Lecture - 5

Digital Signal Processing

ENGN 4537/ 6537

Semester 2, 2023

Announcements

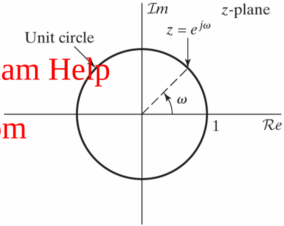
- Quiz-1 will be conducted this week during the tutorial sessions
- Mid-term exam date: 18th September 2023 (Monday, Week-7)

z-Transform: Definition (Revisit)

The z-transform of $x[n]$:

$$\mathscr{Z}\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$z = re^{j\omega}$ is a continuous complex variable



- Sometimes, in order to get the Fourier transform $X(e^{j\omega})$, it is easier to get $X(z)$ in general first and then evaluate it at $z = e^{j\omega}$
  - Restrict z to have unity magnitude; i.e., $|z| = 1$.

Region of Convergence (ROC) (Revisit)

- Condition on convergence of z-transform is $|X(z)| < \infty$, hence

$$\sum_{n=-\infty}^{\infty} |x[n]||z^{-n}| < \infty$$

  ROC consists of all values of z such that the above inequality holds.

- If $z_1 \in$ ROC, then all values of z on the circle defined by $|z| = |z_1|$ will also be in the ROC.

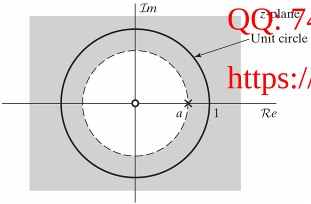
  - The ROC is circular and defined over all phases in the circular convergence region.

- Unit circle ($|z| = 1$) $\in$ ROC $\Leftrightarrow$ FT of the sequence converges.

Lecture 4 (Recap)

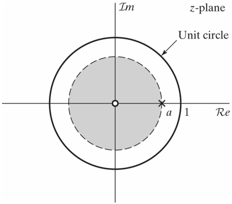
Example 3.1: Right-sided Exponential

- $x[n] = a^n u[n]$
- $X(z) = \frac{z}{z-a}$ for $|$
- Zero: $z = 0$, Pole: $z = a$
- FT converges if $|a| < 1$
- ROC and pole-zero plot for $|a| < 1$:



Example 3.2: Left-sided Exponential sequences

- $x[n] = -a^n u[-n-1]$
- $X(z) = \frac{z}{z-a}$ for $|z| < |a|$
- Zero: $z = 0$, Pole: $z = a$
- FT converges if $|a| > 1$
- ROC and pole-zero plot for $|a| < 1$:

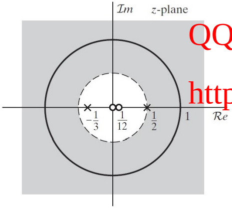


Lecture 4 (Recap)

Example 3.3-3.4: Sum of Two Exponential S

- $x[n] = \left(\frac{1}{2}\right)^n u[n] + ($
- $X(z) = \frac{2z(z-\frac{1}{12})}{(z-\frac{1}{2})(z+\frac{1}{3})}$
- ROC: $|z| > \frac{1}{2}$
- Zeros: $z = 0, \frac{1}{12}$

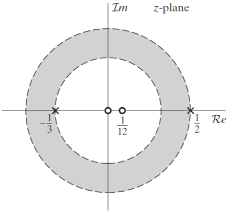
  Poles: $z = \frac{-1}{3}, \frac{1}{2}$
- FT exists.



Example 3.5: Two-Sided Exponential Sequence

- $x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$
- $X(z) = \frac{2z(z-\frac{1}{12})}{(z-\frac{1}{2})(z+\frac{1}{3})}$
- ROC: $\frac{1}{3} < |z| < \frac{1}{2}$
- Zeros: $z = 0, \frac{1}{12}$

  Poles: $z = \frac{-1}{3}, \frac{1}{2}$
- FT does not exist.



Example 3.6: Finite Length Truncated Exponential Series

$x[n$ ... $0 \leqslant n \leqslant N-1$

Otherwise.

Example 3.6: Finite Length Truncated Exponential Series

$$x[n] \quad 0 \leqslant n \leqslant N-1 \quad \text{Otherwise.}$$

Take the z-transform of $x[n]$, as follows:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}}$$

$$= \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}$$

Example 3.6: Finite Length Truncated Exponential Series: Pole Zero Plot

$$\frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}$$

- How many zeros and poles are there? (assume a is real)

Example 3.6: Finite Length Truncated Exponential Series:

Pole Zero Plot

$$\frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}$$

- How many zeros and poles are there? (assume a is real)

- The roots of the numerator polynomial are given by $z^N - a^N = 0$, i.e.,

$$z^N = a^N$$

i.e., an N-th order polynomial complex equation (with N roots).

- Recall: $1 = e^{j2\pi k}$ for an integer $k \in \mathbb{Z}$.

$$z^N = a^N = a^N.1 = a^N e^{j2\pi k}$$

Example 3.6: Finite Length Truncated Exponential Series: Pole Zero Plot

- Hence, the possible solutions of z_k for the equation are

$$z_k = ae^{j(2\pi k/N)}, \quad k = 0, 1, \ldots, N-1.$$

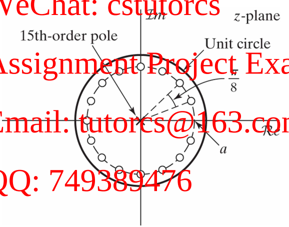


Figure: In the picture, $N = 16$, a is real and $0 < |a| < 1$.

- Thus, depending on N you will get different number of zeros.

Example 3.6: Finite Length Truncated Exponential Series: Pole Zero Plot

$$\frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}$$

- Zeros: roots of the numerator polynomial: $z_k = ae^{j(2\pi k/N)}$ for $k = 0, 1, \ldots, N-1$.
- Poles: roots of the denominator polynomial: $z = a$ and $N-1$ poles at $z = 0$.

Example 3.6: Finite Length Truncated Exponential Series: Pole Zero Plot

$$\frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}$$

- Zeros: roots of the numerator polynomial: $z_k = ae^{j(2\pi k/N)}$ for $k = 0, 1, \ldots, N-1$.
- Poles: roots of the denominator polynomial: $z = a$ and $N - 1$ poles at $z = 0$.
- The zero at $z = a$ (when $k = 0$) cancels the pole at $z = a$. Hence, there will be $N - 1$ zeros at

$$z_k = ae^{j(2\pi k/N)}, \quad k = 1, \ldots, N-1.$$

and $N - 1$ poles at $z = 0$.

Example 3.6: Finite Length Truncated Exponential Series

ROC

- What is the RO $= \sum_{n=0}^{N-1} (az^{-1})^n$?

Example 3.6: Finite Length Truncated Exponential Series

ROC

- What is the RO[illegible] $= \sum_{n=0}^{N-1}(az^{-1})^n$?

- ROC is determined by the set of values of z for which $|X(z)| < \infty$

$$\sum_{n=0}^{N-1} |az^{-1}|^n < \infty.$$

Since there is only a finite number of nonzero terms, the sum will be finite as long as (az^{-1}) is finite, i.e., $|a| < \infty$ and $z \neq 0$.

- Thus, if a is finite, then the ROC includes the entire z-plane except $z = 0$.

Example 3.7: Non overlapping ROC

$$x[n] = \ldots] - \left(-\frac{1}{3}\right)^n u[-n-1]$$

Note that the first term is a right-sided sequence and the second term is a left-sided sequence.

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} \quad \text{(Table 3.1 Prop. 5 and Prop. 6)}$$

What is the ROC of this sequence?

Example 3.7: Non overlapping ROC

$$x[n] = \text{[illegible]}] - \left(-\frac{1}{3}\right)^n u[-n-1]$$

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What is the ROC of this sequence?

The first term converges for $|z| > 1/2$ and the second term converges for $|z| < 1/3$.

Example 3.7: Non overlapping ROC

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$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} \quad \text{(Table 3.1 Prop. 5 and Prop. 6)}$$

What is the ROC of this sequence?

The first term converges for $|z| > 1/2$ and the second term converges for $|z| < 1/3$. Since there is a no-overlap, there is no ROC for $x[n]$ and hence $x[n]$ has no z-Transform.

Table of Z-Transform pairs

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	[illegible]	All z
2. $u[n]$	[illegible]	$\|z\| > 1$
3. $-u[-n-1]$	[illegible]	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
9. $\cos(\omega_0 n)u[n]$	$\dfrac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
10. $\sin(\omega_0 n)u[n]$	$\dfrac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\dfrac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\dfrac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$

Table of Z-Transform pairs

Important!

You need to be able to derive everything in Table 3.1 from first principles and determine the ROC for each sequence.

Properties of the ROC for the z-Transform

Property 1: The RC her be of these three forms

$$0 \leqslant r_R < |z|$$

$$|z| < r_L \leqslant \infty$$

$$0 \leqslant r_R < |z| < r_L \leqslant \infty$$

Properties of the ROC for the z-Transform

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Property 2: The Fourier Transform of $x[n]$ converges if and only if the ROC of the z-transform of $x[n]$ includes the unit circle $\Rightarrow |z| = 1$

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Property 2: The Fourier Transform of $x[n]$ converges if and only if the ROC of the z-transform of $x[n]$ includes the unit circle $\Rightarrow |z| = 1$

Property 3: The ROC cannot contain any poles.

Properties of the ROC for the z-Transform

Property 4: If $x[n]$ … -*duration sequence*, i.e., a sequence that is zero except in a finite interval, then the ROC is the entire z-plane except possibly at $z = 0$ or $z = \infty$.

Properties of the ROC for the z-Transform

Property 4: If $x[n]$ [illegible]-*duration sequence*, i.e., a sequence that is zero except in a finite interval, then the ROC is the entire z-plane except possibly at $z = 0$ or $z = \infty$.

Property 5: If $x[n]$ is a *right-sided sequence*, the ROC extends outwards from the outermost finite pole in $X(z)$ to (and possibly including) $z = \infty$.

Properties of the ROC for the z-Transform

Property 6: If $x[n]$ … *ded sequence*, the ROC extends inwards … *innermost* non-zero pole in $X(z)$ to (and p… …luding) $z = 0$.

Properties of the ROC for the z-Transform

Property 6: If $x[n]$ [illegible]*ded sequence*, the ROC extends inwards [illegible] *innermost* non-zero pole in $X(z)$ to (and p[illegible]luding) $z = 0$.

Property 7: A *two-sided sequence* is an infinite duration sequence that is neither right-sided nor left-sided. If $x[n]$ is a *two-sided sequence*, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and not containing any poles.

Properties of the ROC for the z-Transform

Property 6: If $x[n]$ ... *ded sequence*, the ROC extends inwards ... *innermost* non-zero pole in $X(z)$ to (and p...luding) $z = 0$.

Property 7: A *two-sided sequence* is an infinite duration sequence that is neither right-sided nor left-sided. If $x[n]$ is a *two-sided sequence*, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and not containing any poles.

Property 8: The ROC must be a connected region.

Read and understand Section 3.2 of the textbook.

Non-uniqueness of algebraic expression for z-transform

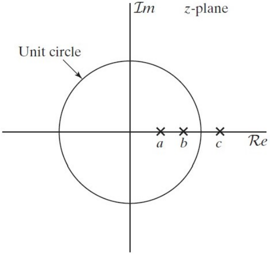
- We have noticed that the algebraic expression or pole-zero pattern does not uniquely specify the z-transform of a sequence.

- ROC must be specified with the above.

- The discussed properties of the ROC limit the possible ROCs that can be associated with a given pole-zero pattern
  - Next slide shows four different possible ROCs for the same pole-zero pattern (algebraic expression $X(z)$).

Non-uniqueness of algebraic expression for z-transform

Typical Exam Question: What are the possible ROCs for the same pole-zero plot of the z-transform.



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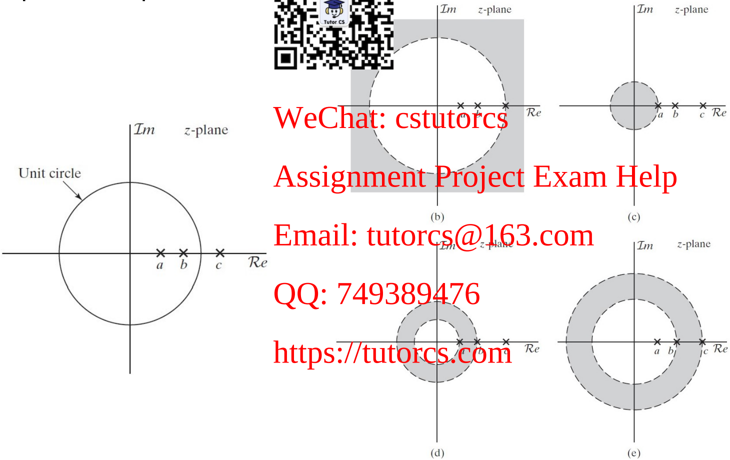


Non-uniqueness of algebraic expression for z-transform (Continued)

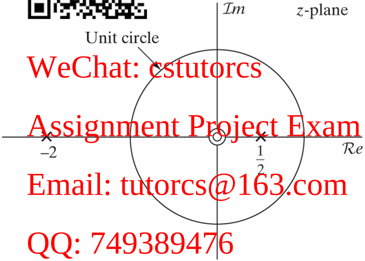
Typical Exam Questi are the corresponding time-domain sequences.

- Ans. b) $x[n] = a^n u[n] + b^n u[n] + c^n u[n]$, where $c^n u[n]$ has the largest pole and $c > b > a$.
- Ans. c) $x[n] = -a^n u[-n-1] - b^n u[-n-1] - c^n u[-n-1]$, where $-a^n u[-n-1]$ has the smallest pole and $c > b > a$.
- Ans. d) $x[n] = a^n u[n] - b^n u[-n-1] - c^n u[-n-1]$, where the ROC for each pole is $|z| > |a|$, $|z| < |b|$, and $|z| < |c|$, plus $c > b$.
- Ans. e) $x[n] = a^n u[n] + b^n u[n] - c^n u[-n-1]$, where the ROC for each pole is $|z| > |a|$, $|z| > |b|$, and $|z| < |c|$, plus $b > a$.

Stability and Causality of a system through ROC

The z-transform of an impulse response $h[n]$ of an LTI system is called System function of the LTI system.

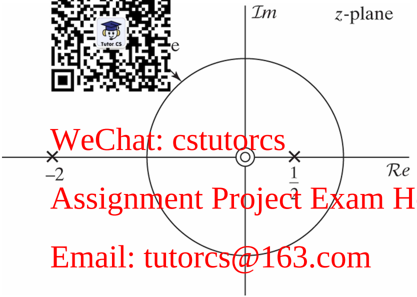
Example 3.8:



- What are the poles and zeros of this system?
- What are the possible options for the ROC?
- Can this system be causal and stable at the same time?

Stability and Causality of a system through ROC

Example 3.8:



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Stability and Causality of a system through ROC

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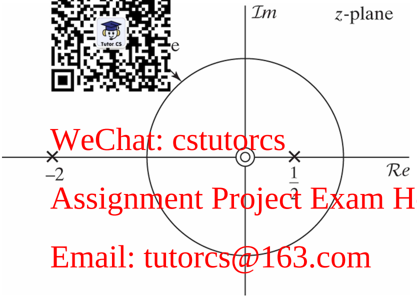

- What are the poles and zeros of this system? There is a double zero at $z = 0$, and poles at $z = \frac{1}{2}$ and $z = -2$.

Stability and Causality of a system through ROC

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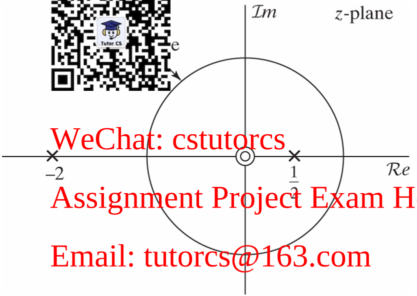

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- What are the possible options for the ROC?

Stability and Causality of a system through ROC

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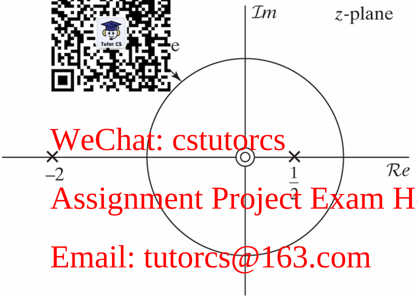

- What are the poles and zeros of this system? There is a double zero at $z = 0$, and poles at $z = \frac{1}{2}$ and $z = -2$.
- What are the possible options for the ROC? There are 3 possible ROCs. (i) $|z| < \frac{1}{2}$; (ii) $\frac{1}{2} < |z| < 2$; (iii) $|z| > 2$.

Stability and Causality of a system through ROC

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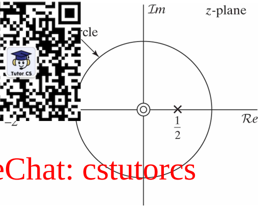

- If the system is stable ($h[n]$ is summable and therefore has a FT) then the ROC should contain the unit circle. Thus, ROC might be $\frac{1}{2} < |z| < 2$.

Stability and Causality of a system through ROC

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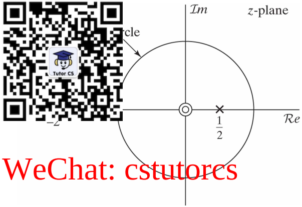

- If the system is stable ($h[n]$ is summable and therefore has a FT) then the ROC should contain the unit circle. Thus, ROC might be $\frac{1}{2} < |z| < 2$.
- But, the ROC $\frac{1}{2} < |z| < 2$ implies that the sequence is two-sided. (A two-sided sequence cannot be causal).

Stability and Causality of a system through ROC

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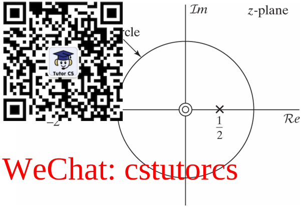

- If the system is stable ($h[n]$ is summable and therefore has a FT) then the ROC should contain the unit circle. Thus, ROC might be $\frac{1}{2} < |z| < 2$.
- But, the ROC $\frac{1}{2} < |z| < 2$ implies that the sequence is two-sided. (A two-sided sequence cannot be causal).
- If we say the system is causal, then the ROC is $|z| > 2$.

Thus, this system cannot be both causal and stable.

Relating Stability of $H(z)$ back to $H(e^{j\omega})$

- The z-transform [illegible]s known as the System function $H(z)$.
- The z-transform [illegible]ralization of the DTFT and allows analysis of unstable systems.
- Thus, the System function $H(z)$ is a generalization of the frequency response $H(e^{j\omega})$.

Stability

An LTI system is STABLE if and only if the impulse response is absolutely summable, i.e, if $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$

- The frequency response $H(e^{j\omega})$ exists if the ROC of the System function $H(z)$ includes the unit circle.

Stability and Causality Summary

	$\lvert a\rvert < 1$	$\lvert a\rvert > 1$
System response: $h_1[n] = a^n u[n]$ ROC : $\lvert z\rvert > \lvert a\rvert$	Causal Stable Unit circle inside the ROC	Causal Not Stable Unit circle outside the ROC
System response: $h_2[n] = -a^n u[-n-1]$ ROC : $\lvert z\rvert < \lvert a\rvert$	Not Causal Not Stable Unit circle outside the ROC	Not Causal Stable Unit circle inside the ROC

z-Transform Properties

- Mathematical properties of z-Transforms are useful in studying discrete-time signals and systems.

- Compared to Fourier Transforms, a special attention needs to be given for ROCs.

Notation:

$$x_1[n] \xleftrightarrow{\mathscr{Z}} X_1(z), \qquad \text{ROC} = R_{x_1}$$

$$x_2[n] \xleftrightarrow{\mathscr{Z}} X_2(z), \qquad \text{ROC} = R_{x_2}$$

Some z-Transform Properties

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n - n_0]$	$z^{-n_0} X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\| R_x$
4	3.4.4	$nx[n]$	$-z \dfrac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Table 3.2 Property 1: Linearity

$$ax_1[n] + bx_2[n] \xleftrightarrow{\mathcal{Z}} aX_1(z) + bX_2(z), \qquad \text{ROC} \supseteq R_{x_1} \cap R_{x_2}$$

- Proof is directly using the definition of z-Transform.

Table 3.2 Property 1: Linearity

$$ax_1[n] + bx_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} + bX_2(z), \qquad \text{ROC} \supseteq R_{x_1} \cap R_{x_2}$$

- Proof is directly using the definition of z-Transform.

- Note that, z must be in both ROCs. Thus, the resulting ROC is **at least** the intersection of individual ROCs.

Table 3.2 Property 1: Linearity

$$ax_1[n] + bx_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} aX_1(z) + bX_2(z), \qquad \text{ROC} \supseteq R_{x_1} \cap R_{x_2}$$

- Proof is directly using the definition of z-Transform.

- Note that, z must be in both ROCs. Thus, the resulting ROC is **at least** the intersection of individual ROCs.

- If the poles of $aX_1(z) + bX_2(z)$ consists of all poles of $X_1(z)$ and $X_2(z)$, then the ROC will be exactly equal to the overlap of the individual ROCs.

Table 3.2 Property 1: Linearity

$$ax_1[n] + bx_2[n] \xleftrightarrow{\mathcal{Z}} aX_1(z) + bX_2(z), \qquad \text{ROC} \supseteq R_{x_1} \cap R_{x_2}$$

- If there are some pole-zero cancellation, then the resulting ROC is larger. This can happen under special circumstances. For example:

$$x_1[n] = a^n u[n], \quad x_2[n] = -a^n u[n-1],$$

$$x_1[n] + x_2[n] = \delta[n], \quad \Rightarrow \text{ROC} : \text{all } z$$

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Table 3.2 Property 2: Time Shifting

$x[n - n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0}X$ [illegible] ROC $= R_x$ (except for possible addition or deletion [illegible] $z = \infty$).

- n_0 is an integer.

- If $n_0 > 0$, then the sequence is shifted to the right and if $n_0 < 0$, then the sequence is shifted to left.

- The factor z^{-n_0} can alter the number of poles at $z = 0$ and $z = \infty$, and hence the ROC.

- Time shifting property can be useful in obtaining inverse z-transform.

Table 3.2 Property 2: Time Shifting

Text Book Example 3.14

$$x[n] = (\frac{1}{4})^n u[n] \longleftrightarrow X(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}, \qquad |z| > \frac{1}{4}$$

If given, $$Y(z) = z^{-1}\frac{1}{1 - \frac{1}{4}z^{-1}}, \qquad |z| > \frac{1}{4}$$

Table 3.2 Property 2: Time Shifting

Text Book Example 3.14

$$x[n] = (\frac{1}{4})^n u[n] \longleftrightarrow X(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}, \qquad |z| > \frac{1}{4}$$

If given, $$Y(z) = z^{-1}\frac{1}{1 - \frac{1}{4}z^{-1}}, \qquad |z| > \frac{1}{4}$$

$$\Rightarrow y[n] = x[n-1] = (\frac{1}{4})^{n-1} u[n-1]$$

The z^{-1} multiplied by $X(z)$ tells us we have one time shift to the right.

Table 3.2 Property 3: Multiplication by an exponential sequence

$$z_0^n x[n] \longleftrightarrow X(z/z_0), \qquad \text{ROC} = |z_0| R_x.$$

- The resulting ROC is R_x scaled by the number $|z_0|$.

Table 3.2 Property 3: Multiplication by an exponential sequence

$$z_0^n x[n] \longleftrightarrow X(z/z_0), \qquad \text{ROC} = |z_0| R_x.$$

- The resulting ROC is R_x scaled by the number $|z_0|$.
- The ROC will increase if $|z_0| > 1$
- The ROC will decrease if $|z_0| < 1$
- If the ROC is a doughnut, R_x is the set of vales of z such that, $r_R < |z| < r_L$, then $|z_0| R_x$ is the set of values of z, such that:

$$|z_0| r_R < |z| < |z_0| r_L.$$

But there is no guarantee that the new system will be stable, as it may or may not include the unit circle.

Table 3.2 Property 3: Multiplication by an exponential sequence

$$z_0^n x[n] \longleftrightarrow X(z/z_0), \qquad \text{ROC} = |z_0| R_x.$$

- All pole and zero locations are also scaled by a factor of $|z_0|$.

- If z_0 is a positive real number, the scaling can be interpreted as shrinking or expanding of the z-plane, i.e., pole-zero locations change along radial lines.

- What happens if z_0 is complex?

Table 3.2 Property 3: Multiplication by an exponential sequence

Text Book Example 3.15

Consider the Transform pair $u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-z^{-1}}, \quad |z| > 1.$

Use the exponential multiplication property to determine the z-transform of

$$x[n] = r^n \cos(\omega_0 n) u[n], \quad r > 0$$

Table 3.2 Property 3: Multiplication by an exponential sequence

Text Book Example 3.15

Consider the Transform pair $u[n] \stackrel{\mathscr{Z}}{\longleftrightarrow} \frac{1}{1-z^{-1}}, \quad |z| > 1.$

Use the exponential multiplication property to determine the z-transform of

$$x[n] = r^n \cos(\omega_0 n) u[n], \quad r > 0$$

$$x[n] = r^n \cos(\omega_0 n) u[n] = \frac{1}{2}\left(r^n e^{j\omega_0 n} u[n] + r^n e^{-j\omega_0 n} u[n]\right), \quad r > 0$$

Table 3.2 Property 3: Multiplication by an exponential sequence

Text Book Example 3.15

Consider the Transform pair $u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-z^{-1}}, \quad |z| > 1.$

Use the exponential multiplication property to determine the z-transform of

$$x[n] = r^n \cos(\omega_0 n) u[n], \quad r > 0$$

$$x[n] = r^n \cos(\omega_0 n) u[n] = \frac{1}{2}\left(r^n e^{j\omega_0 n} u[n] + r^n e^{-j\omega_0 n} u[n]\right), \quad r > 0$$

$$X(z) = \frac{1}{2}\left(\frac{1}{1 - re^{j\omega_0} z^{-1}} + \frac{1}{1 - re^{-j\omega_0} z^{-1}}\right), \quad |z| > r$$

$$X(z) = \frac{1 - r\cos(\omega_0) z^{-1}}{1 - 2r\cos(\omega_0) z^{-1} + r^2 z^{-2}}, \quad |z| > r$$

Table 3.2 Property 4: Differentiation of $X(z)$

$$\xrightarrow{\mathcal{Z}} -z\frac{dX(z)}{dz},$$

ROC $= R_x$.

- This can be easily derived by differentiating the equation which defines the z-transform.

- See also Example 3.16 and derivation of Table 3.1 Transform pairs 7 and 8.

Table 3.2 Property 4: Differentiation of $X(z)$

$$nx[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{dX(z)}{dz}, \qquad \text{ROC} = R_x.$$

Generalized Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \Rightarrow$$

$$-z\frac{d}{dz}X(z) = -z\frac{d}{dz}\left(\sum_{n=-\infty}^{\infty} x[n]z^{-n}\right) = -z\sum_{n=-\infty}^{\infty} x[n]\frac{d}{dz}(z^{-n})$$

$$= -z\sum_{n=-\infty}^{\infty} -nx[n]z^{-n-1} = \sum_{n=-\infty}^{\infty} nx[n]z^{-n} = \mathscr{Z}(nx[n])$$

Table 3.2 Property 4: Differentiation of $X(z)$

Text Book Example 3.17

Determine the z-transform of the sequence $x[n] = na^n u[n]$.
(Table 3.1 Transform pair 7)

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, \quad |z| > |a| \quad \text{(Table 3.1 Transform Pair 5)}$$

$$na^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{d}{dz}\left(\frac{1}{1 - az^{-1}}\right), \quad |z| > |a| \quad \text{(Table 3.2 Property 4)}$$

$$X(z) = -z\left(\frac{-az^{-2}}{(1 - az^{-1})^2}\right) = \frac{az^{-1}}{(1 - az^{-1})^2}, \quad |z| > |a|$$

Table 3.2 Property 5: Complex Conjugation

$$x^*[n] \overset{\mathcal{Z}}{\longleftrightarrow} X^*(z^*), \qquad \text{ROC} = R_x.$$

1. Evaluate z at the complex conjugate z^*.

2. Take the complex conjugate of $X(z^*) = X^*(z^*)$.

$$\mathscr{Z}\{x^*[n]\} = \sum_{n=-\infty}^{\infty} x^*[n]z^{-n} = \left(\sum_{n=-\infty}^{\infty} x[n](z^*)^{-n}\right)^*$$

$$= X^*(z^*)$$

Property not in Table 3.2 → But Important: Time Reversal

$$x[-n] \quad \ldots z), \qquad \text{ROC} = 1/R_x.$$

Property not in Table 3.2 → But Important: Time Reversal

$$x[-n] \longleftrightarrow X(1/z), \qquad \text{ROC} = 1/R_x.$$

- The notation ROC $= 1/R_x$ implies that R_x is inverted.

- If R_x is the set of values of z such that

$$r_R < |z| < r_L$$

the ROC for $X(1/z)$ is the set of values of z such that

$$1/r_L < |z| < 1/r_R$$

- If z_0 is in ROC of $x[n]$ then $1/z_0$ is in ROC for the z-transform of $x[-n]$.

Table 3.2 Property 8: Complex Conjugate Time Reversal

$$x^*[-n] \longleftrightarrow X^*(1/z^*), \qquad \text{ROC} = 1/R_x.$$

$$\mathscr{Z}\{x^*[-n]\} = \sum_{n=-\infty}^{\infty} x^*[-n] z^{-n} = \left(\sum_{n=-\infty}^{\infty} x[-n](z^*)^{-n} \right)^*$$

Table 3.2 Property 8: Complex Conjugate Time Reversal

$$x^*[-n] \longleftrightarrow X^*(1/z^*), \qquad \text{ROC} = 1/R_x.$$

$$\mathscr{Z}\{x^*[-n]\} = \sum_{n=-\infty}^{\infty} x^*[-n] z^{-n} = \left(\sum_{n=-\infty}^{\infty} x[-n] (z^*)^{-n} \right)^*$$

Change $-n = k$

$$\mathscr{Z}\{x^*[-n]\} = \left(\sum_{k=-\infty}^{\infty} x[k] (z^*)^{k} \right)^* = \left(\sum_{k=-\infty}^{\infty} x[k] (1/z^*)^{-k} \right)^*$$

$$= X^*(1/z^*)$$

Table 3.2 Property 9: Convolution of Sequences

$$x_1[n] \star x_2[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X_1(z)X_2(z), \qquad \text{ROC} \supseteq R_{x_1} \cap R_{x_2}.$$

Table 3.2 Property 9: Convolution of Sequences

$$x_1[n] \star x_2[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X_1(z)X_2(z), \qquad \text{ROC} \supseteq R_{x_1} \cap R_{x_2}.$$

- Proof: Use convolution summation.

- ROC is **at least** the intersection of the ROCs of $X_1(z)$ and $X_2(z)$. Some zero-pole cancellations may make it bigger.

Textbook Example 3.19

$$x_1[n] \quad 2\delta[n-1] + \delta[n-2]$$

$$x_2[n] \quad \delta[n-1]$$

Find the convolution between $x_1[n]$ and $x_2[n]$ using z-transform.

Textbook Example 3.19

$$x_1[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$$
$$x_2[n] = \delta[n] - \delta[n-1]$$

Find the convolution between $x_1[n]$ and $x_2[n]$ using z-transform.

$$X_1(z) = 1 + 2z^{-1} + z^{-2}$$

$$X_2(z) = 1 - z^{-1}$$

Textbook Example 3.19

$$x_1[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$$
$$x_2[n] = \delta[n] - \delta[n-1]$$

Find the convolution between $x_1[n]$ and $x_2[n]$ using z-transform.

$$X_1(z) = 1 + 2z^{-1} + z^{-2}$$

$$X_2(z) = 1 - z^{-1}$$

$$X_1(z)X_2(z) = (1 + 2z^{-1} + z^{-2})(1 - z^{-1})$$

$$X_1(z)X_2(z) = 1 + z^{-1} - z^{-2} - z^{-3}$$

Translate it back to the time domain we will get:

$$x[n] = \delta[n] + \delta[n-1] - \delta[n-2] - \delta[n-3]$$

Homework

1. Read and understand Chapter 3 of the textbook
2. Related Problems: 3.7, 3.9, 3.10, 3.11, 3.12, 3.16, 3.19, 3.20, 3.27, 3.54, 3.6, 3.8, 3.54