Lecture – 2

Digital Signal Processing

ENGN 4537 / 6537

Semester 2, 2023

Discrete-time Systems

System: A transform[illegible]perator that maps any input discrete-time signal $x[n]$ into an output discrete-time signal $y[n]$:

$$y[n] = T\{x[n]\}$$

Discrete-time Systems

Read Chapter 2 [illegible] 2 of the Text Book

You need to know:

- Memoryless systems
- Linear Systems
- Time-invariant systems
- *Causality* and *Stability*

Linear Time-Invariant (LTI) Systems

Impulse Response $h[n]$

$h[n]$ is the response of a system to the input $\delta[n]$ (an impulse).

$$\delta[n] \longrightarrow \boxed{h[n]} \longrightarrow h[n]$$

Examples of LTI systems

Ideal Delay

$$h[n] = \delta[n - n_d], \quad n_d \geqslant 0$$

Examples of LTI systems

Accumulator

$$h[n] = \sum_{k=-\infty}^{n} \delta[k] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} = u[n]$$

Examples of LTI systems

Forward Difference

$$h[n] = \delta[n+1] - \delta[n]$$

Examples of LTI systems

Backward Difference

$$h[n] = \delta[n] - \delta[n-1]$$

Linear Time-Invariant (LTI) Systems: Convolution

- Let $y[n]$ is the output of a system whose input is $x[n]$.

- How do we work out $y[n]$?

Recall: What is Convolution sum?

$$\boxed{h[n]} \longrightarrow h[n]$$

$$\delta[n-k] \longrightarrow \boxed{h[n]} \longrightarrow h[n-k]$$

Recall: What is Convolution sum?

$$\boxed{h[n]} \longrightarrow h[n]$$

$$\delta[n-k] \longrightarrow \boxed{h[n]} \longrightarrow h[n-k]$$

$$x[k]\delta[n-k] \longrightarrow \boxed{h[n]} \longrightarrow x[k]h[n-k]$$

Recall: What is Convolution sum?

$$\boxed{h[n]} \longrightarrow h[n]$$

$$\delta[n-k] \longrightarrow \boxed{h[n]} \longrightarrow h[n-k]$$

$$x[k]\delta[n-k] \longrightarrow \boxed{h[n]} \longrightarrow x[k]h[n-k]$$

$$\sum_k x[k]\delta[n-k] \longrightarrow \boxed{h[n]} \longrightarrow \sum_k x[k]h[n-k]$$

What is Convolution Sum?

- Recall: $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$

- Hence, the output of an LTI system to an input $x[n]$ is

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Linear Time-Invariant (LTI) Systems: Convolution

Convolution

Output $y[n]$ to an input $x[n]$ of an LTI system with impulse response $h[n]$ is given by:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Linear Time-Invariant (LTI) Systems: Convolution

Convolution

Output $y[n]$ to an input $x[n]$ of an LTI system with impulse response $h[n]$ is given by:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

This is referred to as the convolution sum and can be represented in operator notation

$$y[n] = x[n] * h[n]$$

Properties of LTI Systems: Commutative Property

Commutative Property

$$x[n] * h[n] = h[n] * x[n]$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Properties of LTI Systems: Commutative Property

Commutative Prope

$$] = h[n] * x[n]$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

By substituting $n - k = m$,

$$y[n] = \sum_{m=\infty}^{-\infty} x[n-m]h[m] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

$$= h[n] * x[n]$$

Properties of LTI Systems: Commutative Property

Commutative Prope

$$x[n] * h[n] = h[n] * x[n]$$

- The roles of $x[n]$ and $h[n]$ in the convolution can be interchanged.
- *The order of the sequences in a convolution operation is unimportant.*
- Hence the system output is the same if the roles of the input and impulse response are reversed.

Properties of LTI Systems: Distributive property

Convolution distributes over addition

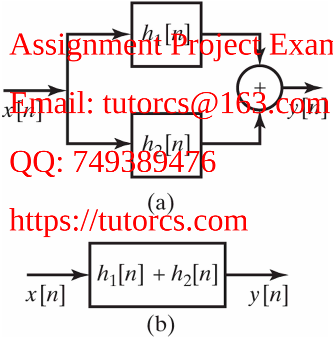
$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

Properties of LTI Systems: Distributive property

Convolution distributes over addition

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

Proof: Use the convolution summation to prove this.



Properties of LTI Systems: Associative Property

Commutative and Associative Property

$$y[n] = x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$

$$y[n] = x[n] * (h_2[n] * h_1[n]) = (x[n] * h_2[n]) * h_1[n]$$

Properties of LTI Systems: Associative Property

Commutative and Associative Property

$$y[n] = x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$

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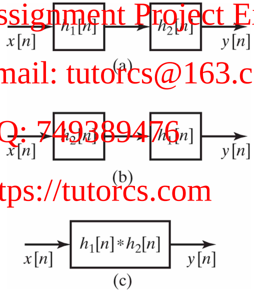


Properties of LTI Systems: Stability and Causality

LTI system can be st causal (Sometimes it is important to know whether a given LTI system is stable/causal or not).

- A stable LTI system will keep everything contained and balanced.

- A causal LTI system means that we cannot predict or exactly know the future.

Properties of LTI Systems: Stability

A stable system: *Every bounded input produces a bounded output.*

Stability

An LTI system is bounded if and only if the impulse response is absolutely summable, i.e., if $B_h = \sum_{k=-\infty}^{\infty} |h[k]| < \infty$.

Properties of LTI Systems: Stability

How to link the def stability to convolution?

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Properties of LTI Systems: Stability

How to link the def stability to convolution?

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- The condition for a system to be stable, is if every bounded input produces a bounded output.
- If the input and output are bounded, we can write:

$$|x[k]| < \infty \text{ or } <= B_x$$
$$|y[n]| < \infty \text{ or } <= B_y$$

Properties of LTI Systems: Stability

How to link the definition of stability to convolution?

- Therefore, we can write the bounded output as:

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[k]h[n-k] \right| < \infty$$

- Since the input is bounded, then the impulse response must satisfy: $\sum_{k=-\infty}^{\infty} |h[n-k]| < \infty$. .

- Due to the time-invariant property of $h[n-k]$, we can remove the time delay and rewrite the impulse response as: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$.

Properties of LTI Systems: Causality

A causal system: *The output $y[n_0]$ depends only on the input samples $x[n]$, for $n \leqslant n_0$.*

Causality of LTI systems

An LTI system is causal if $h[n] = 0$ for $n < 0$.

Thus, you cannot have a non-zero output to a zero input for $n < 0$.

Questions

Which of the LTI systems studied before is stable and/or casual?

a) Ideal Delay
b) Accumulator
c) Forward Difference
d) Backward Difference

Worked Problems

a) Ideal Delay: $h[n] = \delta[n - n_d]$, $n_d \geqslant 0$.

- CAUSAL because $h[n] = 0$ for $n < 0$. If $n_d < 0$ then the ideal delay would be anti-causal.
- STABLE because the sum of all impulse responses is equal to 1, and does not tend towards ∞.

Worked Problems

b) Accumulator: $h[n] = \sum_{k=-\infty}^{n} \delta[k] = u[n]$.

- CAUSAL because $h[n] = 0$ for $n < 0$.

- NOT STABLE because the sum is accumulating to ∞.

Worked Problems

c) Forward Difference: $h[n] = \delta[n+1] - \delta[n]$.

- NOT CAUSAL because $h[n] \neq 0$ for $n < 0$.

- STABLE because the absolute sum is equal to $|1| + |-1| = 2$.

Worked Problems

d) Backward Difference: $h[n] = \delta[n] - \delta[n-1]$.

- CAUSAL because $h[n] = 0$ for $n < 0$.

- STABLE because the absolute sum is equal to $|1| + |-1| = 2$.

Linear Constant-Coefficient Difference Equations

- An important cl[...] systems are given by Nth order constant coeffici[...]ence equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

- Expressing a given LTI system in the form of difference equation is useful in implementing LTI systems.

- *Shows how the past values of the output relate to the past values of the input.*

Linear Constant-Coefficient Difference Equations

Example: Accumula

$$y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

Linear Constant-Coefficient Difference Equations

Example: Accumula

$$y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

By a time shift we have

$$y[n-1] = \sum_{k=-\infty}^{n-1} x[k]$$

Linear Constant-Coefficient Difference Equations

Example: Accumula

$$y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

By a time shift we have

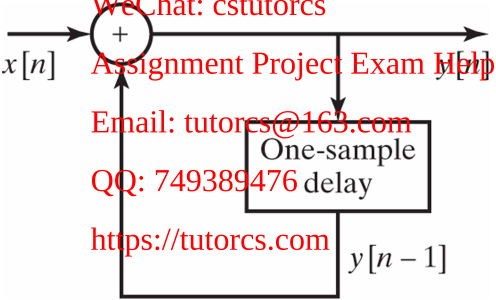
$$y[n-1] = \sum_{k=-\infty}^{n-1} x[k]$$

Thus, combining the above two equations: $y[n] = x[n] + y[n-1]$.

Linear Constant-Coefficient Difference Equations

Example: Accumulator

$$y[n] = x[n] + y[n-1]$$



Eigenfunctions for LTI Systems

- Consider a comp[illegible]ential signal $x[n] = e^{j\omega n}$ for $-\infty < n < \infty$. [illegible]t of an LTI system to this exponential sign[illegible]

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} \\
&= \underbrace{\left[\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right]}_{} \times e^{j\omega n} \\
&= H(e^{j\omega}) \times e^{j\omega n}
\end{aligned}
$$

Eigenfunctions for LTI Systems

- Thus, the outpu[illegible]I system with an impulse response $h[n]$, to a complex exponential input $x[n] = e^{j\omega n}$, is given by

$$y[n] = T\{e^{j\omega n}\} = H(e^{j\omega})e^{j\omega n}$$

where

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}.$$

Eigenfunctions for LTI Systems

$$y[n] = T\{e^{j\omega n}\} = H(e^{j\omega})e^{j\omega n} = \left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right) e^{j\omega n}$$

- The output is equal to the input multiplied by a complex scalar, $H(e^{j\omega})$.

- **Hence, we say $e^{j\omega n}$ is an eigenfunction of the LTI system.**

Eigenfunctions for LTI Systems

$$y[n] = T\{e^{j\omega n}\} \text{ [illegible] } e^{j\omega n} = \left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right) e^{j\omega n}$$

- The corresponding eigenvalue $H(e^{j\omega})$ describes the change in complex amplitude of a complex exponential signal. We call $H(e^{j\omega})$ as the frequency response of the system.
- $y[n] = H(e^{jw})e^{j\omega n}$ tells us what the behavior of the system is at different frequencies.
- $H(e^{j\omega})$ may also be referred to as the Fourier Transform of the impulse response.

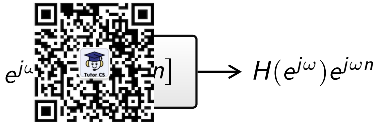
Frequency response $H(e^{j\omega})$ of an LTI system

$$e^{j\omega n} \longrightarrow \boxed{h[n]} \longrightarrow H(e^{j\omega})e^{j\omega n}$$

- In general, $H(e^{j\omega})$ is complex. Can be expressed in terms of real and imaginary parts

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega})$$

Frequency response $H(e^{j\omega})$ of an LTI system



- In general, $H(e^{j\omega})$ is complex. Can be expressed in terms of real and imaginary parts

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega})$$

- $H(e^{j\omega})$ can also be written in terms of magnitude and phase:

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\angle H(e^{j\omega})}$$

Frequency response $H(e^{j\omega})$ of an LTI system

Question

- What frequency [illegible] do we need to set, such that the output is equal [illegible]ut, y[n]=x[n]?

  If

$$x[n] = e^{j\omega n}$$

$$y[n] = H(e^{j\omega})e^{j\omega n}$$

$$\therefore H(e^{j\omega}) = 1$$

- What is the corresponding impulse response?

$$h[n] = \delta[n] = \text{ unit impulse response}$$

$$\therefore H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} = \sum_{k=-\infty}^{\infty} \delta[k]e^{-j\omega k} = e^{-j\omega(0)} = 1$$

Frequency response $H(e^{j\omega})$ of an LTI system

Example

- Find the frequency response of the ideal delay system: $y[n] = x[n - n_d]$ where n_d is a fixed integer (i.e., n_d can be either greater than, less than, or equal to zero).

- This problem can be solved via 2 methods: algebraically or by inspection.

Frequency response $H(e^{j\omega})$ of an LTI system

Example Solution - 1: Algebraically

- The impulse response to $y[n] = x[n - n_d]$ is $h[n] = \delta[n - n_d]$.

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta[n - n_d] e^{-j\omega n}$$

$$\therefore H(e^{j\omega}) = e^{-j\omega n_d}$$

Frequency response $H(e^{j\omega})$ of an LTI system

Example Solution - … 2: By Inspection

- For input $x[n] =$ … output is

$$y[n] = x[n - n_d] = e^{j\omega(n-n_d)} = e^{-j\omega n_d} e^{j\omega n}.$$

- Recall: $y[n] = H(e^{j\omega})e^{j\omega n}$.

- Hence, we can identify $H(e^{j\omega}) = e^{-j\omega n_d}$.

- *We end up with the same result, as the algebraic method.*

Frequency response $H(e^{j\omega})$ of an LTI system

Example Solution

- What are the re[illegible]aginary parts of the frequency response, $H(e^{j\omega}$[illegible]?

- Recall: $e^{-j\theta} = \underbrace{\cos(\theta)}_{\text{real}} - j\underbrace{\sin(\theta)}_{\text{imaginary}}$.

Frequency response $H(e^{j\omega})$ of an LTI system

Example Solution

- What are the real and imaginary parts of the frequency response, $H(e^{j\omega}) = e^{-j\omega n_d}$?

- Recall: $e^{-j\theta} = \underbrace{\cos(\theta)}_{real} - j\,\underbrace{\sin(\theta)}_{imaginary}$.

$$H_R(e^{j\omega}) = \cos(\omega n_d), \quad H_I(e^{j\omega}) = -\sin(\omega n_d)$$

Frequency response $H(e^{j\omega})$ of an LTI system

Example Solution

- What are the real and imaginary parts of the frequency response, $H(e^{j\omega}) = e^{-j\omega n_d}$?

- Recall: $e^{-j\theta} = \underbrace{\cos(\theta)}_{\text{real}} - j\,\underbrace{\sin(\theta)}_{\textit{imaginary}}$.

$$H_R(e^{j\omega}) = \cos(\omega n_d), \quad H_I(e^{j\omega}) = -\sin(\omega n_d)$$

- Identify the magnitude and phase of the frequency response, $H(e^{j\omega}) = e^{-j\omega n_d}$:

$$|H(e^{j\omega})| = 1, \quad \angle H(e^{j\omega}) = -\omega n_d$$

Frequency response $H(e^{j\omega})$ of an LTI system

Extra Problems

Here are some extra problems for you to do in your own time:

Find the frequency response for:

a) Accumulator

b) Forward Difference

c) Backward Difference

LTI systems with complex exponential inputs

$$\longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$e^{j\omega_k n} \longrightarrow \boxed{h[n]} \longrightarrow H(e^{j\omega_k})e^{j\omega_k n}$$

$$\alpha_k e^{j\omega_k n} \longrightarrow \boxed{h[n]} \longrightarrow \alpha_k H(e^{j\omega_k})e^{j\omega_k n}$$

$$\sum_k \alpha_k e^{j\omega_k n} \longrightarrow \boxed{h[n]} \longrightarrow \sum_k \alpha_k H(e^{j\omega_k})e^{j\omega_k n}$$

We use superposition and multiplication (due to linearity) to see the above.

Some facts: Frequency response of LTI systems

- The concept of frequency response is same for both continuous time and discrete time LTI systems.

- However, the frequency response of discrete time LTI systems is always periodic function of the frequency ω with period 2π.

Some facts: Frequency response of LTI systems

Show that $H(e^{j(\omega+2\pi)}) = H(e^{j\omega})$:

$$H(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j(\omega+2\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}e^{-j2\pi n}$$

$$= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = H(e^{j\omega})$$

Some facts: Frequency response of LTI systems

- Since $H(e^{j\omega})$ is [illegible]ith period 2π, we only need to specify $H(e^{j\omega})$ over an interval of length 2π.

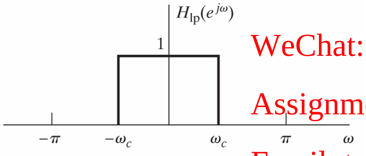
- Low frequencies are frequencies closer to 0 or multiples of 2π.

- High frequencies are those that are odd multiples of π.

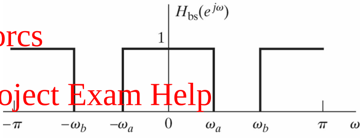
Some facts: Frequency response of LTI systems

Examples: Ideal frequency-selective filters

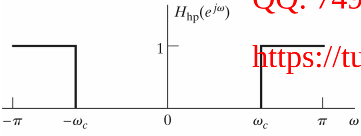
Low-pass filter



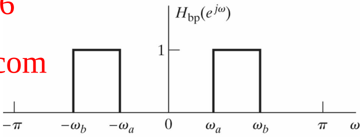
Band-stop filter



High-pass filter



Band-pass filter



Tutorial Problem Set for this week

1. Read and understand Chapter 2 of the textbook
2. Problems Related to Lecture 02: 2.1, 2.2 (a very useful result), 2.3, 2.4, 2.5, 2.7, 2.10, 2.14, 2.15, 2.18, 2.19, 2.20, 2.23, and 2.29.