MONASH University

REVOLUTION ANALYTICS

Rob J Hyndman

Forecasting using R

6. ETS models

OTexts.com/fpp/7/

Outline

1. Exponential smoothing methods so far

2. Holt-Winters' seasonal method

3. Taxonomy of exponential smoothing methods

4. Exponential smoothing state space models

Exponential smoothing methods

- Simple exponential smoothing: no trend.
  `ses(x)`

- Holt's method: linear trend.
  `holt(x)`

- Exponential trend method.
  `holt(x, exponential=TRUE)`

- Damped trend method.
  `holt(x, damped=TRUE)`
- Damped exponential trend method.
  `holt(x, damped=TRUE, exponential=TRUE)`

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Outline

1 Exponential smoothing methods so far

2 **Holt-Winters' seasonal method**

3 Taxonomy of exponential smoothing methods

4 Exponential smoothing state space models

Holt-Winters additive method

- Holt and Winters extended Holt's method to capture seasonality.
- Three smoothing equations—one for the level, one for trend, and one for seasonality.
- Parameters: $0 \le \alpha \le 1$, $0 \le \beta^* \le 1$, $0 \le \gamma \le 1-\alpha$ and $m =$ period of seasonality.

Holt-Winters additive method

$$\hat{y}_{t+h|t} = \ell_t + hb_t + s_{t+h-m(k+1)}$$

$$\ell_t = \alpha(y_t - s_{t-m}) + (1-\alpha)(\ell_{t-1} + b_{t-1})$$

$$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$$

$$s_t = \gamma(y_t - \ell_{t-1} - b_{t-1}) + (1-\gamma)s_{t-m}$$

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Holt-Winters additive method

$$\hat{y}_{t+h|t} = \ell_t + hb_t + s_{t-m+h_m^+}$$

$$\ell_t = \alpha(y_t - s_{t-m}) + (1-\alpha)(\ell_{t-1} + b_{t-1})$$

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Holt-Winters additive method

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\begin{aligned}
\hat{y}_{t+h|t} &= \ell_t + hb_t + s_{t-m+h_m^+} \\
\ell_t &= \alpha(y_t - s_{t-m}) + (1-\alpha)(\ell_{t-1} + b_{t-1}) \\
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$$s_t = \gamma(y_t - \ell_{t-1} - b_{t-1}) + (1-\gamma)s_{t-m}$$

$$h_m^+ = \lfloor (h-1) \mod m \rfloor + 1$$

Holt-Winters multiplicative method

Holt-Winters multiplicative method

$$\hat{y}_{t+h|t} = (\ell_t + hb_t)s_{t-m+h_m^+}$$

$$\ell_t = \alpha(y_t/s_{t-m}) + (1-\alpha)(\ell_{t-1} + b_{t-1})$$

$$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$$

$$s_t = \gamma(y_t/(\ell_{t-1} + b_{t-1})) + (1-\gamma)s_{t-m}$$

- Most textbooks use $s_t = \gamma(y_t/\ell_t) + (1-\gamma)s_{t-m}$
- We optimize for α, β^*, γ, ℓ_0, b_0, s_0, $s_{-1}, \dots, s_{1-m}$.

Holt-Winters multiplicative method

$$
\begin{aligned}
\hat{y}_{t+h|t} &= (\ell_t + hb_t)s_{t-m+h_m^+} \\
\ell_t &= \alpha(y_t/s_{t-m}) + (1-\alpha)(\ell_{t-1} + b_{t-1}) \\
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s_t &= \gamma(y_t/(\ell_{t-1} + b_{t-1})) + (1-\gamma)s_{t-m}
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- We optimize for α, β^*, γ, ℓ_0, b_0, s_0, $s_{-1}, \dots, s_{1-m}$.

Damped Holt-Winters method

Damped Holt-Winters multiplicative method

$$
\begin{aligned}
\hat{y}_{t+h|t} &= [\ell_t + (1 + \phi + \phi^2 + \cdots + \phi^{h-1})b_t]s_{t-m+h_m^+} \\
\ell_t &= \alpha(y_t/s_{t-m}) + (1-\alpha)(\ell_{t-1} + \phi b_{t-1}) \\
b_t &= \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)\phi b_{t-1} \\
s_t &= \gamma[y_t/(\ell_{t-1} + \phi b_{t-1})] + (1-\gamma)s_{t-m}
\end{aligned}
$$

- This is often the single most accurate forecasting method for seasonal data.

A confusing array of methods?

- All these methods can be confusing!
- How to choose between them?
- The ETS framework provides an automatic way of selecting the best method.
- It was developed to solve the problem of automatically forecasting pharmaceutical sales across thousands of products.

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Outline

Exponential smoothing methods

Trend Component		Seasonal Component: N (None)	Seasonal Component: A (Additive)	Seasonal Component: M (Multiplicative)
N	(None)	N,N	N,A	N,M
A	(Additive)	A,N	A,A	A,M
A_d	(Additive damped)	A_d,N	A_d,A	A_d,M
M	(Multiplicative)	M,N	M,A	M,M
M_d	(Multiplicative damped)	M_d,N	M_d,A	M_d,M

Exponential smoothing methods

Trend Component		Seasonal Component N (None)	A (Additive)	M (Multiplicative)
N	(None)	**N,N**	N,A	N,M
A	(Additive)	A,N	A,A	A,M
A_d	(Additive damped)	A_d,N	A_d,A	A_d,M
M	(Multiplicative)	M,N	M,A	M,M
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(N,N): Simple exponential smoothing

Exponential smoothing methods

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A_d	(Additive damped)	A_d,N	A_d,A	A_d,M
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(N,N): Simple exponential smoothing

(A,N): Holt's linear method

Exponential smoothing methods

Trend Component		Seasonal Component N (None)	A (Additive)	M (Multiplicative)
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(N,N): Simple exponential smoothing
(A,N): Holt's linear method
(A,A): Additive Holt-Winters' method

Exponential smoothing methods

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There are 15 separate exponential smoothing methods.

R functions

- `ses()` implements method (N,N)
- `holt()` implements methods (A,N), (A_d,N), (M,N), (M_d,N)
- `hw()` implements methods (A,A), (A_d,A), (A,M), (A_d,M), (M,M), (M_d,M).

R functions

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- `holt()` implements methods (A,N), (A_d,N), (M,N), (M_d,N)

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Outline

1. Exponential smoothing methods so far

2. Holt-Winters' seasonal method

3. Taxonomy of exponential smoothing methods

4. **Exponential smoothing state space models**

Exponential smoothing

- Until recently, there has been no stochastic modelling framework incorporating likelihood calculation, prediction intervals, etc.

- Ord, Koehler & Snyder (JASA, 1997) and Hyndman, Koehler, Snyder and Grose (IJF, 2002) showed that all ES methods (including non-linear methods) are optimal forecasts from innovations state space models.

- Hyndman et al. (2008) provides a comprehensive and up-to-date survey of area.

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- The forecast package implements the state space framework.

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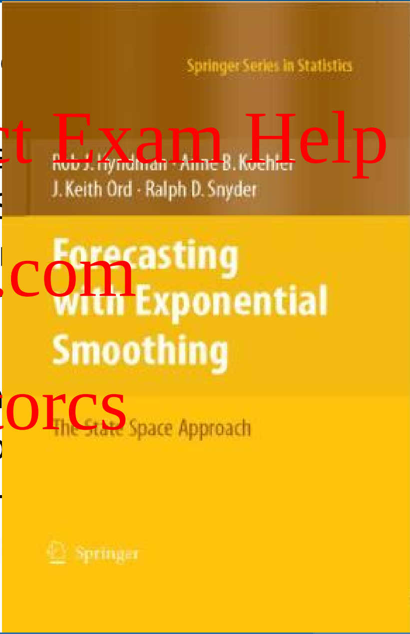

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A	(Additive)	A,N	A,A	A,M
A_d	(Additive damped)	A_d,N	A_d,A	A_d,M
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General notation ETS *(Error,Trend,Seasonal)*

Exponential smoothing

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Exponen**T**ial **S**moothing

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General notation **ETS**(*Error,Trend,Seasonal*)
Exponen**T**ial **S**moothing

ETS(A,N,N): Simple exponential smoothing with additive errors

Exponential smoothing

Trend Component		**Seasonal Component** N (None)	A (Additive)	M (Multiplicative)
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A	(Additive)	**A,N**	A,A	A,M
A_d	(Additive damped)	A_d,N	A_d,A	A_d,M
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General notation **ETS**(*Error,Trend,Seasonal*)

Exponen**T**ial **S**moothing

ETS(A,A,N): Holt's linear method with additive errors

Exponential smoothing

Trend Component		Seasonal Component N (None)	A (Additive)	M (Multiplicative)
N	(None)	N,N	N,A	N,M
A	(Additive)	A,N	**A,A**	A,M
A_d	(Additive damped)	A_d,N	A_d,A	A_d,M
M	(Multiplicative)	M,N	M,A	M,M
M_d	(Multiplicative damped)	M_d,N	M_d,A	M_d,M

General notation **ETS**(*Error*,*Trend*,*Seasonal*)
Exponen**T**ial **S**moothing

ETS(A,A,A): Additive Holt-Winters' method with additive errors

Exponential smoothing

Trend Component		Seasonal Component N (None)	Seasonal Component A (Additive)	Seasonal Component M (Multiplicative)
N	(None)	N,N	N,A	N,M
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A_d	(Additive damped)	A_d,N	A_d,A	A_d,M
M	(Multiplicative)	M,N	M,A	M,M
M_d	(Multiplicative damped)	M_d,N	M_d,A	M_d,M

General notation **ETS**(*Error,Trend,Seasonal*)

Exponen**T**ial **S**moothing

ETS(M,A,M): Multiplicative Holt-Winters' method with multiplicative errors

Exponential smoothing

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N	(None)	N,N	N,A	N,M
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A_d	(Additive damped)	**A_d,N**	A_d,A	A_d,M
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General notation **ETS**(*Error,Trend,Seasonal*)
Exponen**T**ial **S**moothing

ETS(A,A_d,N): Damped trend method with additive errors

Exponential smoothing

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A_d	(Additive damped)	A_d,N	A_d,A	A_d,M
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General notation **ETS**(*Error*,*Trend*,*Seasonal*)

Exponen**T**ial **S**moothing

There are 30 separate models in the ETS framework

Innovations state space models

SES

$$\hat{y}_{t+1|t} = \ell_t$$

$$\ell_t = \alpha y_t + (1 - \alpha)\ell_{t-1}$$

Innovations state space models

SES

$$\hat{y}_{t+1|t} = \ell_t$$

$$\ell_t = \alpha y_t + (1 - \alpha)\ell_{t-1}$$

If $\varepsilon_t = y_t - \hat{y}_{t-1|t}$

$\sim \text{NID}(0, \sigma^2)$, then

ETS(A,N,N)

$$\begin{aligned} y_t &= \ell_{t-1} + \varepsilon_t \\ \ell_t &= \alpha y_t + (1 - \alpha)\ell_{t-1} \\ &= \ell_{t-1} + \alpha\varepsilon_t \end{aligned}$$

Innovations state space models

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$$\hat{y}_{t+1|t} = \ell_t$$
$$\ell_t = \alpha y_t + (1-\alpha)\ell_{t-1}$$

If $\varepsilon_t = y_t - \hat{y}_{t-1|t} \sim \text{NID}(0, \sigma^2)$, then

ETS(A,N,N)

$$y_t = \ell_{t-1} + \varepsilon_t$$
$$\ell_t = \alpha y_t + (1-\alpha)\ell_{t-1}$$
$$= \ell_{t-1} + \alpha\varepsilon_t$$

If $\varepsilon_t = (y_t - \hat{y}_{t-1|t})/\hat{y}_{t-1|t} \sim \text{NID}(0, \sigma^2)$, then

ETS(M,N,N)

$$y_t = \ell_{t-1}(1 + \varepsilon_t)$$
$$\ell_t = \alpha y_t + (1-\alpha)\ell_{t-1}$$
$$= \ell_{t-1}(1 + \alpha\varepsilon_t)$$

Innovations state space models

SES

$$\hat{y}_{t+1|t} = \ell_t$$

$$\ell_t = \alpha y_t + (1-\alpha)\ell_{t-1}$$

If $\varepsilon_t = y_t - \hat{y}_{t-1|t}$ $\sim \text{NID}(0, \sigma^2)$, then

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$$y_t = \ell_{t-1} + \varepsilon_t$$

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If $\varepsilon_t = (y_t - \hat{y}_{t-1|t})/\hat{y}_{t-1|t}$ $\sim \text{NID}(0, \sigma^2)$, then

ETS(M,N,N)

$$y_t = \ell_{t-1}(1 + \varepsilon_t)$$

$$\ell_t = \alpha y_t + (1-\alpha)\ell_{t-1}$$
$$= \ell_{t-1}(1 + \alpha\varepsilon_t)$$

All exponential smoothing methods can be written using analogous state space equations.

Innovations state space models

Example: **Holt-Winters' multiplicative seasonal method**

ETS(M,A,M)

$$
\begin{aligned}
y_t &= (\ell_{t-1} + b_{t-1}) s_{t-m}(1 + \varepsilon_t) \\
\ell_t &= (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t) \\
b_t &= b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t \\
s_t &= s_{t-m}(1 + \gamma\varepsilon_t)
\end{aligned}
$$

where $\beta = \alpha\beta^*$.

Innovations state space models

- All the methods can be written in this state space form.
- Prediction intervals can be obtained by simulating many future sample paths.
- For many models, the prediction intervals can be obtained analytically as well.
- Additive and multiplicative versions give the same point forecasts.
- Estimation is handled via maximizing the likelihood of the data given the model.

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Akaike's Information Criterion

$$\text{AIC} = -2\log(\text{Likelihood}) + 2p$$

where p is the number of estimated parameters in the model.

- *Minimizing* the AIC gives the best model for prediction.

AIC corrected (for small sample bias)

$$\text{AIC}_C = \text{AIC} + \frac{2(p+1)(p+2)}{n-p}$$

Schwartz' Bayesian IC

$$\text{BIC} = \text{AIC} + p(\log(n) - 2)$$

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Exponential smoothing

From Hyndman et al. (2008):

- Apply each of 30 methods that are appropriate to the data. Estimate parameters and initial values using MLE.

- Select best method using AIC.
- Produce forecasts using best method.

- Obtain prediction intervals using underlying state space model.

Method performed very well in M3 competition.

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```r
fit <- ets(ausbeer)
fit2 <- ets(ausbeer,model="AAA",damped=FALSE)
fcast1 <- forecast(fit, h=20)
fcast2 <- forecast(fit2, h=20)
```



```r
ets(y, model="ZZZ", damped=NULL, alpha=NULL,
    beta=NULL, gamma=NULL, phi=NULL,
    additive.only=FALSE,
    lower=c(rep(0.0001,3),0.80),
    upper=c(rep(0.9999,3),0.98),
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Exponential smoothing

```
> fit
ETS(M,Md,M)

  Smoothing parameters:
    alpha = 0.1776
    beta  = 0.0454
    gamma = 0.1947
    phi   = 0.9549

  Initial states:
    l = 263.8531
    b = 0.9997
    s = 1.1856 0.9109 0.8612 1.0423

  sigma:  0.0356

     AIC     AICc      BIC
2272.549 2273.444 2302.715
```

Exponential smoothing

```
> fit2
ETS(A,A,A)

  Smoothing parameters:
    alpha = 0.2079
    beta  = 0.0304
    gamma = 0.2483

  Initial states:
    l = 255.6559
    b = 0.5687
    s = 52.3841 -27.1061 -37.6758 12.3978

  sigma:  15.9053

     AIC     AICc      BIC
2312.768 2313.481 2339.583
```

Exponential smoothing

`ets()` function

- Automatically chooses a model by default using the AIC, AICc or BIC.
- Can handle any combination of trend, seasonality and damping
- Produces prediction intervals for every model
- Ensures the parameters are admissible (equivalent to invertible)
- Produces an object of class `ets`.

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Exponential smoothing

`ets` objects

- **Methods:** `coef()`, `plot()`, `summary()`, `residuals()`, `fitted()`, `simulate()` and `forecast()`

- `plot()` function shows time plots of the original time series along with the extracted components (level, growth and seasonal).

Exponential smoothing

ets objects

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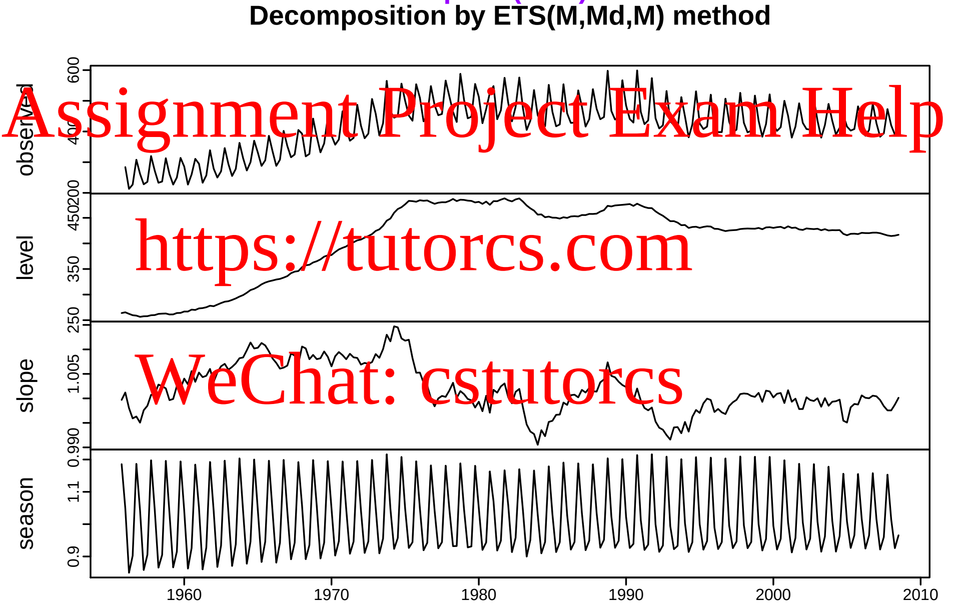
- plot() function shows time plots of the original time series along with the extracted components (level, growth and seasonal).

Exponential smoothing

```
plot(fit)
```



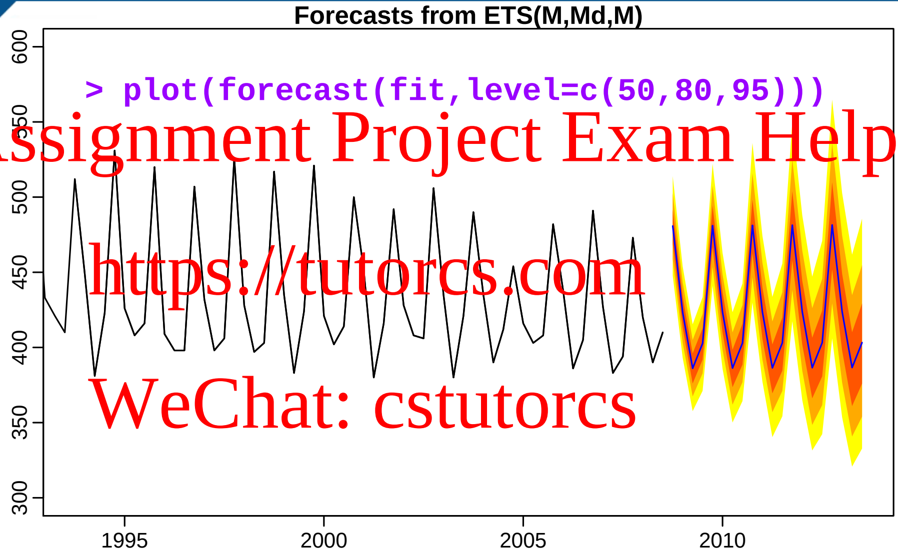
Goodness-of-fit


```
> accuracy(fit)
      ME     RMSE      MAE      MPE     MAPE     MASE
 0.17847 15.48781 11.77800  0.07204  2.81921  0.20705
```

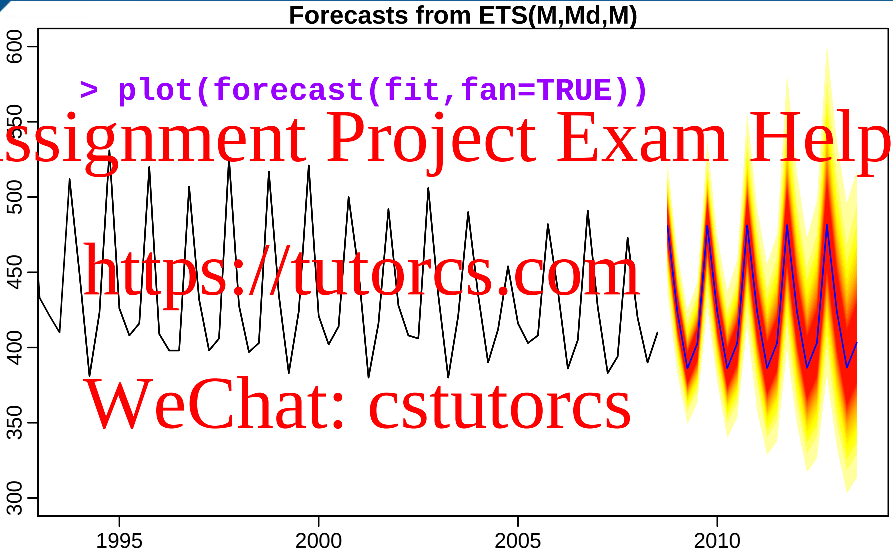


```
> accuracy(fit2)
      ME     RMSE      MAE      MPE     MAPE     MASE
-0.11711 15.90526 12.18930 -0.03765  2.91255  0.21428
```

Forecast intervals



Forecast intervals




```
> plot(forecast(fit,fan=TRUE))
```

Exponential smoothing

`ets()` function also allows refitting model to new data set.


```
> usfit <- ets(usnetelec[1:45])
> test <- ets(usnetelec[46:55], model = usfit)
```



```
> accuracy(test)
      ME      RMSE       MAE       MPE      MAPE      MASE
-3.35419 58.02763 43.85545 -0.07524  1.18483  0.52452
```



```
> accuracy(forecast(usfit,10), usnetelec[46:55])
       ME       RMSE        MAE        MPE       MAPE       MASE
  40.7034    61.2075    46.3246     1.0980     1.2620     0.6776
```

Unstable models

- ETS(M,M,A)
- ETS(M,M_d,A)
- ETS(A,N,M)
- ETS(A,A,M)
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- ETS(A,M_d,A)
- ETS(A,M_d,M)

In practice, the models work fine for short- to medium-term forecasts provided the data are strictly positive.

Forecastability conditions

```
ets(y, model="ZZZ", damped=NULL, alpha=NULL,
    beta=NULL, gamma=NULL, phi=NULL,
    additive.only=FALSE,
    lower=c(rep(0.0001,3),0.80),
    upper=c(rep(0.9999,3),0.98),
    opt.crit=c("lik","amse","mse","sigma"),
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```

The magic `forecast()` function

- `forecast` returns forecasts when applied to an `ets` object (or the output from many other time series models).

- If you use `forecast` directly on data, it will select an ETS model automatically and then return forecasts.

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