Lecture - 4

Digital Signal Processing

ENGN 4537/ 6537

Semester 2, 2023

z-Transform: Definition (Revisit)

The z-transform of

$$\sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- z is considered to be a complex variable
  - $z = re^{j\omega}$, where r is the radius, and ω is the phase

- Notation $\mathscr{Z}\{x[n]\} = X(z)$

- How can you evaluate the FT from the z-transform?
  - Evaluate the z-transform at $z = e^{j\omega}$ *(unit circle on complex z-plane)*, to get the FT.

Region of Convergence (ROC)

- For a given sequence, the set of z values for which the z-transform converges is called the *Region of Convergence (ROC)* of the z-transform.
- Condition on convergence is

$$|X(z)| \leq \sum_{n=-\infty}^{\infty} |x[n]||z^{-n}| < \infty$$

- ROC of the z-transform consists of all values of z such that the above inequality holds.

- If for some value of z, say $z = z_1$, is in the ROC ($z_1 \in$ ROC), then all values of z on the circle defined by $|z| = |z_1|$ will also be in the ROC.

Poles and Zeros of z-Transform

- Let the z-transf[illegible]presented inside ROC as

$$X(z) = \frac{P(z)}{Q(z)}$$

where $P(z)$ and $Q(z)$ are polynomials in z.

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where $P(z)$ and $Q(z)$ are polynomials in z.

- The zeros are the roots of the numerator polynomial (i.e., $P(z) = 0$) and the poles are the roots of the denominator polynomial (i.e., $Q(z) = 0$).

Reminder: Sum of a Finite Power Series: $S_N = \sum_{n=0}^{N} b^n$

$$S_N = 1 + b + b^2 + b^3 + \ldots + b^N \quad (1)$$

$$bS_N = b + b^2 + b^3 + \ldots + b^N + b^{N+1} \quad (2)$$

By subtracting (2) from (1), we have,

$$(1-b)S_N = 1 - b^{N+1}$$

$$S_N = \frac{1-b^{N+1}}{1-b} \quad \text{provided } b \neq 1$$

Reminder: Sum of a Infinite Power Series: $S_N = \sum_{n=0}^{\infty} b^n$

Similarly, $S_\infty = \sum_{n=0}^{\infty} b^n$

$$S_\infty = 1 + b + b^2 + b^3 + \dots$$

$$bS_\infty = b + b^2 + b^3 + \dots$$

$$S_\infty = \frac{1}{1-b} \qquad \text{provided } |b| < 1$$

Reminder: Sum of Power Series: $S = \sum_{n=N_1}^{N_2} b^n$

$$S = b^{N_1} + b^{N_1+1} + \ldots + b^{N_2} \tag{3}$$

$$bS = b^{N_1+1} + b^{N_1+2} + \ldots + b^{N_2} + b^{N_2+1} \tag{4}$$

By subtracting (4) from (3), we have,

$$(1-b)S = b^{N_1} - b^{N_2+1}$$

$$S = \frac{b^{N_1} - b^{N_2+1}}{1-b} \quad \text{provided } b \neq 1$$

Example 3.1 Right-sided Exponential sequences

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For convergence, we need the series absolutely summable, i.e.,

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty$$

- Above condition holds if $az^{-1} < 1$. Thus, the ROC is the range of values for which $|az^{-1}| < 1$ or $|z| > |a|$. i.e., outside of the circle with radius $|a|$ in the z-plane.

Example 3.1: Right-sided Exponential sequences

Inside the ROC, $|z| >$ [illegible] series converges to

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1 - az^{-1}} \quad \text{(Geometric series)}$$

$$= \frac{z}{z - a}$$

Example 3.1: Right-sided Exponential sequences

- Thus, $x[n] = a^n u[n]$ has a closed form z-Transform:

$$\frac{z}{z-a} \quad \text{for } |z| > |a|$$

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- Recall: The zeros are the roots of the numerator, and the poles are the roots of the denominator.

- Note that there is a zero at $z = 0$ and a pole at $z = a$.

- If $|a| < 1$ then ROC contains the unit circle, and hence, the FT of the sequence converges to:

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} \quad \text{(Table 2.3 Prop. 4)}$$

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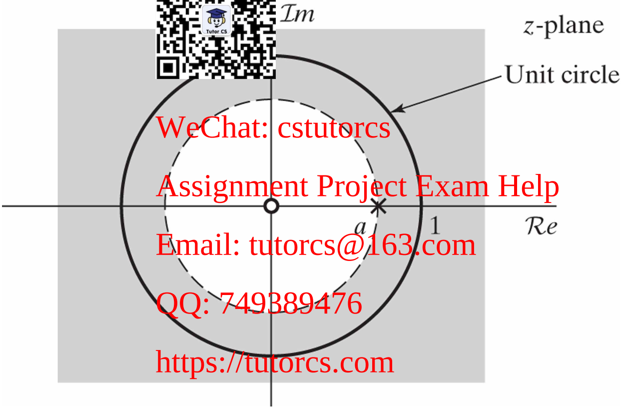
- However, if $|a| \geqslant 1$ then ROC does not contain the unit circle, and hence, the FT of the sequence does not converge.

Example 3.1: Right-sided Exponential sequences

ROC and pole-zero plot for $|a| < 1$



The symbol '$\circ$' denotes the zero and '$\times$' denotes the pole.

Example 3.2: Left-sided Exponential sequences

$$x[n] = -a^n u[-n-1] = \begin{cases} -a^n & n \leqslant -1 \\ 0 & n > -1. \end{cases}$$

Since the sequence is nonzero for $n \leqslant -1$, this is a left-sided sequence.

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Since the sequence is nonzero for $n \leqslant -1$, this is a left-sided sequence. The z- transform:

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

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We need to change the sequence to start from zero and increase to infinity.

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We need to change the sequence to start from zero and increase to infinity.

Change of index $n = -m$, we have

$$X(z) = -\sum_{m=\infty}^{1} a^{-m} z^{m} = -\sum_{m=1}^{\infty} (a^{-1}z)^m = 1 - \sum_{m=0}^{\infty} (a^{-1}z)^m$$

Example 3.2: Left-sided Exponential sequences

The z-Transform:

$$X(z) = 1 - \sum_{m=0}^{\infty} (a^{-1}z)^m$$

The series converges if $|(a^{-1}z)| < 1$, i.e., $|z| < |a|$ which is the ROC.

Example 3.2: Left-sided Exponential sequences

Using the formula for sum of the geometric series

$$X(z) = 1 - \sum_{m=0}^{\infty} (a^{-1}z)^m = 1 - \frac{1}{1 - a^{-1}z} = 1 - \frac{a}{a - z} = \frac{z}{z - a}$$

We can see that there is a zero at $z = 0$ and a pole at $z = a$.

Example 3.2: Left-sided Exponential sequences

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- Note that if $|a| < 1$, then the unit circle is not in ROC and the FT of the sequence does not exist.

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$$X(z) = \frac{z}{z-a}$$

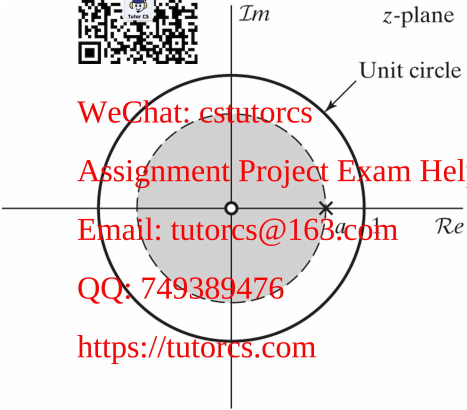
- Note that if $|a| < 1$, then the unit circle is not in ROC and the FT of the sequence does not exist.

- But if $|a| > 1$ then the unit circle is in the ROC and the FT is

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

Example 3.2: Left-sided Exponential sequences

ROC and pole-zero p < 1



Example 3.2: Left-sided Exponential sequences

- Important: Note that the algebraic expressions for Example 3.1 and 3.2 (right-sided and left-sided sequences) are the same. Also, their pole-zero locations are the same. However, the regions of convergence are different.

- Thus, the ROC needs to be found, in order to understand the convergence of each sequence.

Example 3.3: Sum of Two exponential Series $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$

The z-transform is

$$X(z) = \sum_{n=-\infty}^{\infty} \left(\left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \right) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{3}\right)^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{3} z^{-1}\right)^n$$

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The z-transform is

$$X(z) = \underbrace{\sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n}_{\substack{|\frac{1}{2} z^{-1}| < 1 \\ |z| > |\frac{1}{2}|}} + \underbrace{\sum_{n=0}^{\infty} \left(-\frac{1}{3} z^{-1}\right)^n}_{\substack{|-\frac{1}{3} z^{-1}| < 1 \\ |z| > |\frac{1}{3}|}}$$

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The z-transform is

$$X(z) = \underbrace{\sum_{n=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^n}_{\substack{|\frac{1}{2}z^{-1}|<1 \\ |z|>|\frac{1}{2}|}} + \underbrace{\sum_{n=0}^{\infty} \left(-\frac{1}{3}z^{-1}\right)^n}_{\substack{|-\frac{1}{3}z^{-1}|<1 \\ |z|>|\frac{1}{3}|}}$$

Convergence requires both summations to converge, i.e., $|z| > \frac{1}{2}$ and $|z| > \frac{1}{3}$.

Thus the ROC is the region of overlap, which is $|z| > \frac{1}{2}$.

Example 3.3: Sum of Two exponential Series

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] + \left(-\tfrac{1}{3}\right)^n u[n]$$

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{3} z^{-1}\right)^n$$

$$= \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 + \frac{1}{3} z^{-1}} \quad (\text{Geometric Series})$$

$$= \frac{z}{z - \frac{1}{2}} + \frac{z}{z + \frac{1}{3}}$$

$$= \frac{2z\left(z - \frac{1}{12}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)}$$

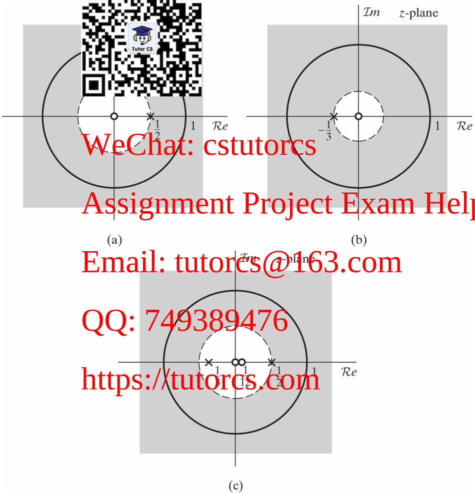
Example 3.3: Sum of Two exponential Series

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$

What are the poles and zeros of $X(z) = \frac{2z(z-\frac{1}{12})}{(z-\frac{1}{2})(z+\frac{1}{3})}$?

Example 3.3: Sum of Two exponential Series

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$



Example 3.5: Two-sided Exponential Sequence

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

Verify this as an exer

Figure 3.6 Pole–zero plot

Homework

1. Read and understand Chapter 3 (Sections 1 to 2) of the textbook
2. Related Problems: 3.1, 3.2, 3.3, 3.4, 3.5, 3.10, 3.19, 3.20, 3.54