MONASH University

Information Technology



# FIT1006 Business Information Analysis



## Lecture 10 Probability (cont…)

# Topics covered:

- Independent and co events.
- Probability trees.
- Bayes' Theorem.
- *Notes on background concepts for Bayes' Theorem.
- *Notes on background mathematics for probability distributions.

https://flux.qa (Feedcode: 3J6KGV)

# Question 1. Medical testing problem

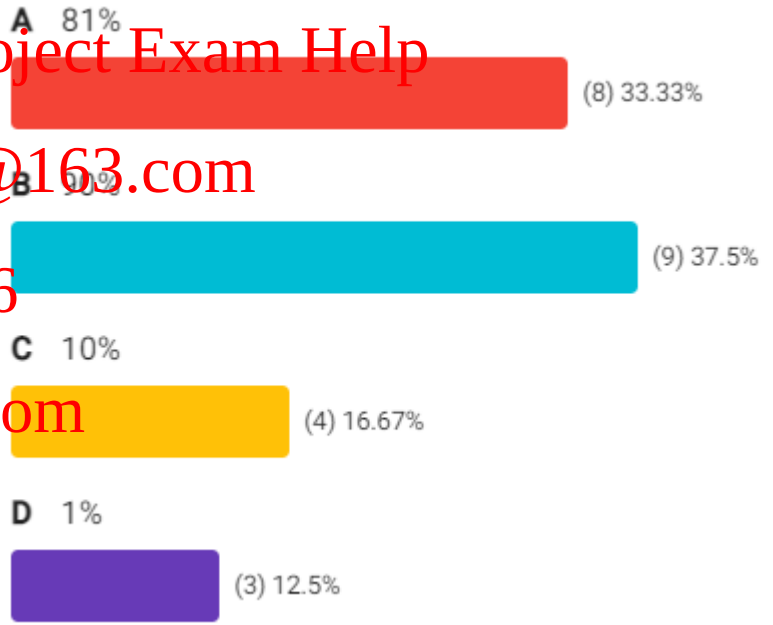
- The probability of d[illegible] the population is 1%
- If a person has disease X the probability they will test positive is 90%
- If a person doesn't have disease X the probability they will test positive is 9%

(Adapted from: Gigerenzer, G. et al, Knowing your chances. *Sci Am Mind*, April/May 2009)

- I have just been tested and the test is positive. What is the probability I have disease X?

Who is correct?

We will answer this later (Slide 13)



# Today's lecture



- How do we incorporate additional information in to our estimates of probability?
- For the motivating problem, without knowing the test results there is a 0.01 chance that a person selected at random has disease X.
- If that person tested positive then we would expect the probability the person had the disease to increase. Similarly, it should decrease for a negative test.
- Bayes' theorem gives us a tool for calculating these probabilities.

# Bayes' Theorem

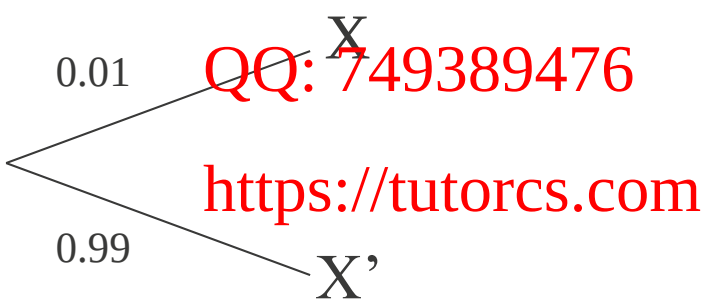


- Bayes' Theorem is a method for updating the probability of an event when the occurrence of that event is affected (conditional) on another event.
- The stages of a Bayesian problem:
  1. Start with the *Prior* probability – this is the probability of an event in the absence of any other information. Sometimes called the *state of nature*.
  2. Receive additional information as *conditional probabilities*.
  3. Update the Prior probability using the additional information to determine the *Posterior* probability.

# Prior probability

- Draw the mot problem as a probability tree.
- Use X to indicate person has disease,
- ’ denotes complement.
- The first stage of the tree reflects the *state of nature*.



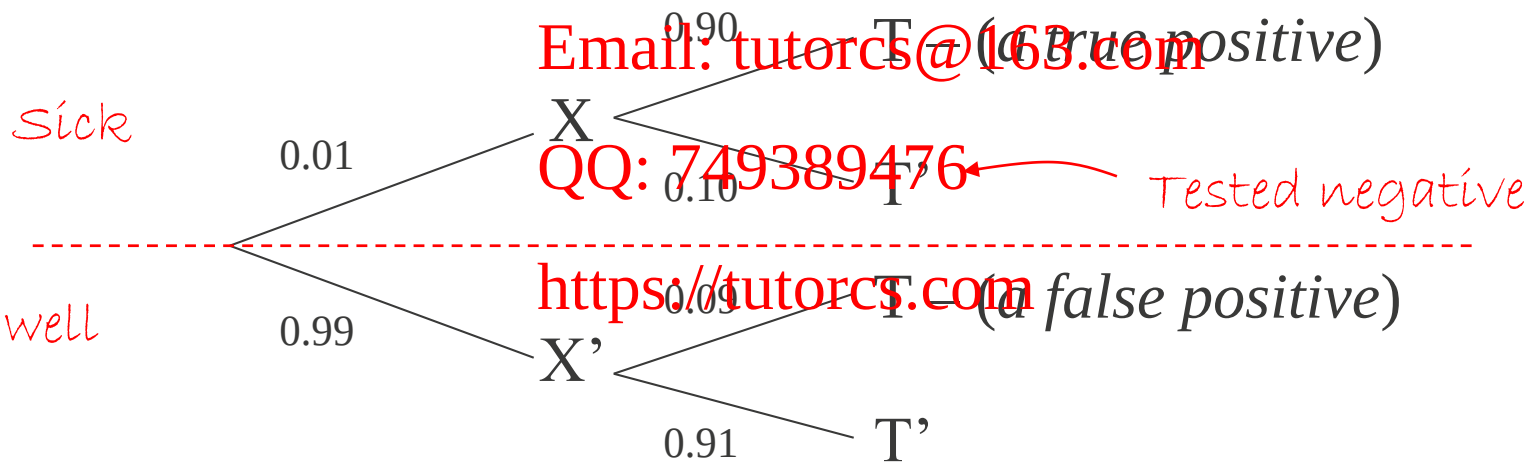


# Conditional probabilities



- Use T to indic[illegible]ositive test result.
- Recall that: P[illegible] 0.90 and P(T|X') = 0.09.
- That is, the probability of positive result is conditional on a person's disease state.



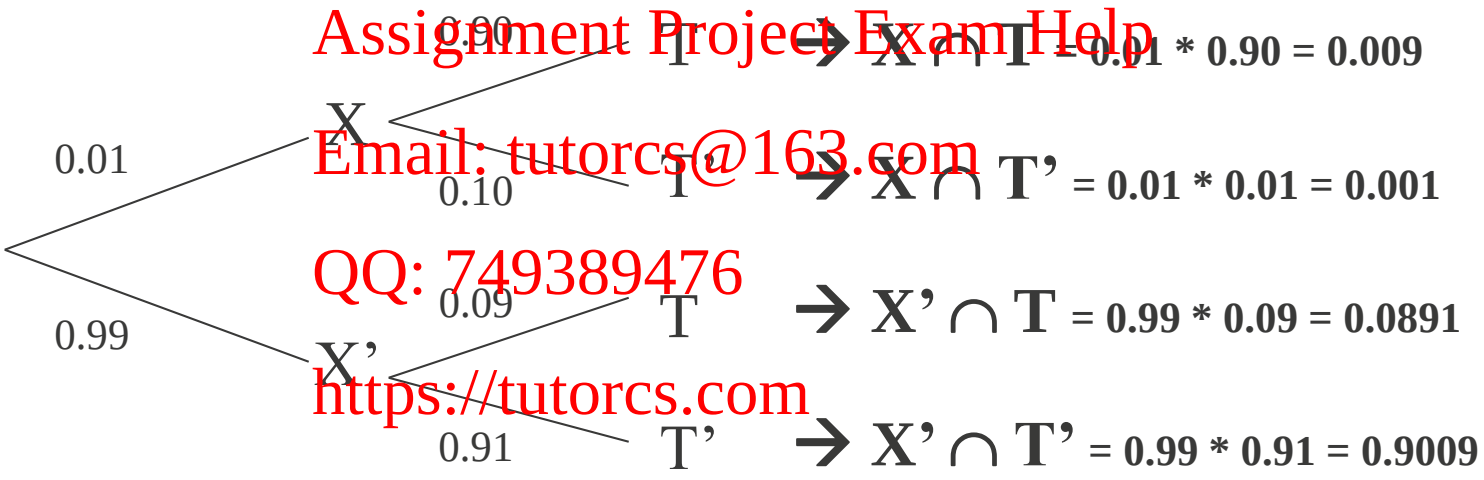


# Joint probabilities



- Probabilities for each of the 4 situations corresponding to disease status *and* test outcome are evaluated. These are the *joint* probabilities.

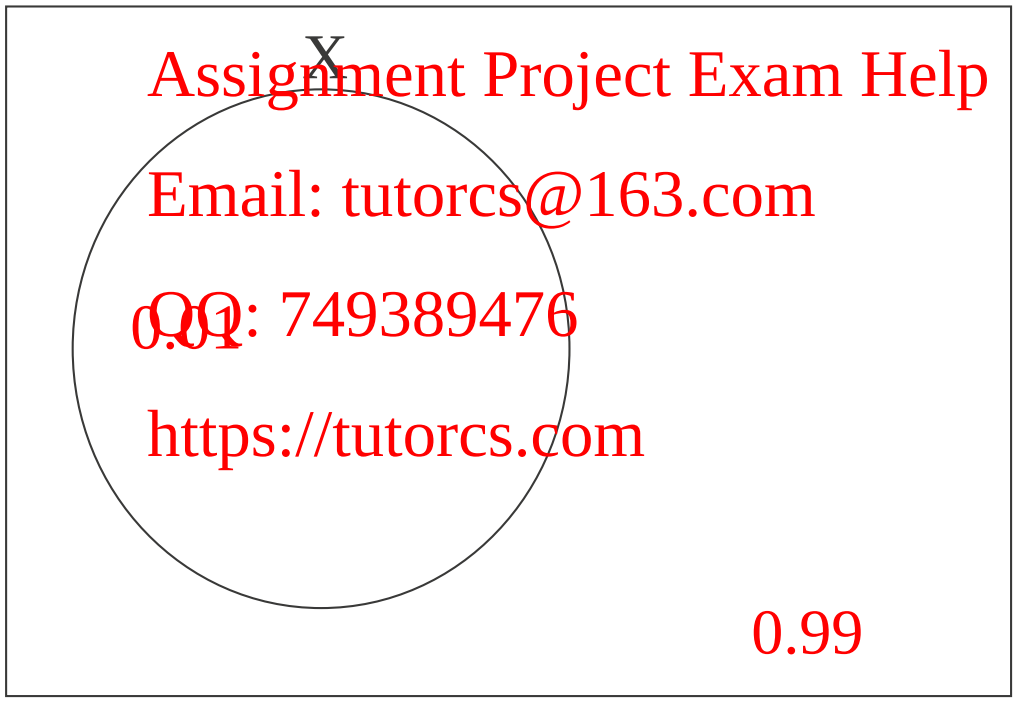




# As a Venn Diagram

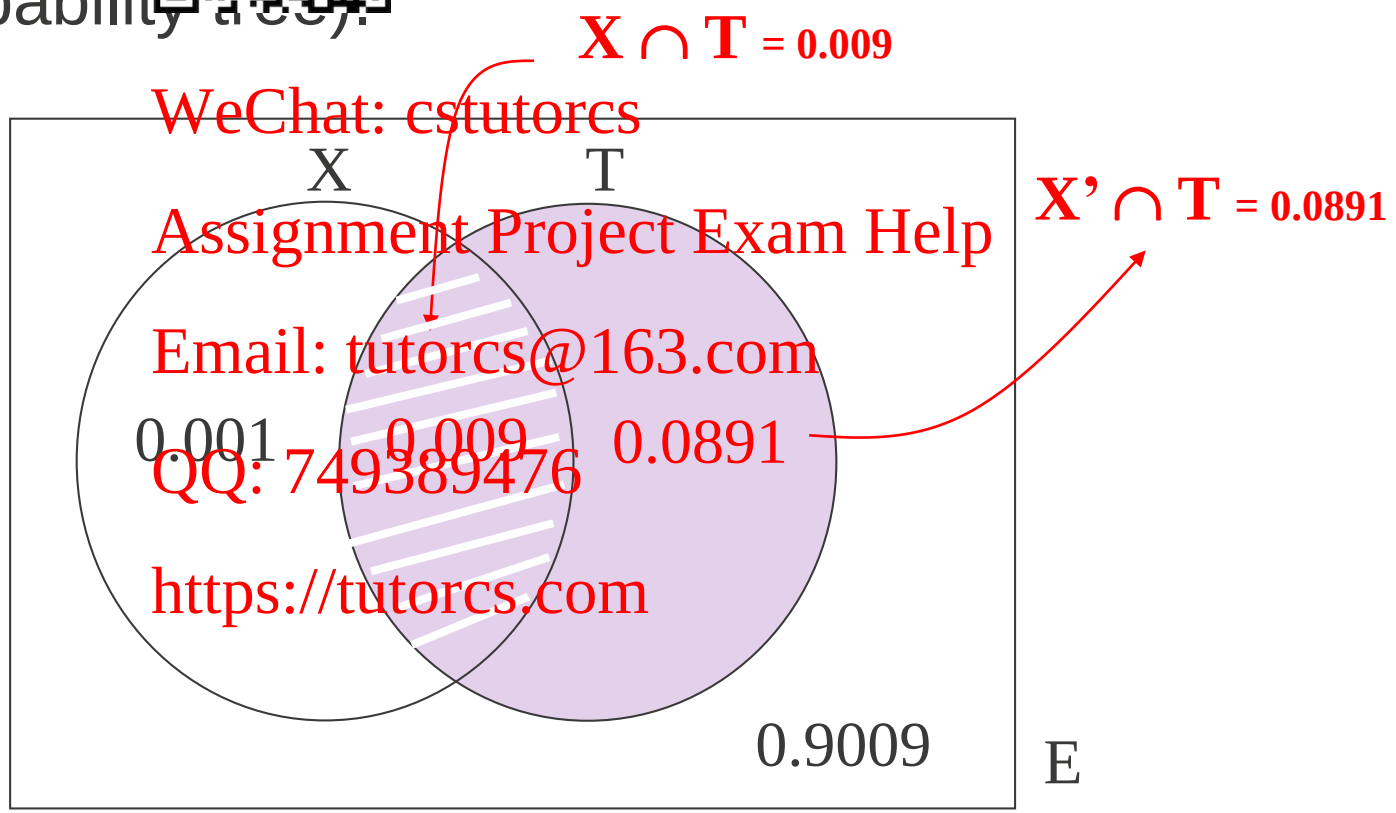
- Venn Diagram showing disease status. Without a test, the probability a person chosen at random has disease is 0.01 .
- This is the *prior* probability of X.





# Redraw… showing test results

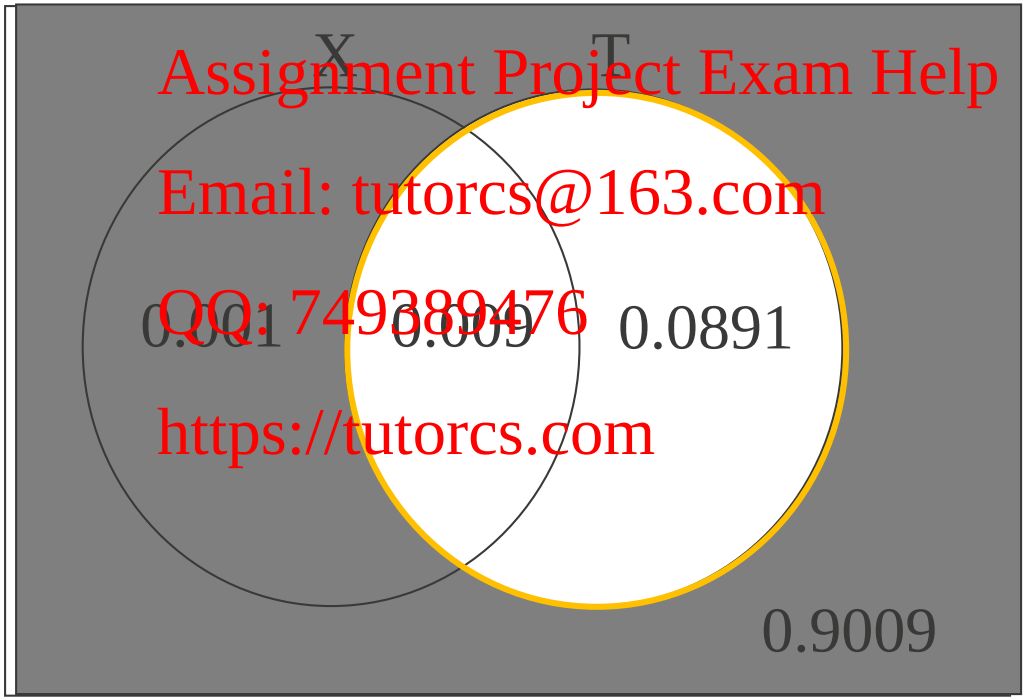
As a Venn Diag wing disease status and test result after on of joint probabilities (using probability axioms):



# … positive test results only

- As a Venn Di[illegible]howing disease status and test result.
- Conditioning on a positive test result means that only a subset of the original problem is relevant.





# Posterior probability



0.009 + 0.0891

The probability [illegible]itive test result (T) is 0.0981.

The probability [illegible]g disease after positive test is P(X|T) = 0.0917. The *posterior* probability of X.



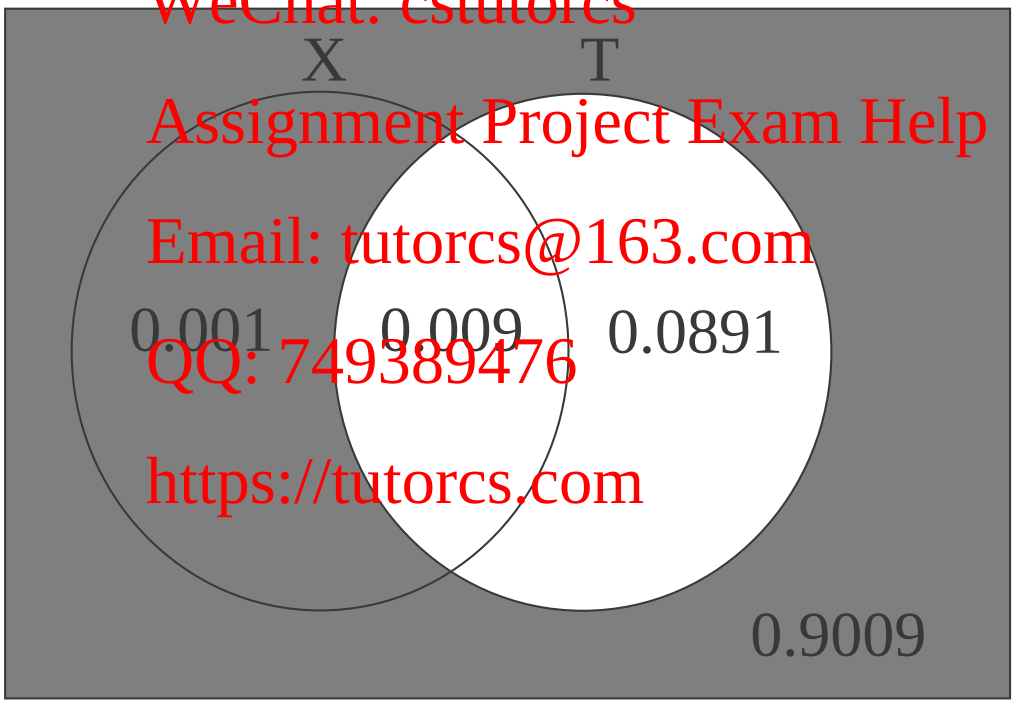
0.009 / 0.0981 = 0.0917

# … calculated

P(T) is 0.009 + (0.0981.

P(X|T) = 0.009 / = 0.0917. Approx 10% chance of having disease after a positive test result.





# Summary



Begin with state

re

P(X) = 0.01 *prior probability*

Incorporate additional information



P(T|X) = 0.9 *conditional probability*



P(T|X') = 0.09 *conditional (false positive)*



Calculate



$P(X \cap T)$ and $P(X' \cap T)$ *joint probabilities*



$P(T) = P(X \cap T) + P(X' \cap T)$

$P(X|T) = P(X \cap T)/P(T)$ *posterior probability*

# Summary



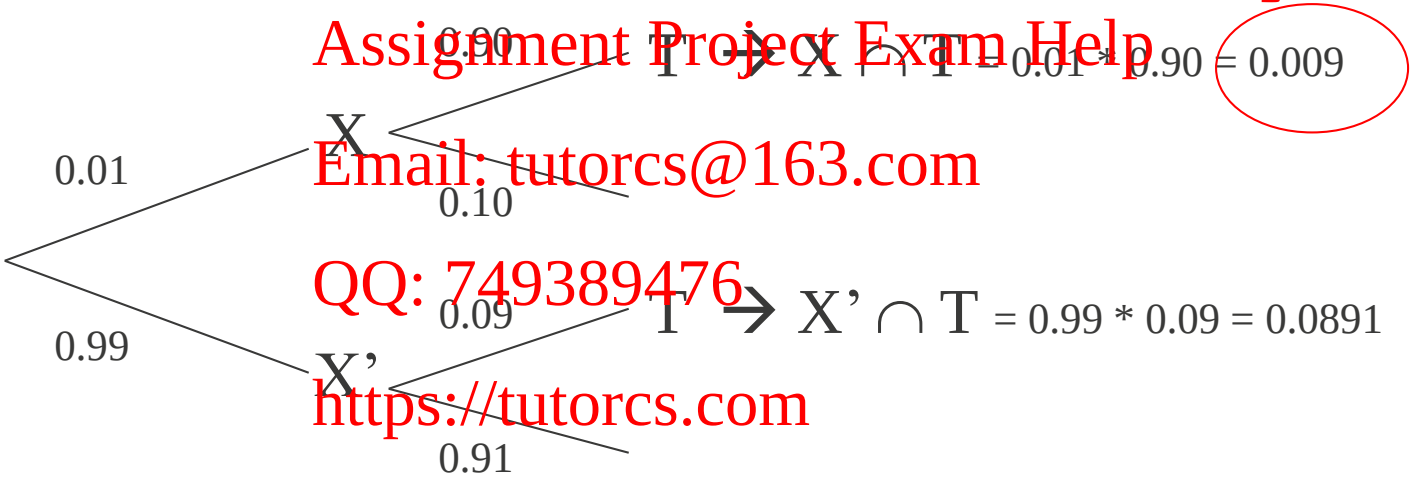
As a probability

P(T) is 0.009 + ( 0.0981.

P(X|T) = 0.009 / 0.0981 = 0.0917.

P(X ∩ T) + P(X' ∩ T)





# Summary



As a table:

P(T) is 0.009 + (0.0981.

P(X|T) = 0.009 / 0.0981 = 0.0917.



| Disease | Prior | Test | Conditional | Joint | Posterior after +ve Test |
|---|---|---|---|---|---|
| Has | 0.01 | +ve | 0.9 | 0.0090 | 0.0917 |
| | | -ve | 0.1 | 0.0010 | |
| Not | 0.99 | +ve* | 0.09 | 0.0891 | 0.9083 |
| | | -ve | 0.91 | 0.9009 | |
| | | | Pr +ve = | 0.0981 | |

*Note: we could adapt this method to calculate posterior probability after a negative test.*

https://flux.qa (Feed code: SJ6KGV)

## Question 2

Using the Venn diagram below P(B) =



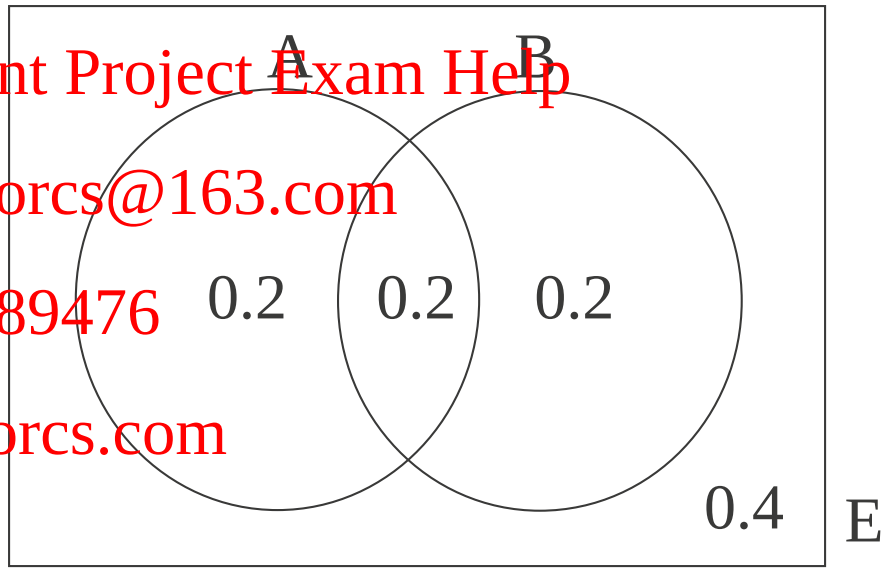

A. 0.2

✓ B. 0.4

C. 0.5

D. 0.6

E. None of these

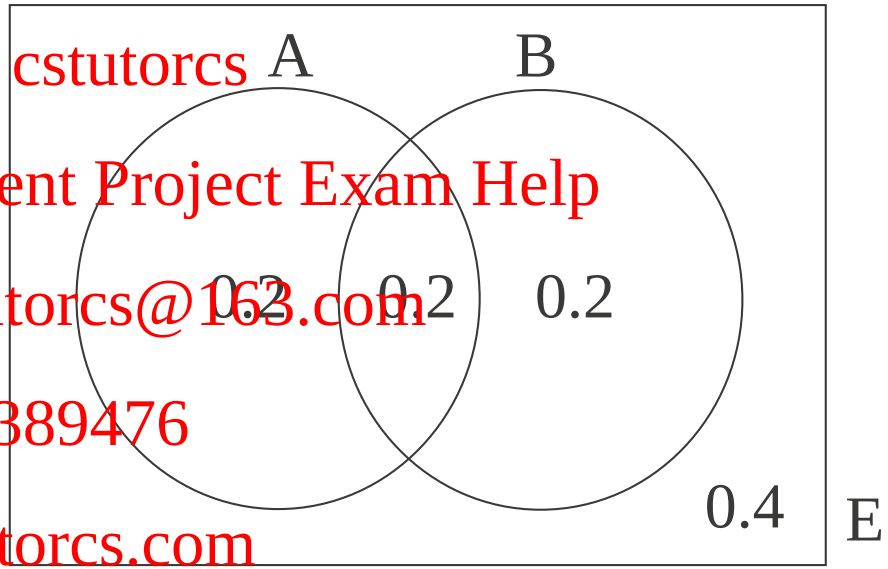


https://flux.qa (Feed code: SJ6KGV)



## Question 3

Using the Venn diagram below P(A|B) =




A. 0.2

B. 0.4

✓ C. 0.5

D. 0.6

E. None of these.



$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad = \frac{0.2}{0.4} = 0.5$$

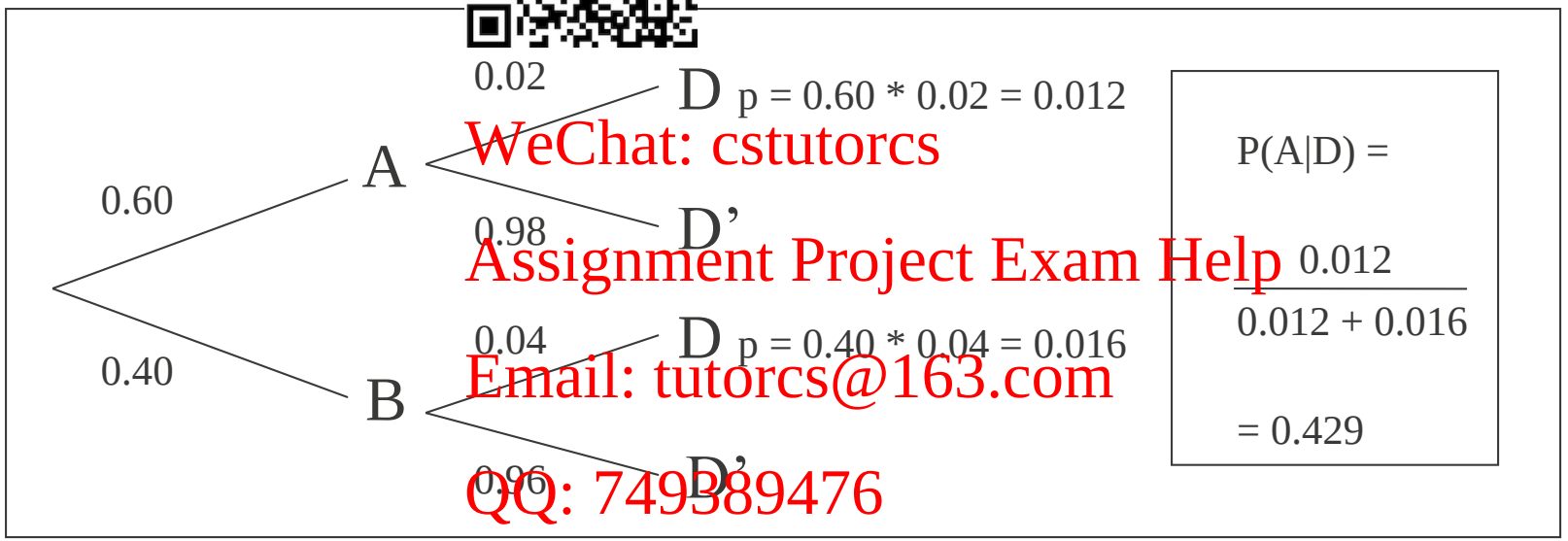
# Example 1



- A plant has two machines.
- Machine A produces 60% of the output with the fraction defective being 0.02.
- Machine B produces 40% of the output with the fraction defective being 0.04.
- The quality control inspector finds a defective part awaiting pack and ship.
- What is the probability that it was produced by Machine A?

# Answer





# Example 2

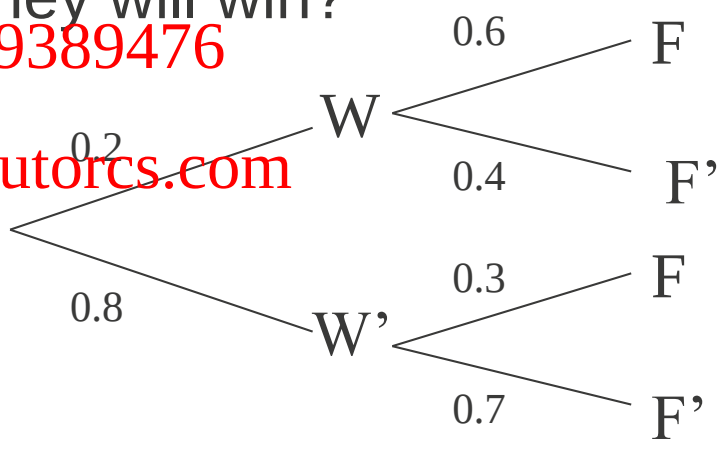


- The local footba as a 20% chance of winning any given match.
- The local commentator is 'quite' good at predicting winners: he correctly forecasts the team will win (F) in 60% of cases and correctly predicts the team will lose in 70% of cases.
- What is the probability the team will win when the commentator predicts they will win?

P(F|W) =0.6

P(w|F)??





https://flux.qa (Feed code: SJ6KGV)



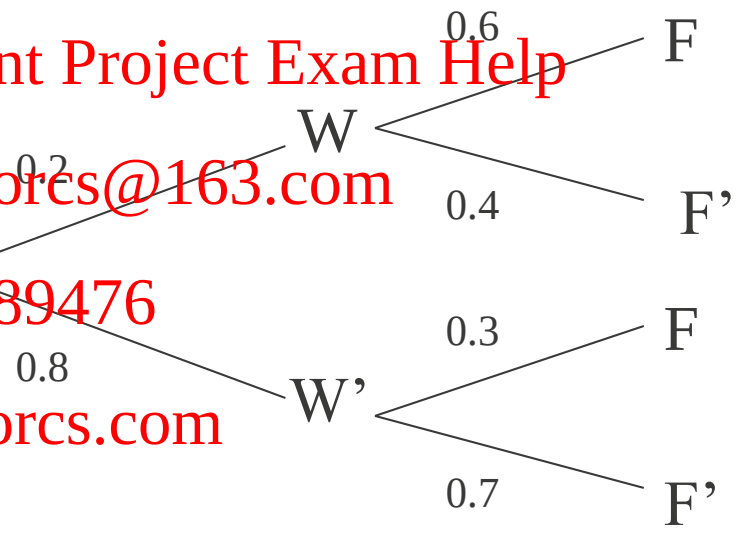
## Question 4

For the football team shown below, without knowing the forecast, the team's probability of winning is:



P(W)??

- ✓ A. 0.2
- B. 0.3
- C. 0.6
- D. 0.8
- E. None of these.



https://flux.qa (Feed code: SJ6KGV)



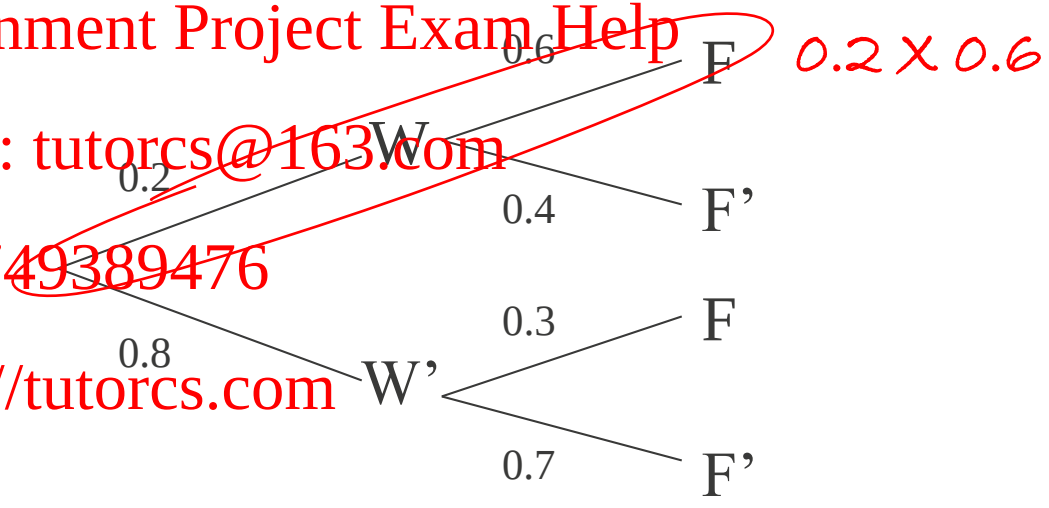
## Question 5

For the football [illegible]m shown below, the probability the team wins and the commentator forecasts they win is:



P(W ∩ F)??

- ✓ A. 0.12
- B. 0.08
- C. 0.24
- D. 0.56
- E. None of these.




0.2 X 0.6

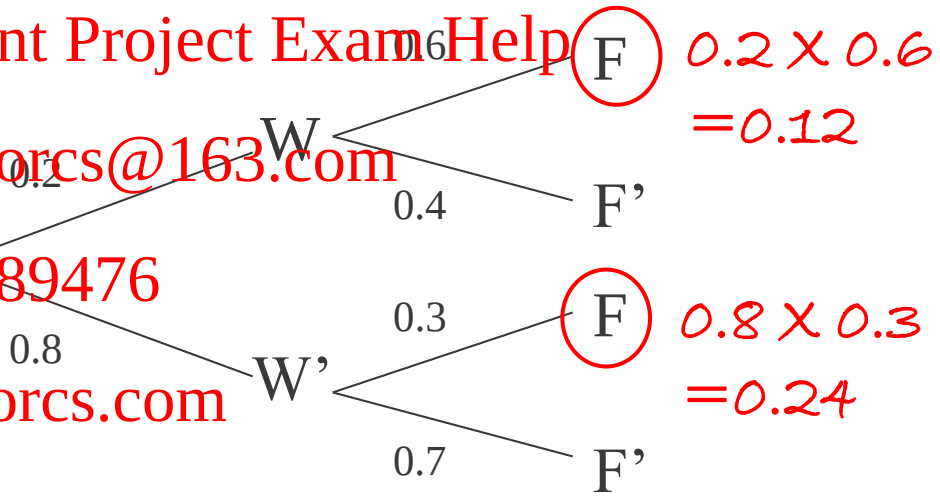
https://flux.qa (Feed code: SJ6KGV)



## Question 6

For the football team shown below, the probability the commentator forecasts the team will win is:






0.2 X 0.6 =0.12

0.8 X 0.3 =0.24

A. 0.12

B. 0.20

C. 0.24

✓ D. 0.36

0.12 + 0.24

E. None of these.

# Question 7

For the football team shown below, the probability the team wins when the commentator forecasts they win is:



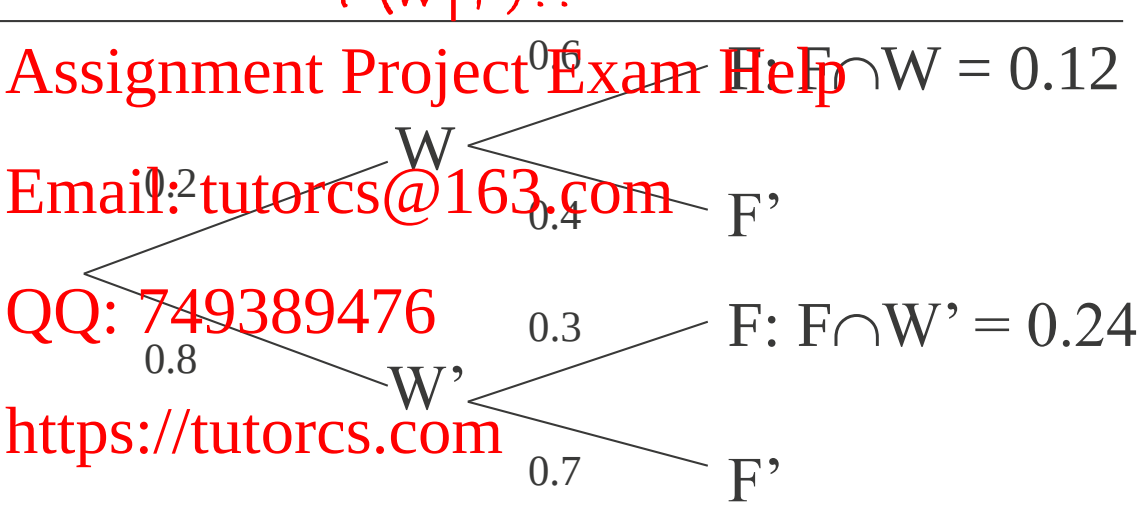
A. 0.12

B. 0.20

C. 0.24

✓ D. 0.33

E. 0.67.




$$P(\mathrm{W} \mid \mathrm{F}) = \frac{P(\mathrm{F} \cap \mathrm{W})}{P(\mathrm{F})} = \frac{0.12}{0.12+0.24} = 0.33$$

# Answer



| Match | Prior | | Conditional | Joint | Posterior after predicting win. |
|---|---|---|---|---|---|
| Win | 0.2 | Win | 0.6 | 0.1200 | 0.3333 |
| | | Lose | ~~0.4~~ | ~~0.0800~~ | |
| Lose | 0.8 | Win | 0.3 | 0.2400 | 0.6667 |
| | | Lose | ~~0.7~~ | ~~0.5600~~ | |
| | | | **Pr Win** | 0.3600 | |

# Reading:

- For you to read
- Formal statement of Bayes' Theorem.
- The following slides take you through the components of the formal statement.

# Independent Events

- If two events are independent then the probability of one event occurring has no effect on the other.
- Thus, for independent events A and B,
  - P(B|A) = P(B) and P(A|B) = P(A)
  - P(A∩B) = P(A)*P(B)
- For tosses of a coin, let A be the outcome of a head with the first toss and B the outcome of a head with the second toss.
- Then P(A∩B) = P(A)*P(B) = 0.5*0.5 = 0.25

# Mutually Exclusive Events



- Mutually exclusive events are events which cannot all occur in the same
  - For example, a person tosses a coin, the outcome of head and tails is mutually exclusive.
  - A person may choose product A or B or C.
  - A person may be infected or not infected.
- For mutually exclusive events A and B $P(A \cap B) = 0$.

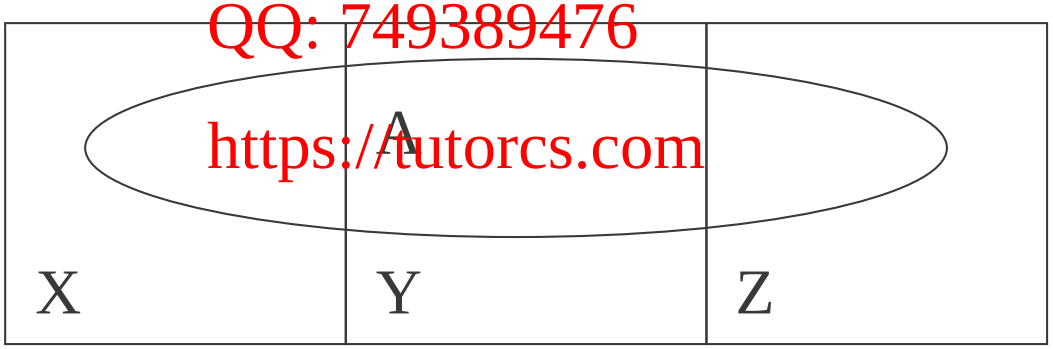
# Collectively Exhaustive Events

- Collectively exh events cover the whole sample space.
  - For example Head or Tails
  - Infected or Not Infected.

# The Law of Total Probability

- X, Y and Z are *mutually exclusive* and *collectively exhaustive* because they do not intersect and together they cover the total sample space (or universe).
- Let A = (A∩X)∪(A∩Y)∪(A∩Z). Then by the law of total probability:

$$P(A) = P(A \cap X) + P(A \cap Y) + P(A \cap Z).$$



# Conditional Probabilities

- From earlier work on Venn and Tree Diagrams the following is true:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

*thus*

$$P(A \cap B) = P(B|A)P(A)$$

# Bayes' Theorem

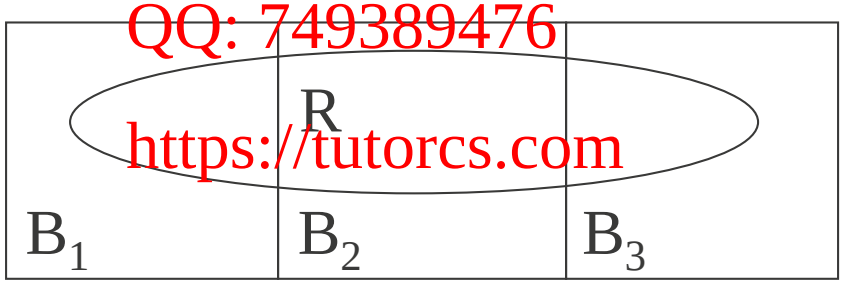


- Formal statement

For an event with outcomes $B_1, B_2 \ldots B_n$, and event $R$. what is the probability that outcome $B_x$ occurred given that event $R$ has occurred?

$$P(B_x \mid R) = \frac{P(B_x)P(R \mid B_x)}{\sum_{j=1}^{n} P(B_j)P(R \mid B_j)} = \frac{P(B_x \cap R)}{P(R)}$$





# Probability Distributions

- For many commonly occurring situations we don't have to create a probability distribution from scratch but instead use well understood mathematical models. Next week we cover three of the most important:
- Binomial and Poisson Distributions. Both are discrete, where the random variable can take on natural (counting) number values, and the
- Normal Distribution, the most important continuous distribution.

# Background for next week…



- *If you intend do culations by hand you will need to know s sic mathematical functions:*




- Exponential
- Factorial
- Combinatorial

# Exponents



We describe the no as $a$ raised to the power of $b$.

This is defined form $a^b = \underbrace{(a \cdot a \cdot a \cdot a \cdot a \cdot a \cdots a)}_{b \text{ times}}$



$a$ and $b$ can take on non - integer values and we often use the number '$e$' as a base. $e \approx 2.7182\ldots$



Using a calculator you should be able to calculate expressions such as: $6^2$, $e^2$, $e^{-0.5}$, $0.3^{10}$, $0.994^{10}$, $e^{-0.44}$, $8^8$, $-0.001^2$, $e^{-20}$.





You should be able to use the $\left[y^x\right]$ and $\left[e^x\right]$ or $\left[\exp\right]$ keys on your calculator.

# Factorial



Factorial notation, st to understand with an example.

$5! = 5 \times 4 \times 3 \times 2 \times 1! = 1$, *and* $0! = 1$ by convention.



Formally, $n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdots 3 \cdot 2 \cdot 1$

# Combinations



We use the notation ${}^{n}C_{x}$ to describe the number of different ways we can select $x$ a time from a group of $n$ objects.

$${}^{n}C_{x} = \frac{n!}{x!(n-x)!}$$



The number of ways that we can select 4 students from a class of 10 students is given by ${}^{10}C_{4} = \frac{10!}{4!(10-4)!} = \frac{10!}{4!\,6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210$

The number of ways that we can select 3 cards from a deck of 52 cards is given by ${}^{52}C_{3} = \frac{52!}{3!(52-3)!} = \frac{52!}{3!\,49!} = \frac{52 \cdot 51 \cdot 50}{3 \cdot 2 \cdot 1} = 22100$

# Reading/Questions

- Reading:
  - 7th Ed. Sect[illegible]2 – 6.6.
- Questions:
  - 7th Ed. Questions 6.18, 6.22, 6.32, 6.45, 6.52, 6.77, 6.79, 6.81, 6.84.