# FIT2014 Theory of Computation



## cture 3

## cate Logic



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# Lecture overview

- Statements with variables
- Predicates
- Definitions and terminology
- Existential quantifier
- Universal quantifier
- Doing logic with quantifiers

# Statements with variables

Consider these statements:

- $W$ is negative.
- $X$ passed this subject.
- $Y = Z$.

These do not yet have truth values.

The variables are **free**, in that no value is (yet) given to them.

You can, if you wish, assign values to them.

Each set of values you give to the variables creates a different specific proposition.

# Statements with variables



For example, in the statement ative, the variable $W$ is free.
If we assign values to it, we car pecific propositions:

| | |
|---|---|
| $\vdots$ | $\vdots$ |
| $-2$ is negative | True |
| $-1$ is negative | True |
| $0$ is negative | False |
| $1$ is negative | False |
| $2$ is negative | False |
| $\vdots$ | $\vdots$ |

# Predicates

**Definitions**



A **predicate** is a statement with variables such that, for any values of the variables, it is either True or False.

- i.e., it becomes a propositi

We treat each variable as ranging over some **domain**.



- For the predicate $W$ is negative, we've just been using the domain $\mathbb{Z}$.



The variables of a predicate are also called its **arguments**.





A predicate is called $k$-**ary** if it has $k$ arguments. Special cases: unary, binary, ternary, ...



Some alternative terminology:

| # arguments | terminology |
|---|---|
| 1 | *property* |
| $\geq 2$ | *relation* |

# Predicates: examples



| # args. | example | domain |
|---|---|---|
| 1 | isNegative($X$) | $\mathbb{N}$ |
| 1 | isNegative($X$) | $\mathbb{Z}$ |
| 2 | $=$ *[always* … *]* | objects |
| 2 | $X < Y$ | numbers |
| 2 | isMotherOf($X, Y$), meaning "$X$ is the mother of $Y$" | people |
| 3 | gives($X, Y, Z$), meaning "$X$ gives $Y$ to $Z$" | $X, Z$ are people, $Y$ is a gift |
| $\vdots$ | $\vdots$ | $\vdots$ |



Predicates may be thought of as *truth-valued functions*, i.e., functions whose value is always in {True, False}.

# Functions



We'll also use functions whose values aren't necessarily just True or False.

| # args. | example | domain | codomain |
|---|---|---|---|
| 1 | $\sqrt{X}$ | non[illegible]numbers | numbers |
| 1 | motherOf($X$) | people | people |
| 2 | $X + Y$ | numbers | numbers |
| $\vdots$ | $\vdots$ | $\vdots$ | |



Functions with no arguments are called **constants**.

- Examples: 5, Annie, ...

A function's arguments can be: constants; variables; functions.

# Existential quantifier



There's a fly in my soup. [illegible] is a fly) $\land$ ($X$ is in my soup).



There exists $W$ : $W$ is negative. $\exists W : W < 0$



If domain of $W$ is $\mathbb{N}$: it's False. $\exists W \in \mathbb{N} : W < 0$



If domain of $W$ is $\mathbb{Z}$: it's True. $\exists W \in \mathbb{Z} : W < 0$

# Existential quantifier

Someone did it. $\exists X : X$ did it.



It's *sort-of* like a disjunction . .

$\cdots\cdots \lor$ (Annie did it) $\lor$ (Edwar (Henrietta did it) $\lor$ (Radhanath did it) $\lor \cdots\cdots$

. . . *but:*

- often the domain of a variable is infinite;
- keep the variables, they're useful.



The variables are now **bound**.
You can no longer give specific values to the variables to create specific propositions.
The quantifier has turned the statement into a single proposition about the entire domains of the variables.



Quantifiers can only be used with *variables*.
Using them with constant objects *makes no sense:* $\exists 5$, $\exists$Annie.

# Existential quantifier

Some computer is human. *i.e.,* There exists a human computer.



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If the domain of $X$ is { computers }
Predicate:

- human($X$): $X$ is human.



$$\exists X : \text{human}(X)$$



But what if the domain of $X$ is { everything on Earth } ?



Predicates:

- human($X$): $X$ is human.
- computer($X$): $X$ is a computer.

# Existential quantifier

Some computer is human. *i.e.*, There exists a human computer.



---

If the domain of $X$ is $\{$ everything on earth $\}$

Predicates:

- human($X$): $X$ is human.
- computer($X$): $X$ is a computer.



*Correct:*

$$\exists X : \text{computer}(X) \wedge \text{human}(X)$$

- "There exists something that is both computer and human."
- "There exists a human computer."
- "Some computer is human."

*Incorrect:*

$$\exists X : \text{computer}(X) \Rightarrow \text{human}(X)$$

- "There exists something which is not a computer or is human."
- "There exists something which is not both a computer and non-human."
- "Not everything is a nonhuman computer."

# Universal quantifier



Everyone can pass this subject. r every $X$ : $X$ can pass this subject. $\forall X :$ canPass$(X)$.

All numbers are interesting. $\forall X :$ $X$ is interesting. $\forall X :$ isInteresting$(X)$.

- True — and we'll prove it!



For all $W$ : $W$ is negative $\forall W : W < 0$. False.



Again, the variables are now **bound**.

# Universal quantifier

Every computer is human.



---

If the domain of $X$ is $\{$ compu
Predicate:

- human($X$): $X$ is human.



$$\forall X : \text{human}(X)$$



But what if the domain of $X$ is $\{$ everything on Earth $\}$ ?



Predicates:

- human($X$): $X$ is human.
- computer($X$): $X$ is a computer.

# Universal quantifier

Every computer is human.



---

If the domain of $X$ is $\{$ everyt[illegible]arth $\}$

Predicates:

- human($X$): $X$ is human.
- computer($X$): $X$ is a computer.



*Incorrect:*

$\forall X : \text{computer}(X) \land \text{human}(X)$

- "Everything is both computer and human."
- "Everything is a human computer."

*Correct:*

$\forall X : \text{computer}(X) \Rightarrow \text{human}(X)$

- "For everything, if it's a computer, then it's human."
- "Everything that's a computer is also human."
- "Every computer is human."

# Multiple quantifiers



Thinking of graphs . . .

Suppose we have a predicate a meaning that vertices $X$ and $Y$ are adjacent.

Some two vertices are not adj

$$\exists X\, \exists Y : \neg(X = Y) \wedge \neg \mathrm{adj}(X, Y).$$
$$\exists (X, Y) : \neg(X = Y) \wedge \neg \mathrm{adj}(X, Y).$$



Every pair of vertices is adjacent.



$$\forall X\, \forall Y : \neg(X = Y) \Rightarrow \mathrm{adj}(X, Y).$$
$$\forall (X, Y) : \neg(X = Y) \Rightarrow \mathrm{adj}(X, Y).$$



Some vertex is adjacent to all other vertices.

$$\exists X\, \forall Y : \neg(X = Y) \Rightarrow \mathrm{adj}(X, Y).$$



Every vertex has a neighbour.

$$\forall X\, \exists Y : \mathrm{adj}(X, Y).$$

# Multiple quantifiers



**Six degrees of separation**

Suppose we have a predicate knows$(X, Y)$ meaning that person $X$ knows person $Y$.



It has been claimed that, in the human social network,
the distance between any two people is at most 6.



Exercise: write this claim in predicate logic, using just the predicate knows.

# Doing logic with quantifiers



If we know that

$$\ldots \text{blah}(X)$$

and obj is any specific object (i[illegible]nain of $X$),
then we can deduce that

$$\text{blah}(\text{obj})$$



We have:

$$(\forall X \ \text{blah}(X)) \Rightarrow \text{blah}(\text{obj})$$



Also:

$$\text{blah}(\text{obj}) \Rightarrow (\exists X \ \text{blah}(X))$$

# Doing logic with quantifiers



$\forall X\, (p(X) \wedge q(X))$ is logically equivalent to $(\forall X\ p(X)) \wedge (\forall X\ q(X))$

$\exists X\, (p(X) \vee q(X))$ is logically equivalent to $(\exists X\ p(X)) \vee (\exists X\ q(X))$



What about the logical relationship between ...



$\forall X\, (p(X) \vee q(X))$ and $(\forall X\ p(X)) \vee (\forall X\ q(X))$ ...?



...etc

# Relationship between quantifiers



$\neg\,\forall Y$ means the same as $\exists Y\,\neg$

"Not all dogs are happy." is the same as . . . "There exists an unhappy dog."



$$
\begin{array}{rll}
 & \neg\,\forall X\,(\text{dog}(X) \Rightarrow \text{happy}(X)) & \text{Not all dogs are happy} \\
= & \exists X\,\neg(\text{dog}(X) \Rightarrow \text{happy}(X)) & \\
= & \exists X\,\neg(\neg\text{dog}(X) \vee \text{happy}(X)) & \text{(see last lecture)} \\
= & \exists X\,(\neg\neg\text{dog}(X) \wedge \neg\text{happy}(X)) & \text{(by De Morgan)} \\
= & \exists X\,(\text{dog}(X) \wedge \neg\text{happy}(X)) & \text{There exists an unhappy dog}
\end{array}
$$

# Relationship between quantifiers



Similarly,

$\neg\, \exists Y$ means the same as $\forall Y\, \neg$



$\neg\, \forall Y\, \neg$ means the same as ______



$\neg\, \exists Y\, \neg$ means the same as ______