Lecture - 8

Digital Signal Processing

ENGN 4537/ 6537

Semester 2, 2023

Transform Analysis of Linear Time-Invariant (LTI) Systems

We will study how to analyze a system given the Z-Transform and/or Fourier Transform

- Analysis of LTI systems described by Difference Equations
- Analysis of LTI systems described by Fourier Transform
- All Pass and Minimum Phase Systems

Frequency Response of LTI Systems: Magnitude and Phase response

- Frequency respo[illegible]) of an LTI system is a complex gain that the system [illegible] a complex exponential input $e^{j\omega n}$:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

- System input/output Fourier transforms are related by:

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$|Y(e^{j\omega})| = |H(e^{j\omega})||X(e^{j\omega})|$$

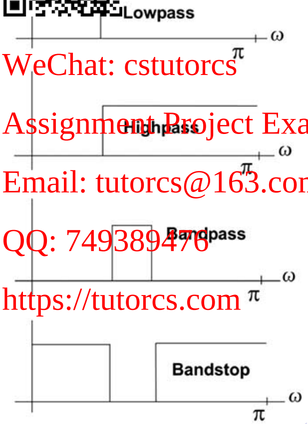
$$\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega})$$

$|H(e^{j\omega})|$- Magnitude response

$\angle H(e^{j\omega})$ - Phase response

Frequency Response: Magnitude

Magnitude response $|H(e^{j\omega})|$ defines how different frequencies of the input signal gets [illegible] by the system



Frequency Response Magnitude: Influence of poles and zeros

An LTI system described by a difference equation has a system function

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0 \prod_{k=1}^{M} (1 - c_k z^{-1})}{a_0 \prod_{k=1}^{N} (1 - d_k z^{-1})}$$

If the system is stable, ROC includes the unit circle and will have a Fourier response

$$H(e^{j\omega}) = \frac{b_0 \prod_{k=1}^{M} (1 - c_k e^{-j\omega})}{a_0 \prod_{k=1}^{N} (1 - d_k e^{-j\omega})}$$

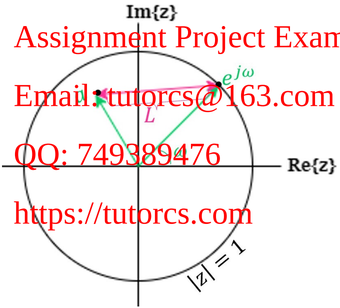
Frequency Response Magnitude: Influence of poles and zeros

$$H(e^{j\omega}) = \frac{b_0 \prod_{k=1}^{M}(1 - c_k e^{-j\omega})}{a_0 \prod_{k=1}^{N}(1 - d_k e^{-j\omega})}$$

$$|H(e^{j\omega})| = \frac{|b_0| \prod_{k=1}^{M} |1 - c_k e^{-j\omega}|}{|a_0| \prod_{k=1}^{N} |1 - d_k e^{-j\omega}|} = \frac{|b_0| \prod_{k=1}^{M} |e^{j\omega} - c_k|}{|a_0| \prod_{k=1}^{N} |e^{j\omega} - d_k|}$$

Frequency Response Magnitude: Influence of poles and zeros

$|H(e^{j\omega})|$ depends on ... the form $|e^{j\omega} - a|$. Therefore if we understand how that ...aves at poles and zeros of the system, we can chara... intuitively understand the magnitude response of the system.



Frequency Response Magnitude: Influence of poles and zeros

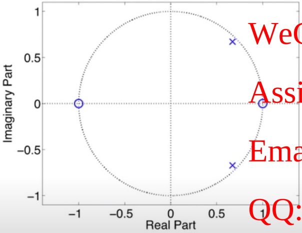
$$|H(e^{j\omega})| = \frac{|b_0| \prod_{k=1}^{M} |e^{j\omega} - c_k|}{|a_0| \prod_{k=1}^{N} |e^{j\omega} - d_k|}$$

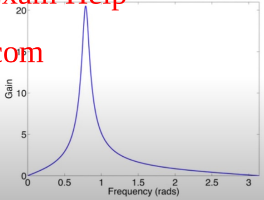
$$|H(e^{j\omega})| = \frac{|b_0| \prod_{k=1}^{M} \text{"distance from } e^{j\omega} \text{ to zeros"}}{|a_0| \prod_{k=1}^{N} \text{"distance from } e^{j\omega} \text{ to poles"}}$$

- When $e^{j\omega}$ is close to a zero, $|H(e^{j\omega})|$ is small- zeros pull magnitude response down.
- When $e^{j\omega}$ is close to a pole, $|H(e^{j\omega})|$ is large - poles push magnitude response up.

Frequency Response Magnitude: Influence of poles and zeros

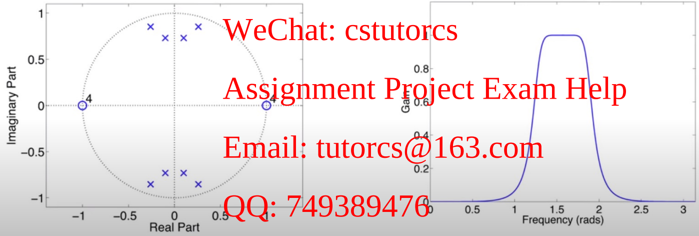
Example 1:





Frequency Response Magnitude: Influence of poles and zeros

Example 2:



Summary: Given the poles and zeros of a system, we can infer the magnitude characteristics Fourier response.

Frequency Response: Phase

Given the Frequency Response $H(e^{j\omega})$ of a system, $\angle H(e^{j\omega})$ is its "Phase response".

- Due to the periodic behavior of ω the phase is not unique.
- We define phase in two ways, "principle value (wrapped phase)" and "continuous phase (unwrapped phase").

- The principal value or wrapped phase is:

$$-\pi < \mathrm{ARG}[H(e^{j\omega})] \leq \pi$$

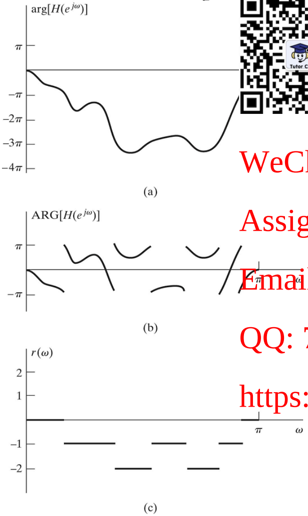
- The continuous phase can be represented in terms of the principal value as:

$$\arg[H(e^{j\omega})] = \mathrm{ARG}[H(e^{j\omega})] + 2\pi r(\omega)$$

where $r(\omega)$ is an integer that is somewhat arbitrary.

Frequency Response: Phase

$$\arg[H(e^{j\omega})] = \text{ARG}[H(e^{j\omega})] + 2\pi r(\omega)$$



(a) The continuous phase exceeds the range $-\pi$ to π.

(b) The principal value is restricted within the range.

(c) Integer multiples of 2π to be added to $\text{ARG}[H(e^{j\omega})]$ to obtain $\arg[H(e^{j\omega})]$.

Frequency Response Phase: Group Delay

- Group delay is a useful representation of phase. It measures the linearity of the phase and predicts the system's delay to different frequency components:

$$\tau(\omega) = \text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$

- Group delay is represented by *grd*
- The units of group delay is samples.

Frequency Response Phase: Group Delay

Let us look at an ide ystem:

$$h_{id}[n] = \delta[n - n_d]$$

- Frequency response

$$H_{id}(e^{j\omega}) = e^{-j\omega n_d}$$

$$\angle H_{id}(e^{j\omega}) = -\omega n_d$$

- Time delay is associated with phase that is linear with frequency.

Frequency Response Phase: Group Delay

The ideal delay syste

$$) = e^{-j\omega n_d},$$

where $|H_{id}(e^{j\omega})| = 1$ and $\angle H_{id}(e^{j\omega}) = -\omega n_d$.

Frequency Response Phase: Group Delay

The ideal delay system

$$H_{id}(e^{j\omega}) = e^{-j\omega n_d},$$

where $|H_{id}(e^{j\omega})| = 1$ and $\angle H_{id}(e^{j\omega}) = -\omega n_d$.

$$\tau(\omega) = -\frac{d}{d\omega}\{-\omega n_d\} = n_d$$

- The group delay: $\tau(\omega) = n_d$ samples
- This is a constant group delay = Phase is linear (perfect scenario).
- Group delay is independent of the frequency in an ideal delay system.

Frequency Response Phase: Group Delay

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- If the group delay is constant with frequency, then each narrowband component undergoes the same time delay.

Frequency Response Phase: Group Delay

We use the group de asure the linearity of the phase.

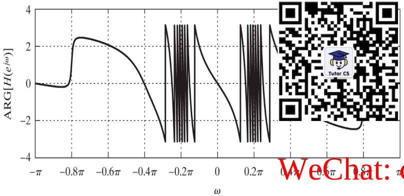
- a broadband signal can be thought of as a superposition of narrrowband signals

- If the group delay is constant with frequency, then each narrowband component undergoes the same time delay.

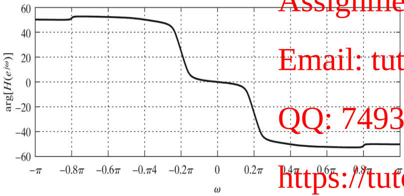
- If group delay varies with frequency (is not constant), then each narrowband component undergoes different time delays - time dispersion of the output energy.

Effects of Group Delay and Attenuation

Consider an example system with the phase response shown below:



(a) Principle Value of Phase Response



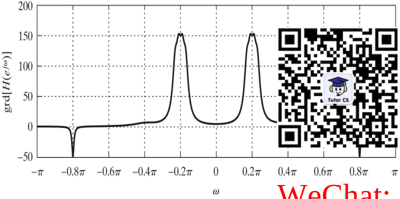
(b) Unwrapped Phase Response

(a) The principal value phase response exhibits multiple discontinuities, due to the modulo 2π computation of the wrapped phase.

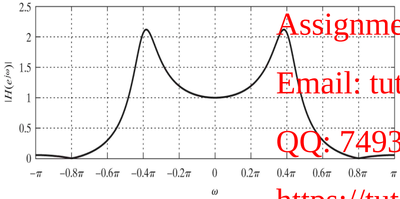
(b) The unwrapped (continuous) phase is smoother by removing the jumps of size 2π.

- Is the phase linear?
- What does this mean in terms of group delay?

Effects of Group Delay and Attenuation



(a) Group delay of $H(z)$



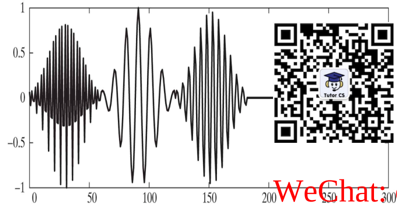
(b) Magnitude of Frequency Response

(a) The group delay is not a constant, or the phase is not linear phase, negative at $\omega = \pm 0.8\pi$ and a large positive peak at $0.17\pi < |\omega| < 0.23\pi$.

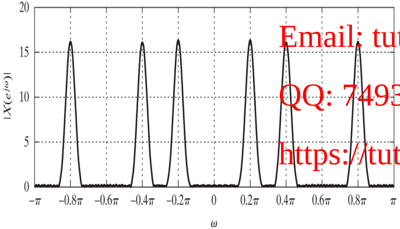
(b) The magnitude response $(|H(e^{j\omega})|)$.

- What do the peaks mean in the group delay?
- What happens when the magnitude is zero?

Effects of Group Delay and Attenuation



(a) Waveform of signal $x[n]$



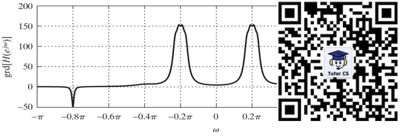
(b) Magnitude of DTFT of $x[n]$

(a) An input signal $x[n]$ consisting of three narrowband pulses separated in time.

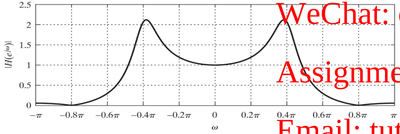
(b) The corresponding DTFT magnitude $|X(e^{j\omega})|$ shows the frequency components at $0.2\pi, 0.4\pi$ and 0.8π.

Effects of Group Delay and Attenuation

System:

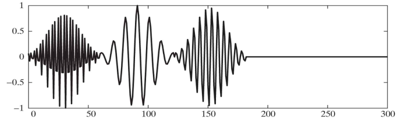


(a) Group delay of $H(z)$



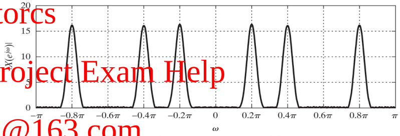
(b) Magnitude of Frequency Response

Input:



Sample number (n)

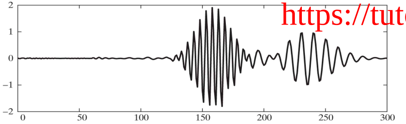
(a) Waveform of signal $x[n]$



(b) Magnitude of DTFT of $x[n]$

Output:

Waveform of signal $y[n]$

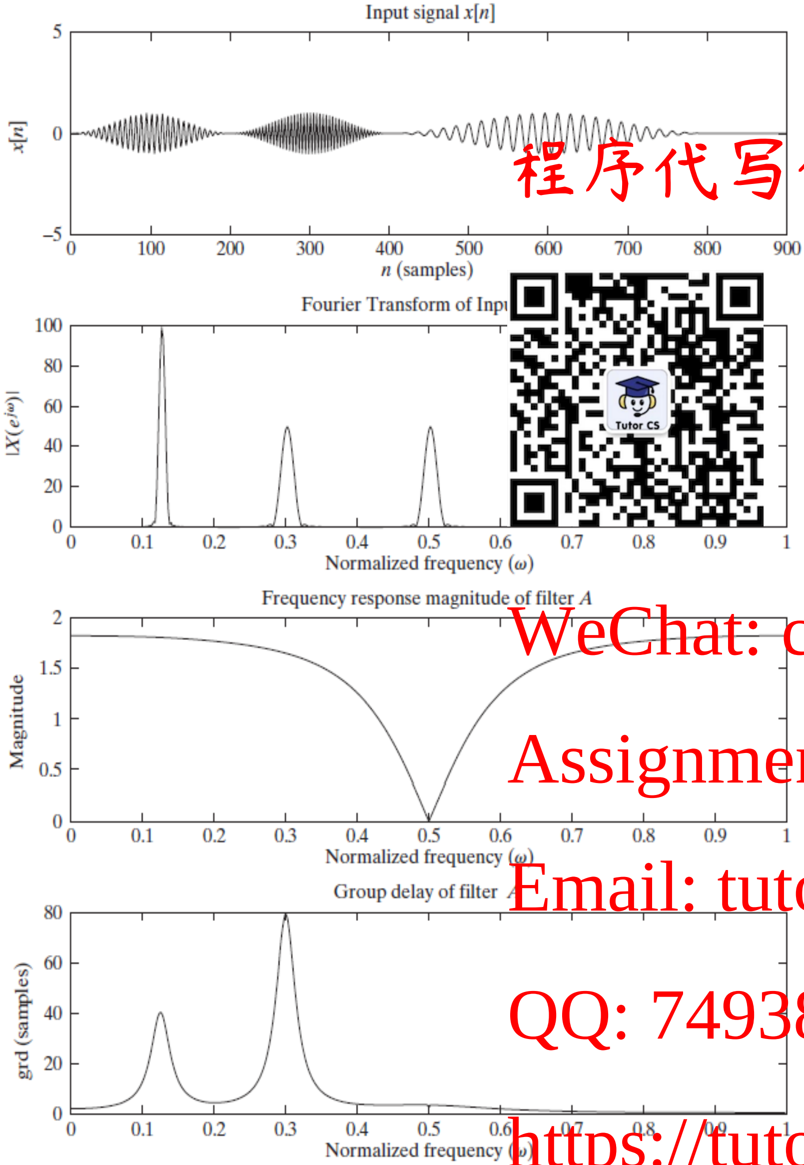


Sample number (n)

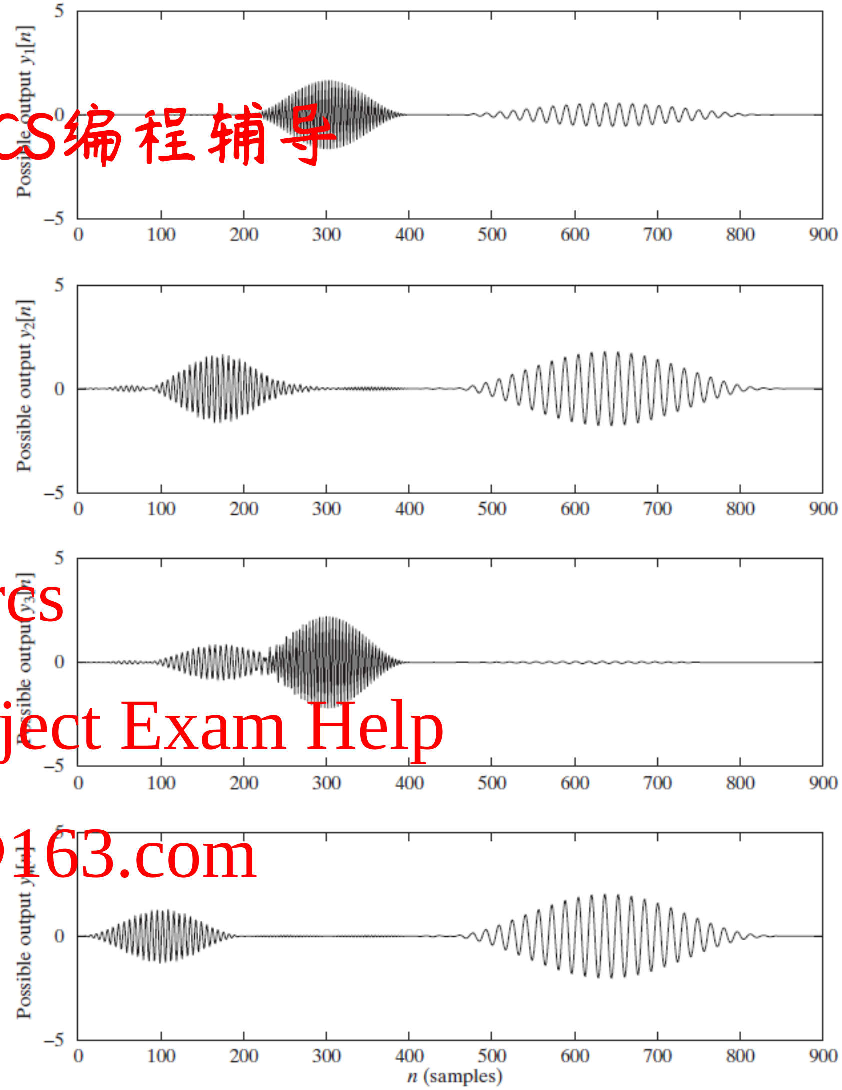
The output signal $y[n]$

- $\omega = 0.8\pi$ eliminated.
- $\omega = 0.2\pi$ pulse experience the most delay.

Problem 5.34:



Determine which one of the following signals is the output of filter A when the input is $x[n]$:



Homework

- Read and understand Section 5.1 of the textbook.
- Related Problems: 5.1, 5.2, 5.3, 5.4, 5.5, 5.6 and 5.7