Lecture – 9

Digital Signal Processing

ENGN 4537/ 6537

Semester 2, 2023

Announcements

- Teaching Feedback Survey-1 closes on Friday, 25th August.
- Mid-term Examination (worth 20%)
  - Online Wattle Examination on 18/09/2023 (Monday, Week-7) 10am to 11:30pm
  - Reading time: 15 min, Writing time: 60 min, Uploading answers to Wattle: 15 min
  - Assessable contents: Lectures 1 to 10, Tutorials 1 to 5
  - Permitted materials: Lecture/Tutorial notes, Calculator (non-programmable)
  - Parallel zoom session: Attendance not compulsory, join for asking questions.

Transform Analysis of Linear Time-Invariant (LTI) Systems

We will study how to analyze a system given the Z-Transform and/or Fourier Transform

- Analysis of LTI systems described by Difference Equations
- Analysis of LTI systems described by Fourier Transform
- All Pass and Minimum Phase Systems

Relationship between Magnitude and Phase

Question : When we [illegible]v the magnitude response of an LTI system, can we infer [illegible] response?

Relationship between Magnitude and Phase

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- In general for LTI systems, no relationship exists between phase and magnitude.

- However, for systems described by linear constant coefficient difference equations (rational system functions), there is some constraint between magnitude and phase

Relationship between Magnitude and Phase

- For systems described by linear constant coefficient difference equations (rational transfer functions):
  - Knowing the magnitude and number of poles and zeros, a finite number possibilities exists for the phase responses.
  - Knowing the phase and number of poles and zeros, a finite number possibilities exists for the magnitude within a scale factor. (*Not the topic of this lecture*)

Relationship between Magnitude and Phase

- For systems des[illegible] linear constant coefficient difference equations (ratio[illegible] functions):
  - Knowing the magnitude and number of poles and zeros, a finite number possibilities exists for the phase responses.
  - Knowing the phase and number of poles and zeros, a finite number possibilities exists for the magnitude within a scale factor. (*Not the topic of this lecture*)
- For the special case of minimum phase systems the frequency-response magnitude specifies the phase uniquely.

Relationship between Magnitude and Phase

A general rational system can be represented by its Fourier Transform as

$$H(e^{j\omega}) = \frac{b_0 \prod_{k=1}^{M}(1 - c_k e^{-j\omega})}{a_0 \prod_{k=1}^{N}(1 - d_k e^{-j\omega})}$$

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Given $|H(e^{j\omega})|^2$ replace z for $e^{j\omega}$ to get $C(z)$:

$$C(z) = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})(1 - c_k^* z)}{\prod_{k=1}^{N}(1 - d_k z^{-1})(1 - d_k^* z)}$$

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Note that

$$C(z) = H(z)H^*(1/z^*)|_{z=e^{j\omega}}$$

Relationship between Magnitude and Phase

$$C(z) = H(z)H^*(1/z^*) = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})(1 - c_k^* z)}{\prod_{k=1}^{N}(1 - d_k z^{-1})(1 - d_k^* z)}$$

What can C(z) tell us?

Relationship between Magnitude and Phase

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What can C(z) tell us?

- For each pole d_k of $H(z)$, there is a conjugate reciprocal pair of poles at d_k and $(d_k^*)^{-1}$ for $C(z)$.

- Similarly, for each zero c_k of $H(z)$, there is a conjugate reciprocal pair of zeros c_k and $(c_k^*)^{-1}$ for $C(z)$

Relationship between Magnitude and Phase

$$C(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})(1 - c_k^* z)}{\prod_{k=1}^{N}(1 - d_k z^{-1})(1 - d_k^* z)}$$

- If one element of each pole pair or zero pair is inside the unit circle, the the other (conjugate reciprocal) will be outside the unit circle.

Relationship between Magnitude and Phase

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- If one element of each pole pair or zero pair is inside the unit circle, the the other (conjugate reciprocal) will be outside the unit circle.

- Assume $H(z)$ a causal, stable system, all its poles must lie insider the unit circle (therefore, poles are identified).

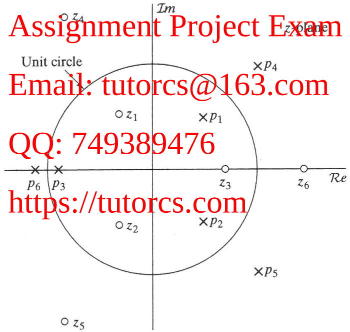
$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

Relationship between Magnitude and Phase

Example 5.10

Given the C(z) zero-[illegible] below, find the system function $H(z)$ that corresponds to a[illegible]able with real-valued time domain impulse response with 3 zeros.

Note: For real-valued signals, pole & zeros are in complex conjugates.



Relationship between Magnitude and Phase

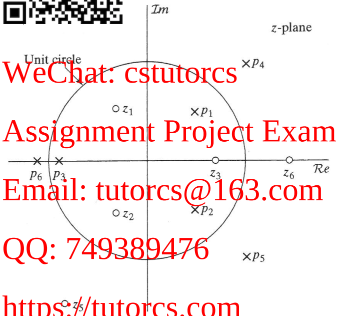
Example 5.10

Pole pairs in $H(z)$ and $H^*(1/z^*)$: (p_1, p_4), (p_2, p_5), (p_3, p_6)

Zero pairs in $H(z)$ and $H^*(1/z^*)$: (z_1, z_4), (z_2, z_5), (z_3, z_6)

Relationship between Magnitude and Phase

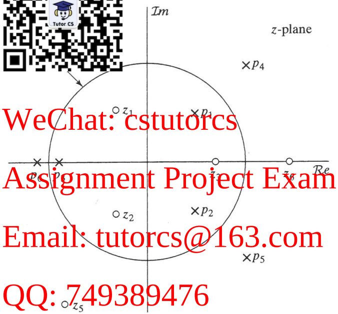
Example 5.10



Choose poles inside the unit circle: p_1, p_2, p_3 for stability.

Relationship between Magnitude and Phase

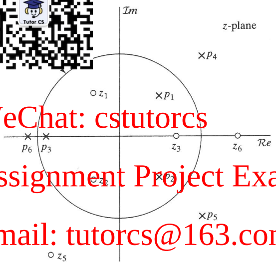
Example 5.10



Zero options: **Assuming the system has three zeros and its impulse response is real valued, we can choose 3 out of these 4 options.** z_3, or z_6 **and** (z_1, z_2), or (z_4, z_5). See next page for the option details.

Relationship between Magnitude and Phase

Example 5.10



Zero options: z_3, or z_6 **and** (z_1, z_2), or (z_4, z_5).

Option 1: z_3 and (z_1, z_2).
Option 2: z_3 and (z_4, z_5).
Option 3: z_6 and (z_1, z_2).
Option 4: z_6 and (z_4, z_5).

Relationship between Magnitude and Phase

Example 5.10

By looking at the Z transform (or $C(z)$) of the Fourier transform's Squared magnitude (or $|H(e^{j\omega})|^2$), we can predict a limited number of possibilities for H(z).

Thus, we can conclude that given the magnitude of a rational system, we can predict a certain number of possibilities for the system's phase.

Relationship between Magnitude and Phase

Example 5.9

Different systems can have the same C(z)

$$H_1(z) = \frac{(1 - z^{-1})(1 + 0.5z^{-1})}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})}$$

$$H_2(z) = \frac{(1 - z^{-1})(1 + 2z^{-1})}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})}$$

Relationship between Magnitude and Phase

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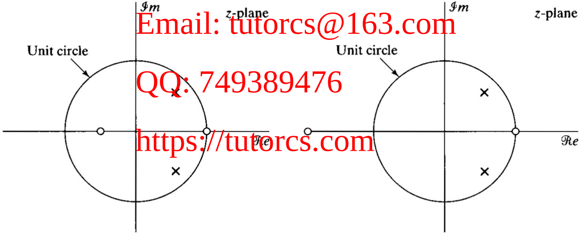

(a) $H_1(z)$ (b) $H_2(z)$

Relationship between Magnitude and Phase

Example 5.9

$$C_1(z) = \frac{(1 - z^{-1})2(1 + 0.5z^{-1})(1 - z)2(1 + 0.5z)}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z)(1 - 0.8e^{j\pi/4}z)}$$

$$C_2(z) = \frac{(1 - z^{-1})(1 + 2z^{-1})(1 - z)(1 + 2z)}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z)(1 - 0.8e^{j\pi/4}z)}$$

Relationship between Magnitude and Phase

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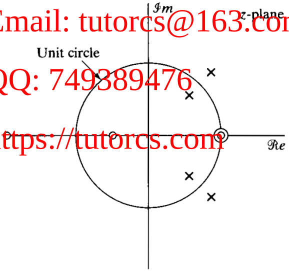
Given $4(1+0.5z^{-1})(1+0.5z) = (1+2z^{-1})(1+2z) \Rightarrow C_1(z) = C_2(z)$



All-Pass Systems and Minimum Phase Systems - Context

All-Pass Systems

A system with frequen[illegible]nse magnitude that is constant, passing all of the fre[illegible]mponents of its input with constant gain or attenuation, [illegible]n all-pass system.

All-Pass Systems

A system with frequency-response magnitude that is constant, passing all of the frequency components of its input with constant gain or attenuation, is referred to as an all-pass system.

- The simplest all-pass system is a stable system with system function of the form $|a| < 1$

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

- Frequency response:

$$H_{ap}(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} = e^{-j\omega}\frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}}$$

- What are the poles and zeros?

All-Pass Systems

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- What are the poles and zeros?
  Ans. Pole at $z = a$ and zero at $z = \frac{1}{a^*}$

All-Pass Systems

- The simplest all-pass system is a stable system with system function of the form

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

- **Important:** If a is a pole of $H_{ap}(z)$, then $1/a^*$ is a zero of $H_{ap}(z)$ and vice versa.

- If $a = re^{j\theta}$, where are the poles and zeros on the z-plane?

All-Pass Systems

- The simplest all-pass system is a stable system with frequency response:

$$H_{ap}(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} = e^{-j\omega}\frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}}$$

- How can you tell if $H(e^{j\omega})$ is an all pass system?

All-Pass Systems

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$$H_{ap}(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} = e^{-j\omega}\frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}}$$

- How can you tell if $H(e^{j\omega})$ is an all pass system?

- $|e^{-j\omega}| = 1$ and numerator and denominator are complex conjugates of each other so have the same magnitude. $\Rightarrow |H_{ap}(e^{j\omega})| = 1$.

- The magnitude of $H_{ap}(e^{j\omega})$ in dB scale is equal to 0 dB.

All-Pass Systems

- The general form of the frequency response of **real-valued (in time domain)** all-pass systems

$$H_{ap}(e^{j\omega}) = A \prod_{k=1}^{M_r} \frac{e^{-j\omega} - d_k}{1 - d_k e^{-j\omega}} \prod_{k=1}^{M_c} \frac{(e^{-j\omega} - e_k^*)(e^{-j\omega} - e_k)}{(1 - e_k e^{-j\omega})(1 - e_k^* e^{-j\omega})},$$

All-Pass Systems

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- where A is a positive constant, d_k are real poles and e_k and e_k^* are complex poles (paired in complex conjugates).
- Hence, $1/d_k$ are real zeros and $1/e_k^*$ and $1/e_k$ are complex zeros (paired in complex conjugates)

All-Pass Systems

$$H_{ap}(e^{j\omega}) = A\prod_{k=1}^{M_r}\frac{e^{-j\omega}-d_k}{1-d_ke^{-j\omega}}\prod_{k=1}^{M_c}\frac{(e^{-j\omega}-e_k^*)(e^{-j\omega}-e_k)}{(1-e_ke^{-j\omega})(1-e_k^*e^{-j\omega})},$$

- Example: $M_r = 2$ and $M_c = 1$

All-Pass Systems

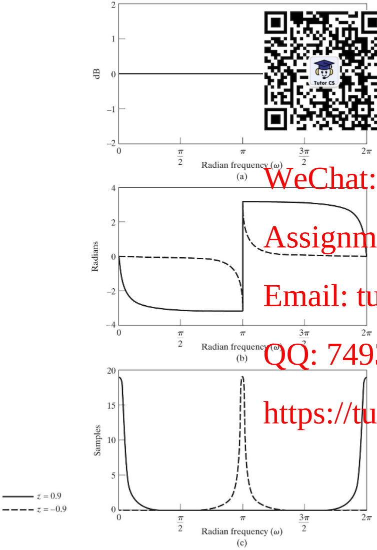
$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})},$$

- Pole is paired with conjugate reciprocal zero i.e. a pole at $e_k \Rightarrow$ a zero at $(e_k^*)^{-1}$
- If all the poles are inside the unit circle (causal and stable), the zeros will be outside the unit circle (conjugate reciprocal positions).

All-Pass Systems

- In general, the group delay of all casual stable rational all-pass systems is non-negative.
- In general, the **unwrapped phase** of all casual stable rational all-pass systems is non-positive for $0 \leqslant \omega \leqslant \pi$.

All-Pass Systems



Example 5.11:

A stable causal all-pass system ($|a| = r \leqslant 1$):

(a) Log magnitude: 0 dB

(b) Principal Phase: non-positive for $0 \leqslant \omega \leqslant \pi$

(c) Group delay: non-negative

Homework

- Read and understand Sections 5.4 and 5.5 of the textbook.