Lecture – 3

Digital Signal Processing

ENGN 4537/ 6537

Semester 2, 2023

Representation of discrete-time signals by Fourier Transform

- Fourier Transform of a discrete-time signal $x[n]$:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \text{ (Analysis Equation)}$$

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- Fourier Transform of a discrete-time signal $x[n]$:

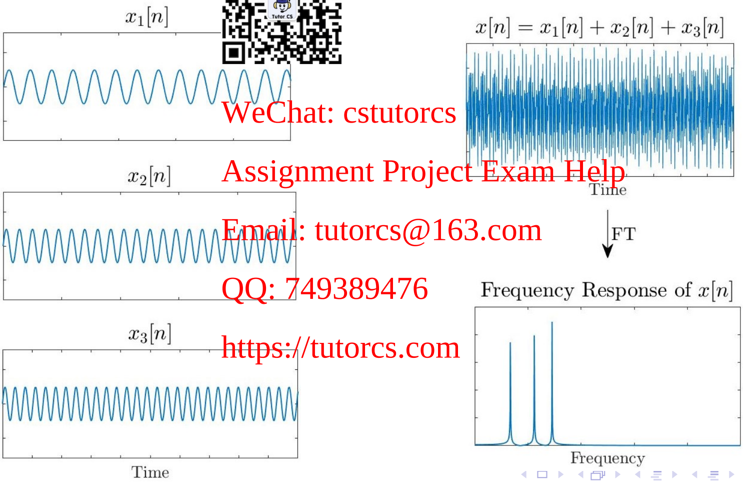
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad (\text{Analysis Equation})$$

- Inverse Fourier Transform

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega. \quad (\text{Synthesis Equation})$$

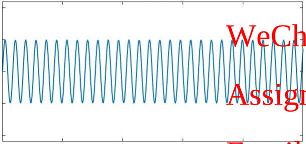
Fourier Transform: Use cases

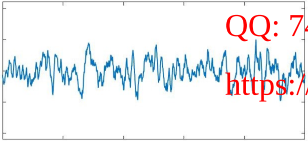
Example 1: Decompose signal into frequency components

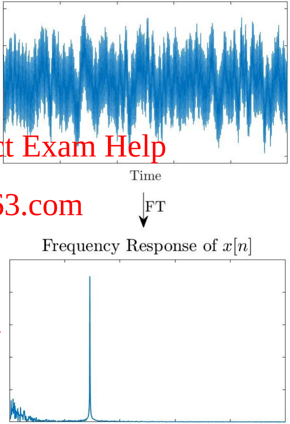


Fourier Transform: Use cases

Example 2: Noise r







Fourier Transform of $x[n]$: $X(e^{j\omega})$

- Can be written w … and Imaginary parts

$$X_R(e^{j\omega}) + jX_I(e^{j\omega})$$

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$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\angle X(e^{j\omega})}$$

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$$X(e^{j\omega}) = |X(e^{j\omega})| e^{j\angle X(e^{j\omega})}$$

- The phase $\angle X(e^{j\omega})$ is not unique since any integer multiplier of 2π may be added to $\angle X(e^{j\omega})$ without affecting the result.

- $\mathrm{ARG}[X(e^{j\omega})]$ is the phase $\angle X(e^{j\omega})$ restricted to values between $-\pi$ and π.

Example Fourier Transform pair: Impulse response and frequency response of an LTI system

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}. \quad \text{(Analysis Equation)}$$

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega. \quad \text{(Synthesis Equation)}$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$X(e^{j\omega}) \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow Y(e^{j\omega})$$

Comparison with Fourier Transform of a Continuous Time Signal

Discrete Time

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega.$$

Continuous Time

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$$

$$x(t) = \int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega.$$

Comparison with Fourier Transform of a Continuous Time Signal

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$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

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Continuous Time

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$$

$$x(t) = \int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega.$$

- $X(e^{j\omega})$ is a continuous function of ω and a periodic function with period 2π.
- Generally, $X(j\Omega)$ is not periodic but a continuous function $-\infty < \Omega < \infty$.

Convergence of Fourier Transform

- Convergence: $|X$ ∞ for all ω.

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- A **sufficient** con convergence

$$|X(e^{j\omega})| = \left| \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right|$$

$$\leqslant \sum_{n=-\infty}^{\infty} |x[n]||e^{-j\omega n}| = \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

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$$\leqslant \sum_{n=-\infty}^{\infty} |x[n]| |e^{-j\omega n}| = \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- Thus, if $x[n]$ is absolutely summable then $X(e^{j\omega})$ exists.

Convergence of Fourier Transform

- Convergence: $|X(e^{j\omega})| < \infty$ for all ω.
- A **sufficient** condition for convergence

$$|X(e^{j\omega})| = \left| \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right|$$

$$\leqslant \sum_{n=-\infty}^{\infty} |x[n]||e^{-j\omega n}| = \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- Thus, if $x[n]$ is absolutely summable then $X(e^{j\omega})$ exists.
- Since a stable system, by definition, has an impulse response $h[n]$ that's absolutely summable, all stable systems have corresponding Fourier Transforms $H(e^{jw}) \Rightarrow$ finite and continuous frequency response.

Symmetry properties of Fourier Transform

TABLE 2.1 SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{Re}\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{Im}\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{Re}\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{Im}\{X(e^{j\omega})\}$
	The following properties apply only when $x[n]$ is real:
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even)
11. Any real $x[n]$	$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

Fourier Transform Theorems

TABLE 2.2 ORM THEOREMS

	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

9. $$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

Table 2.2: Theorem 2

- **Sequence**: $x[n - n_d]$ where n_d is an integer
- **Fourier Transform**: $e^{-j\omega n_d} X(e^{j\omega})$

$$\left(\textit{time shift} \xrightarrow{FT} \textit{phase shift} \right)$$

$$\mathrm{FT}\{x[n - n_d]\} = \sum_{n=-\infty}^{\infty} x[n - n_d] e^{-j\omega n}$$

$$\mathrm{FT}\{x[n - n_d]\} = \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega(m+n_d)} \quad (\text{Change } m = n - n_d)$$

$$\mathrm{FT}\{x[n - n_d]\} = e^{-j\omega n_d} \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega m} = e^{-j\omega n_d} X(e^{j\omega})$$

Table 2.2: Theorem 3

- **Sequence**: $e^{j\omega_0 n}x[n]$
- **Fourier Transform**: $X(e^{j(\omega-\omega_0)})$

$$\mathrm{FT}\{e^{j\omega_0 n}x[n]\} = \sum_{n=-\infty}^{\infty} e^{j\omega_0 n}x[n]e^{-j\omega n}$$

$$\mathrm{FT}\{e^{j\omega_0 n}x[n]\} = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega-\omega_0)n} \quad (\text{Rearrange})$$

$$\Rightarrow \mathrm{FT}\{e^{j\omega_0 n}x[n]\} = X(e^{j(\omega-\omega_0)})$$

Fourier Transform Pairs

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	[illegible]
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\|a\|<1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\|a\|<1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\|r\|<1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \|\omega\|<\omega_c, \\ 0, & \omega_c<\|\omega\|\le\pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0\le n\le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n+\phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$

Table 2.3: Transform Pair 1

- **Sequence**: $\delta[n]$
- **Fourier Transf**

Note: $\delta[n] = 0$ for all n apart from at $n = 0$, where $\delta[0] = 1$.

$$\begin{aligned} \mathrm{FT}\{\delta[n]\} &= \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} \\ &= e^{-j\omega(0)} \\ &= 1 \end{aligned}$$

Table 2.3: Transform Pair 2

- **Sequence**: $\delta[n$
- **Fourier Transf** n_0

$$
\begin{aligned}
\mathrm{FT}\{\delta[n-n_0]\} &= \sum_{n=-\infty}^{\infty} \delta[n-n_0] e^{-j\omega n} \\
&= e^{-j\omega n_0}
\end{aligned}
$$

$$
\therefore |\mathrm{FT}\{\delta[n-n_0]\}| = |e^{-j\omega n_0}| = 1
$$

$$
\therefore \angle \mathrm{FT}\{\delta[n-n_0]\} = -\omega n_0
$$

Note: A time-delay in the time domain is equal to a phase shift in the frequency domain.

Table 2.3: Transform Pair 4

- **Sequence**: $a^n u[n]$, $(|a| < 1)$
- **Fourier Transform**: $\frac{1}{1 - ae^{-j\omega}}$

$$\text{FT}\{a^n u[n]\} = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$\Rightarrow X(e^{j\omega}) = 1 + ae^{-j\omega} + \ldots + a^{\infty} e^{-j\omega\infty} \quad \text{(Expand summation)}$$

$$\Rightarrow X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} \quad \text{(Geometric series for } |a| < 1\text{)}$$

Table 2.3: Transform Pair 8

- **Sequence**: $\frac{\sin(\omega_c n)}{\pi n}$
- **Fourier Transform**:

$$X(e^{j\omega}) = \begin{cases} 1, & |\omega| < |\omega_c|, \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

Ideal Low Pass Filter

Take the inverse FT of the Fourier Transform:

$$\text{IFT}\{X(e^{j\omega})\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$\Rightarrow \text{IFT}\{X(e^{j\omega})\} = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega$$

Table 2.3: Transform Pair 8 (Continued)

$$\begin{aligned}
\text{IFT}\{X(e^{j\omega})\} &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega \\
&= \frac{1}{2\pi}\left[\left(\frac{1}{nj}\right) e^{j\omega n}\right]_{-\omega_c}^{\omega_c} \\
&= \frac{1}{\pi n}\left(\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right)
\end{aligned}$$

$$\Rightarrow \text{IFT}\{X(e^{j\omega})\} = \frac{\sin(\omega_c n)}{\pi n} \quad (\text{Trig. \& Exp. Identity})$$

Table 2.3: Transform pair not in the Table

High Pass Filter

$$X_{HP}(e^{j\omega}) = \begin{cases} 0, & |\omega| < |\omega_c|, \\ 1, & \omega_c < |\omega| \le \pi \end{cases}$$

$$\therefore X_{HP}(e^{j\omega}) = 1 - X_{LP}(e^{j\omega})$$

$$\Rightarrow \mathrm{IFT}\{X_{HP}(e^{j\omega})\} = \delta[n] - \frac{\sin(\omega_c n)}{\pi n}$$

Table 2.3: Transform Pair 9

- **Sequence**:

$$x[n] = \begin{cases} 1, & 0 \leqslant n \leqslant M \\ 0, & \text{otherwise} \end{cases}$$

- **Fourier Transform**: $\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$

Take the FT of the sequence:

$$\mathrm{FT}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$\Rightarrow \mathrm{FT}\{x[n]\} = \sum_{n=0}^{M} e^{-j\omega n}$$

Table 2.3: Transform Pair 9 (Continued)

$$\} = \sum_{n=0}^{M} e^{-j\omega n}$$

$$\Rightarrow \mathrm{FT}\{x[n]\} = 1 + e^{-j\omega} + \ldots + e^{-j\omega M} \quad \text{(Expand summation)}$$

$$\Rightarrow \mathrm{FT}\{x[n]\} = \frac{1 - e^{-j\omega(M+1)}}{1 - e^{-j\omega}} \quad \text{(Finite Geometric Series)}$$

In order to get to the Fourier transform expression, we need to manipulate the above equation by factoring out $e^{-j\omega\frac{M+1}{2}}$ from the numerator, and factoring out $e^{-j\omega\frac{1}{2}}$ from the denominator.

Table 2.3: Transform Pair 9 (Continued)

$$\Rightarrow \mathrm{FT}\{x[n]\} = \frac{e^{-j\omega\frac{M+1}{2}}\left[e^{j\omega\frac{M+1}{2}} - e^{-j\omega\frac{M+1}{2}}\right]}{e^{-j\omega/2}\left[e^{j\omega/2} - e^{-j\omega/2}\right]}$$

$$\Rightarrow \mathrm{FT}\{x[n]\} = \frac{\sin\left(\omega\frac{M+1}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\omega M/2} \quad \text{(Trig. \& Exp. Identities)}$$

Table 2.3 - Transform Pair 3

- **Sequence**: 1, $-\infty < n < \infty$
- **Fourier Transform**: $\sum_{k=-\infty}^{\infty} 2\pi\delta\left(\omega + 2\pi k\right)$

$$\mathrm{FT}\{1\} = \sum_{n=-\infty}^{\infty} e^{-j\omega n} \quad (\text{DTFT})$$

$$\Rightarrow \mathrm{FT}\{1\} = \sum_{k=-\infty}^{\infty} 2\pi\delta\left(\omega + 2\pi k\right)$$

Note: The sum of this sequence is infinite, but it has a Fourier Transform.

Table 2.3 - Transform Pair 3 (Continued)

What is happening in the Fourier domain?

$\text{FT}\{1\} = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$

- On the RHS is the Dirac delta function $\rightarrow$ Take the IFT.
- Recall: $\text{IFT}\{X(e^{j\omega})\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$

$$\text{IFT}\{X(e^{j\omega})\} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)e^{j\omega n}d\omega$$

$$= \sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi}\delta(\omega + 2\pi k)e^{j\omega n}d\omega$$

Table 2.3 - Transform Pair 3 (Continued)

Hence,

$$\int_{-\pi}^{\pi} \delta(\omega + 2\pi k) e^{j\omega n} d\omega = \begin{cases} e^{j0n} = 1, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

Take note of the bounds of the integral, what does this function look like in these bounds?

Table 2.3 - Transform Pair 3 (Continued)

Hence,

$$\int_{-\pi}^{\pi} \delta(\omega + 2\pi k) e^{j\omega n} d\omega = \begin{cases} e^{j0n} = 1, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

Take note of the bounds of the integral, what does this function look like in these bounds?

Therefore,

$$\text{IFT}\{X(e^{j\omega})\} = \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} \delta(\omega + 2\pi k) e^{j\omega n} d\omega = 1$$

Table 2.3 - Transform Pair 5

- **Sequence**: $u[n] = \sum_{m=-\infty}^{n} \delta[m]$
- **Fourier Transform**: $\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta\left(\omega + 2\pi k\right)$
- This is not a straightforward proof, and it requires some fancy manipulation of the sequence and a rather long derivation.
- Note that this sequence is also not summable yet it has a frequency response. Sometimes this is possible with the introduction of delta functions, but the FT in such cases are not continuous over ω
- Also note that this sequence represents the impulse response of an Accumulator. Is it stable?

Table 2.3 - Transform Pair 6 -Homework

- **Sequence**: $(n+1)a^n u[n], \quad (|a| < 1)$
- **Fourier Transform**: $\dfrac{1}{(1 - ae^{-j\omega})^2}$

$$\text{FT}\{(n+1)a^n u[n]\} = \text{FT}\{(na^n u[n] + a^n u[n]\}$$

$$\text{FT}\{(n+1)a^n u[n]\} = j\frac{d}{d\omega}\{\text{FT}\{a^n u[n]\}\} \quad (\text{Table 2.2 Theorem 5})$$
$$+ \text{FT}\{a^n u[n]\}$$

$$\text{FT}\{(n+1)a^n u[n]\} = j\frac{d}{d\omega}\left\{\frac{1}{1 - ae^{-j\omega}}\right\} + \frac{1}{1 - ae^{-j\omega}} \quad (\text{Table 2.3 Pair 3})$$

(Apply Quotient rule of differentiation + some algebra)

$$\Rightarrow \text{FT}\{(n+1)a^n u[n]\} = \frac{1}{(1 - ae^{-j\omega})^2}$$

Table 2.3 - Transform Pair 7 - Homework

- **Sequence**: $\frac{r^n \sin \omega_p(n+1)}{\sin \omega_p} u[n], \ (|r| < 1)$
- **Fourier Transform**: $\frac{1}{1 - 2r\cos(\omega_p) e^{-j\omega} + r^2 e^{-j2\omega}}$

Hint: Use

$$\sin \omega_p(n+1) = \frac{e^{j\omega_p(n+1)} - e^{-j\omega_p(n+1)}}{2j}$$

and

$$\cos(\omega_p) = \frac{e^{j\omega_p} + e^{-j\omega_p}}{2}, \quad \sin(\omega_p) = \frac{e^{j\omega_p} - e^{-j\omega_p}}{2j}$$

Plus also use Table 2.2 Prop. 3 and Table 2.3 Prop. 4 and some algebraic manipulations to arrive at the answer.

Example: Determining a Fourier transform using Tables 2.2 and 2.3

Find the Fourier transf sequence $x[n] = a^n u[n-5]$, $(|a| < 1)$

Example: Determining a Fourier transform using Tables 2.2 and 2.3

Find the Fourier transform of the sequence $x[n] = a^n u[n-5]$, $(|a| < 1)$

$$x_1[n] = a^n u[n] \xrightarrow{\text{FT}} \frac{1}{1 - ae^{-j\omega}} \quad \text{(Table 2.3 Transform Pair 4)}$$

$$x_1[n-5] = a^{n-5} u[n-5] \xrightarrow{\text{FT}} \frac{e^{-j5\omega}}{1 - ae^{-j\omega}} \quad \text{(Table 2.2 Theorem 2)}$$

$$x[n] = a^5 x_1[n-5] = a^n u[n-5] \xrightarrow{\text{FT}} \frac{a^5 e^{-j5\omega}}{1 - ae^{-j\omega}} \quad \text{(Table 2.2 Theorem 1)}$$

Example: Determining an Inverse Fourier transform using Tables 2.2 and 2.3

Find the Inverse Fourie... of $X(e^{j\omega}) = \frac{1}{(1-ae^{-j\omega})(1-be^{-j\omega})}$

Example: Determining an Inverse Fourier transform using Tables 2.2 and 2.3

Find the Inverse Fourier transform of $X(e^{j\omega}) = \frac{1}{(1-ae^{-j\omega})(1-be^{-j\omega})}$

Using partial fraction expansion,

$$X(e^{j\omega}) = \frac{1}{(1-ae^{-j\omega})(1-be^{-j\omega})} = \frac{a/(a-b)}{(1-ae^{-j\omega})} - \frac{b/(a-b)}{(1-be^{-j\omega})}$$

From (Table 2.2 Theorem 1) and (Table 2.3 Transform Pair 4), it follows that

$$x[n] = \left(\frac{a}{a-b}\right) a^n u[n] - \left(\frac{b}{a-b}\right) b^n u[n]$$

Course Outline

- Discrete-time Signals and Systems
- The Z-Transform
- Transform Analysis of Linear Time-Invariant (LTI) Systems
- Sampling of Continuous-Time Signals
- The Discrete Fourier Transform (DFT)
- Computation of DFT using FFT algorithms
- Structure of Discrete-time Systems
- Filter Design Techniques
- Fourier Analysis of Signals using DFT

The z-Transform: Background and Motivation

- The z-transform i[illegible]or sequences (discrete-time signals).
- The z-transform o[illegible]time signals is a counterpart of the Laplace transform for continuous time signals.

- It is a generalization of the Fourier transform, and applies to signals for which DTFT doesn't exist.

- Z-transform encompasses a broader class of signals, thus the z-transform is more inclusive (recall: the FT does not converge for all sequences)

- It gives a broader understanding of a system in terms of causality and stability, therefore is a very useful analysis tool for systems.

z-Transform: Definition

The Fourier transform of a sequence $x[n]$ is:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

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The Fourier transform of a sequence $x[n]$ is:

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The z-transform of $x[n]$ is:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

z-Transform: Definition

The z-transform of

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- z is considered to be a complex variable
- Notation $\mathscr{Z}\{x[n]\} = X(z)$
- How can you evaluate the FT from the z-transform?

z-Transform: Definition

The z-transform of

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- z is considered to be a complex variable

- Notation $\mathscr{Z}\{x[n]\} = X(z)$

- How can you evaluate the FT from the z-transform?
  - Evaluate the z-transform at $z = e^{j\omega}$, to get the FT.

z-Transform: Example

Evaluate the z-transf … $n]$:

Recall the Fourier transform of $\delta[n]$ is: $\mathrm{FT}\{\delta[n]\} = 1$.

Thus,

$$\mathscr{Z}\{\delta[n]\} = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n}$$

$$= z^{-0}$$

$$= 1$$

The z-Transform

The z-transform of a [illegible] $x[n]$ is:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

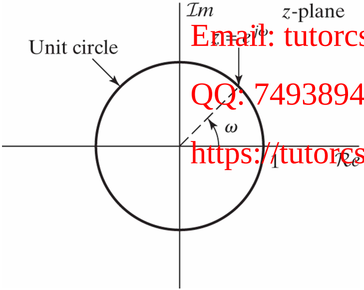
We can express the complex variable z in the polar form, as follows:

$$z = re^{j\omega}$$

where r is the radius, and ω is the phase.

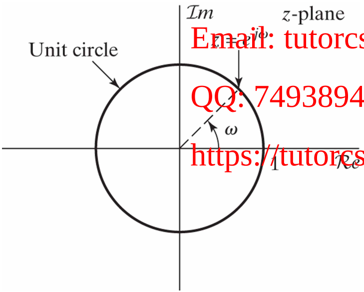
Complex z-plane

- The z-transform function of a co variable, it is co to describe and interpret it using the complex z-plane.



Complex z-plane

- The z-transform is a function of a complex variable, it is convenient to describe and interpret it using the complex z-plane.
- $|z| = 1$ corresponding to a circle of unit radius.



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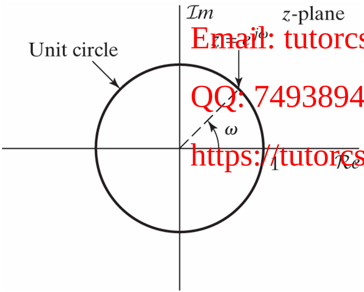

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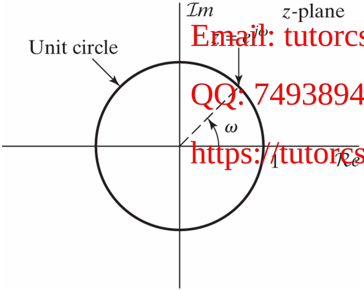
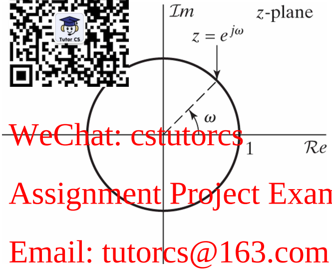


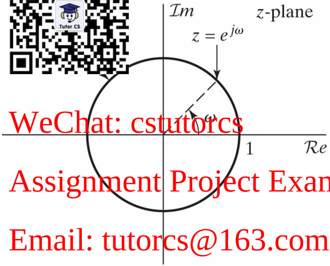
- $|z| = 1$ corresponding to a circle of unit radius.
- This contour is termed as the unit circle and corresponds to a set of points $z = e^{j\omega}$ for $0 \leqslant \omega \leqslant 2\pi$.
- The z-transform evaluated on the unit circle corresponds to the Fourier transform.

Complex z-plane



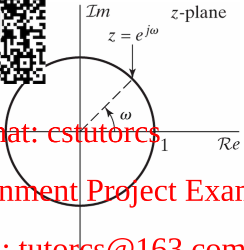
- If we evaluate $X(z)$ at points on the unit circle in the z-plane, beginning at $z = 1$ (i.e., $\omega = 0$) through $z = j$ (i.e., $\omega = \pi/2$) to $z = -1$ (i.e., $\omega = \pi$), we obtain the Fourier transform for $0 \leqslant \omega \leqslant \pi$.

Complex z-plane



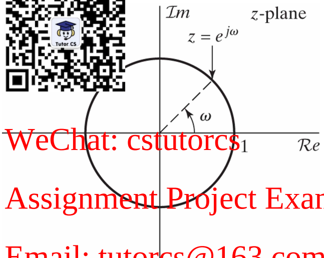
- Continuing around the unit circle would correspond to examining the FT from $\omega = \pi$ to $\omega = 2\pi$ (or $\omega = -\pi$ to $\omega = 0$).

Complex z-plane



- Previously, the FT was displayed on a linear frequency axis. Interpretation of FT on the unit circle corresponds to wrapping the linear frequency axis around the unit circle with $\omega = 0$ at $z = 1$ and $\omega = \pi$ at $z = -1$.

Complex z-plane



- Note that the inherent periodicity in frequency of the FT in the z-plane corresponds to traversing the unit circle once and returning to exactly the same place.
- $\therefore z = re^{j\omega} = re^{j(\omega+2\pi k)}$

Region of Convergence (ROC)

- Recall that FT c[illegible]onverge for all sequences. A **sufficient** condi[illegible]nvergence is

$$|X(e^{j\omega})| = |\sum_{n=-\infty} x[n]e^{-j\omega n}|$$

$$\leqslant \sum_{n=-\infty}^{\infty} |x[n]||e^{-j\omega n}| = \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

Region of Convergence (ROC)

- Recall that FT does not converge for all sequences. A **sufficient** condition for convergence is

$$|X(e^{j\omega})| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right|$$

$$\leqslant \sum_{n=-\infty}^{\infty} |x[n]||e^{-j\omega n}| = \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- Similarly, the z-transform does not converge for all sequences. For a given sequence, the set of z values for which the z-transform converges is called the *Region of Convergence* (ROC) of the z-transform.

Region of Convergence (ROC)

- For convergence [illegible] transform, when $z = re^{j\omega}$, we need the power series [illegible]solutely summable:

$$|X(z)| = |X(re^{j\omega})| = \left| \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} \right|$$

$$\leqslant \sum_{n=-\infty}^{\infty} |x[n]||r^{-n}||e^{-j\omega n}|$$

$$= \sum_{n=-\infty}^{\infty} |x[n]||r^{-n}| < \infty$$

- Thus, r determines what sequences will converge of the z-transform.
- Compare this with the condition on convergence of FT.

Region of Convergence (ROC)

- Condition on co[illegible] of the above series is $|X(z)| < \infty$, hence

$$|X(z)| \leqslant \sum_{n=-\infty}^{\infty} |x[n]||z^{-n}| < \infty$$

- ROC of the z-transform consists of all values of z such that the above inequality holds.

Region of Convergence (ROC)

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$$|X(z)| \leqslant \sum_{n=-\infty}^{\infty} |x[n]||z^{-n}| < \infty$$

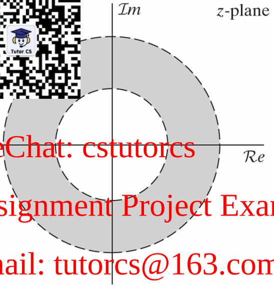
- ROC of the z-transform consists of all values of z such that the above inequality holds.

- If for some value of z, say $z = z_1$, is in the ROC ($z_1 \in$ ROC), then all values of z on the circle defined by $|z| = |z_1|$ will also be in the ROC.

Region of Convergence (ROC)

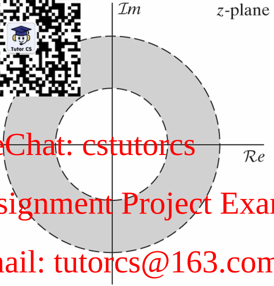
- The ROC will consist of a ring in the z-plane centered about the origin. Its outer boundary will be a circle (the ROC may extend outward to infinity), and its inner boundary will be a circle (it may extend inward to include the origin).

Region of Convergence (ROC)



- Important: If the ROC includes $|z| = 1$, the unit circle, then this implies convergence of z-transform on the unit circle, or equivalently, the FT of the sequence converges.
- If the FT exists $\Rightarrow$ Unit circle belongs to the ROC.

Region of Convergence (ROC)



- Important: Conversely, if the ROC does not include the unit circle, the FT does not converge.
- If the FT does not exists $\Rightarrow$ ROC does not include the unit circle.

Poles and Zeros of z-Transform

- The z-transform [illegible] useful when it can be expressed in closed form (i.e. [illegible] can be expressed as a simple mathematical expression).

Poles and Zeros of z-Transform

- The z-transform [illegible] useful when it can be expressed in closed form (i.e. [illegible] can be expressed as a simple mathematical expression).

- Let the z-transform be represented inside the ROC as a rational function, as follows:

$$X(z) = \frac{P(z)}{Q(z)}$$

where $P(z)$ and $Q(z)$ are polynomials in z.

Poles and Zeros of z-Transform

- Let the z-transf[illegible]presented inside ROC as

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Poles and Zeros of z-Transform

- Let the z-transform be represented inside ROC as

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where $P(z)$ and $Q(z)$ are polynomials in z.

- The zeros are the roots of the numerator polynomial (i.e., $P(z) = 0$) and the poles are the roots of the denominator polynomial (i.e., $Q(z) = 0$).

Homework

1. Problems Related to this lecture: 2.6, 2.9, 2.11, 2.17, 2.31, 2.34, 2.36, 2.38, 2.41, 2.46, 2.47