2\delta[n] - 9(\frac{1}{2})^n u[n] - 8u[-n-1]$

Method 2: Partial Fraction Expansion Generalisation

How to determine wh[illegible]l fraction expansion is right-sided or left-sided for the RO[illegible]?

- To decide whether each term within $X(z)$,i.e.,

$$\frac{A_k}{1 - d_k z^{-1}}$$

corresponds to $(d_k)^n u[n]$ or $-(d_k)^n u[-n-1]$, we need to determine the relationship of d_k position with respect to a given ROC.

Method 2: Partial Fraction Expansion Generalisation

To decide whether each term within $X(z)$, i.e., $\frac{A_k}{1-d_k z^{-1}}$, is a right-sided or left-sided sequence in the time domain:

- Lets assume the ROC is of the form $r_R < |z| < r_L$.

Method 2: Partial Fraction Expansion Generalisation

To decide whether ea[illegible]within $X(z)$, i.e., $\frac{A_k}{1-d_k z^{-1}}$, is a right-sided or left-sid[illegible]ce in the time domain:

- Lets assume the ROC is of the form $r_R < |z| < r_L$.
- If a pole at d_k is $d_k < r_R$ then this will correspond to a right-sided sequence $(d_k)^n u[n]$.

Method 2: Partial Fraction Expansion Generalisation

To decide whether ea[illegible] within $X(z)$, i.e., $\frac{A_k}{1-d_k z^{-1}}$, is a right-sided or left-sid[illegible]ce in the time domain:

- Lets assume the ROC is of the form $r_R < |z| < r_L$.

- If a pole at d_k is $d_k < r_R$ then this will correspond to a right-sided sequence $(d_k)^n u[n]$.

- If a pole at d_k is $d_k > r_L$ then this will correspond to a left-sided sequence $-(d_k)^n u[-n-1]$.

Method 3: Power Series Expansion

We can write a given z-transform in the form of the z-transform definition:

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \cdots + x[-2]z^{2} + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots \end{aligned}$$

We can determine any particular value of the sequence by finding the coefficient of the appropriate power of z^{-1}.

Method 3: Power Series Expansion

Example

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + \cdots$$

Example: $X(z) = z^2 - 0.5z - 1 + 0.5z^{-1}$. Then, by inspection we can evaluate $X(z)$ to $x[n]$

$$x[n] = \begin{cases} 1, & n = -2 \\ -0.5, & n = -1 \\ -1, & n = 0 \\ 0.5, & n = 1 \\ 0 & \text{othewrise} \end{cases}$$

Therefore $x[n]$ can be written as a summation of delta functions:

$$x[n] = \delta[n+2] - 0.5\delta[n+1] - \delta[n] + 0.5\delta[n-1]$$

Method 3: Power Series Expansion

- Sometimes, a given $X(z)$ can be converted to a power series using *Taylor Expansion*. See Example 3.12 in the textbook.
  Taylor series expansion is NOT assessable in the exams.

- When $X(z)$ is a ratio of polynomials, a long division can be used to convert $X(z)$ to a power series.

Method 3: Power Series Expansion Example

If the ROC is $|z| > a$, we expand $X(z)$ as:

$$X(z) = \frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots$$

Then,

$$x[n] = a^n u[n]$$

Method 3: Power Series Expansion Example

ROC is $|z| < a$ or $|a^{-1}z| < 1$, we expand $X(z)$ as:

$$X(z) = \frac{1}{1 - az^{-1}} = -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - \cdots$$

Then,

$$x[n] = -a^n u[-n-1]$$

A Final Example

- The z-transform of $x[n]$ is

$$X(z) = \frac{1 - 2z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

for $|z| > \frac{1}{2}$. Find $x[n]$ using any valid method.

A Final Example (Continued)

This is one approach [illegible]d be used to solve this problem:

$$X(z) = \frac{1 - 2z^{-1}}{1 - \frac{1}{2}z^{-1}} = 4 + \frac{-3}{1 - \frac{1}{2}z^{-1}}$$

Since ROC is $|z| > \frac{1}{2}$, we have

$$x[n] = 4\delta[n] - 3(\frac{1}{2})^n u[n]$$

A Final Example (Continued)

Here is another approach that could be used to solve this problem:

$$X(z) = [illegible] = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{-2z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

For the first term we have

$$x_1[n] = (\frac{1}{2})^n u[n]$$

Using the time shift property for the second term we have

$$x_2[n] = -2(\frac{1}{2})^{n-1} u[n-1] = -4(\frac{1}{2})^n u[n-1]$$

Verify that:

$$x_1[n] + x_2[n] = 4\delta[n] - 3(\frac{1}{2})^n u[n]$$

A Final Example (Continued)

Proof of the second approach to solve this problem:

$$= (\frac{1}{2})^n u[n]$$

Using the time shift prop[illegible] second term we have

$$x_2[n] = -2(\frac{1}{2})^{n-1} u[n-1] = -4(\frac{1}{2})^n u[n-1]$$

Note that for $n = 0$, we have

$$x_1[n]+x_2[n] = (\frac{1}{2})^0 u[0] - 4(\frac{1}{2})^0 u[0-1] = (\frac{1}{2})^0 u[0] = 1 = 4-3 = 4\delta[n] - 3(\frac{1}{2})^0 u[0]$$

Note that for $n \geqslant 1$ we have

$$\begin{aligned} x_1[n] + x_2[n] &= (\frac{1}{2})^n u[n] - 4(\frac{1}{2})^n u[n-1] \\ &= (\frac{1}{2})^n - 4(\frac{1}{2})^n = -3(\frac{1}{2})^n = 0 - 3(\frac{1}{2})^n = 4\delta[n] - 3(\frac{1}{2})^n u[n] \end{aligned}$$

for $n < 0$, $x_1[n] + x_2[n] = 0$. So overall for all n, we have the result.

Homework

1. Read and understand Chapter 3 of the textbook
2. Related Problems: 3.18, 3.21, 3.30, 3.31, 3.32