Monash University
Faculty of Information Technology

FIT2014 Theory of Computation

Lecture 2

Propositional Logic

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Lecture overview

- Propositions
- Logical operations
- Tautologies, logical equivalence
- Disjunctive Normal Form
- Conjunctive Normal Form
- Representing logical statements

Propositions

Definition: A **proposition** is a statement which is either *true* or *false*.

Examples

$1 + 1 = 2$ — a proposition which is true.

The earth is flat. — a proposition which is false.

It will rain tomorrow. — a proposition.

'Twas brillig, and the slithy toves
did gyre and gimble in the wabe — *not* a proposition.

From: Lewis Carroll, *Through the Looking Glass, and What Alice Found There*, Macmillan, London, 1871.

Come and work for us! — *not* a proposition.

This statement is false. — *not* a proposition.

For brevity, a proposition may be given a name, which has a **truth value**, True or False.
For example, let X be the proposition $1 + 1 = 2$. Then the truth value of X is True.

Logical operations

Not	$\neg$	$(\sim, \overline{\ }, -)$
And	$\wedge$	$(\&)$
Or	$\vee$	
Implies	$\Rightarrow$	$(\rightarrow)$
Equivalence	$\Leftrightarrow$	$(\leftrightarrow)$

A **connective** is a binary logical operation. E.g.: $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$.

Negation

P: You have prepared for r ’s tutorial.
$\neg P$: You have not prepared week’s tutorial.

Other notation: $\sim P, \quad \overline{P}, \quad -P$

Truth table:

P	$\neg P$
F	**T**
T	**F**

Conjunction

P	Radhanath was a cor
Q	Radhanath was a per
$P \wedge Q$	Radhanath was a computer and a person.

Radhanath Sikdar (1813–1870)
`http://news.bbc.co.uk/2/hi/south_asia/3193576.stm`

Truth table:

P	Q	$P \wedge Q$
F	F	**F**
F	T	**F**
T	F	**F**
T	T	**T**

Disjunction

P	I will study FIT3155 Advanced Data Structures & Algorithms.
Q	I will study MTH3170 Network Mathematics.
$P \vee Q$	I'll study FIT3155 **or** MTH3170.
	I'll study *at least one* of FIT3155 and MTH3170.

Disjunction is sometimes called *inclusive-OR*, and sometimes written as $+$.

Truth table:

P	Q	$P \vee Q$
F	F	**F**
F	T	**T**
T	F	**T**
T	T	**T**

De Morgan's Laws

$$\neg(P \vee Q) = \text{[illegible]}\, Q$$
$$\neg(P \wedge Q) = \text{[illegible]}\, Q$$

Can be proved using truth tables:

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$
F	F	F	T	T	T	T
F	T	T	F	T	F	F
T	F	T	F	F	T	F
T	T	T	F	F	F	F

Augustus De Morgan (1806–1871)

https://mathshistory.st-andrews.ac.uk/Biographies/De_Morgan/

Conditional

P	Stars are visible.
Q	The sun has set.

$P \Rightarrow Q$	**If** stars are visible **then** the sun has set.
	Stars being visible **implies** the sun has set.
	Stars are visible **only if** the sun has set.
	Stars are visible is **sufficient** for the sun to have set.
$Q \Leftarrow P$	same as $P \Rightarrow Q$

Also called *implication*.

Conditional

Truth table:

P	Q	$P \Rightarrow Q$
F	F	**T**
F	T	**T**
T	F	**F**
T	T	**T**

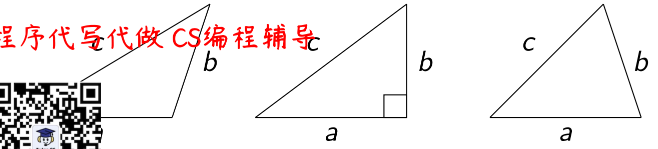
Grace Hopper (1906–1992)
https://www.cs.vassar.edu/history/hopper

P Grace is a COBOL expert.
Q Grace can program.

$P \Rightarrow Q$ **If** Grace is a COBOL expert **then** she can program.

Biconditional

P	The triangle is right-angled.
Q	The side lengths satisfy $a^2 + b^2 = c^2$.



$a^2 + b^2 < c^2$ $\quad$ $a^2 + b^2 = c^2$ $\quad$ $a^2 + b^2 > c^2$

$P \Leftrightarrow Q$ The triangle is right-angled **if and only if** $a^2 + b^2 = c^2$.

The triangle being right-angled is a **necessary and sufficient condition** for $a^2 + b^2 = c^2$.

$Q \Leftrightarrow P$ $a^2 + b^2 = c^2$ is a **necessary and sufficient condition** for the triangle being right-angled.

Truth table:

P	Q	$P \Leftrightarrow Q$
F	F	**T**
F	T	**F**
T	F	**F**
T	T	**T**

Tautologies, logical equivalence

Definitions

A **tautology** is a statement that is always true.

In other words, the right-hand column of its truth table has every entry True.

Two statements P and Q are **logically equivalent** if their truth tables are identical.

In other words, $P \Leftrightarrow Q$ is a tautology.

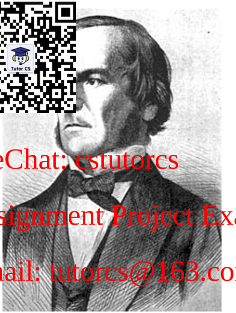
Examples

$$\begin{array}{rll}
\neg\neg P & \text{is logically equivalent to} & P \\
\neg(P \vee Q) & \text{is logically equivalent to} & \neg P \wedge \neg Q \\
\neg(P \wedge Q) & \text{is logically equivalent to} & \neg P \vee \neg Q \\
P \Rightarrow Q & \text{is logically equivalent to} & \neg P \vee Q \\
P \Leftrightarrow Q & \text{is logically equivalent to} & (P \Rightarrow Q) \wedge (P \Leftarrow Q) \\
 & \text{and to} & (\neg P \vee Q) \wedge (P \vee \neg Q)
\end{array}$$

These can all be proved using truth tables.

We usually denote logical equivalence by "$=$". So we write $\neg\neg P = P$, etc.

History



George Boole (1815–1864)

https://mathshistory.st-andrews.ac.uk/Biographies/Boole/

Distributive Laws

$$
\begin{aligned}
P \wedge (Q \vee R) &= (P \wedge Q) \vee (P \wedge R) \\
P \vee (Q \wedge R) &= (P \vee Q) \wedge (P \vee R)
\end{aligned}
$$

Compare with ordinary algebra:

$$p \times (q + r) = (p \times q) + (p \times r)$$

but

$$p + (q \times r) \neq (p + q) \times (p + r)$$

Laws of Boolean algebra

$$\neg\neg P = P$$

$$
\begin{aligned}
\neg\text{True} &= \text{False} & \neg\text{False} &= \text{True}\\
P \wedge Q &= Q \wedge P & P \vee Q &= Q \vee P\\
(P \wedge Q) \wedge R &= P \wedge (Q \wedge R) & (P \vee Q) \vee R &= P \vee (Q \vee R)\\
P \wedge P &= P & P \vee P &= P\\
P \wedge \neg P &= \text{False} & P \vee \neg P &= \text{True}\\
P \wedge \text{True} &= \text{P} & P \vee \text{False} &= \text{P}\\
P \wedge \text{False} &= \text{False} & P \vee \text{True} &= \text{True}
\end{aligned}
$$

Distributive Laws

$$P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R) \qquad P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$$

De Morgan's Laws

$$\neg(P \vee Q) = \neg P \wedge \neg Q \qquad \neg(P \wedge Q) = \neg P \vee \neg Q$$

Disjunctive Normal Form (DNF)

X	Y	P	
F	F	**T**	$\neg X \wedge \neg Y$
F	T	**T**	$\neg X \wedge Y$
T	F	**F**	
T	T	**T**	$X \wedge Y$

$$P = \underbrace{\overbrace{(\neg X \wedge \neg Y)}^{\text{conjunction}} \vee \overbrace{(\neg X \wedge Y)}^{\text{conjunction}} \vee \overbrace{(X \wedge Y)}^{\text{conjunction}}}_{\text{disjunction}}$$

Exercise: simplify this as much as possible, using Boolean algebra.

Disjunctive Normal Form (DNF)

X	Y	Z	P	
F	F	F	**T**	$\neg X \land \neg Y \land \neg Z$
F	F	T	**F**	
F	T	F	**T**	$\neg X \land Y \land \neg Z$
F	T	T	**F**	
T	F	F	**F**	
T	F	T	**T**	$X \land \neg Y \land Z$
T	T	F	**T**	$X \land Y \land \neg Z$
T	T	T	**F**	

$$P = (\neg X \land \neg Y \land \neg Z) \lor (\neg X \land Y \land \neg Z) \lor (X \land \neg Y \land Z) \lor (X \land Y \land \neg Z)$$

Disjunctive Normal Form (DNF)

$$P \;=\; (\neg X \land \neg Y \land \neg Z) \lor (\text{[illegible]} \land \neg Z) \lor (\,X \land \neg Y \land \;Z) \lor (\,X \land \;Y \land \neg Z)$$

- A **literal** is an appearance of a variable in which it is either unnegated or negated just once.
  - Example: there are 12 literals in the above expression.
- A logical expression is in **DNF** if it is a **dis**junction of conjunctions of literals.
- Every logical expression is equivalent to one in DNF.
  - To see this: "just" use the truth table.
  - BUT this can be exponentially large (in # of variables).
- In effect, DNF enumerates all situations in which P is True.
- There is another Normal Form that is much more useful for us . . .

Conjunctive Normal Form (CNF)

- A logical expression is in **CNF** if it is a **con**junction of disjunctions of literals.
  - E.g.:

$$\underbrace{\overbrace{(P \vee Q)}^{\text{disjunction}} \wedge \overbrace{(P \vee \neg Q)}^{\text{disjunction}}}_{\text{conjunction}}$$

  - Each disjunction of literals is called a **clause**.
- Every logical expression is equivalent to one in CNF.
- One way to see this:

  Given P,
  find the DNF of *its negation*, $\neg P$,
  then negate it
  and use De Morgan's Laws.

- BUT it is usually *much faster*, and *much less error-prone*, to work directly from the stated conditions that P must satisfy.
- In this unit, CNF will be *much* more important than DNF.

Representing logical statements

Example:

You are planning a dinner party. Your guest list must have:

- at least one of: Harry, Ron, Hermione, Ginny

$$\text{Harry} \lor \text{Ron} \lor \text{Hermione} \lor \text{Ginny}$$

- Hagrid *only if* it also has Norberta

$$\text{Hagrid} \Rightarrow \text{Norberta} \qquad \ldots \textit{can rewrite as:} \qquad \neg\text{Hagrid} \lor \text{Norberta}$$

- none, or both, of Fred and George

$$\text{Fred} \Leftrightarrow \text{George} \qquad \ldots \textit{can rewrite as:} \qquad (\neg\text{Fred} \lor \text{George}) \land (\text{Fred} \lor \neg\text{George})$$

- no more than one of: Voldemort, Bellatrix, Dolores.

$$(\textit{not both } \text{Voldemort \& Bellatrix}) \land (\textit{not both } \text{Voldemort \& Dolores}) \land (\textit{not both } \text{Bellatrix \& Dolores})$$

$$(\neg\text{Voldemort} \lor \neg\text{Bellatrix}) \land (\neg\text{Voldemort} \lor \neg\text{Dolores}) \land (\neg\text{Bellatrix} \lor \neg\text{Dolores})$$

Representing logical statements

$$
\begin{aligned}
&(\text{Harry} \lor \text{Ron} \lor \text{Hermione} \lor \text{Gin} \ldots\\
&\quad\land\quad (\neg\text{Hagrid} \lor \text{Norberta})\\
&\quad\land\quad (\neg\text{Fred} \lor \text{George}) \quad\land\quad (\text{Fre} \ldots \text{orge})\\
&\quad\land\quad (\neg\text{Voldemort} \lor \neg\text{Bellatrix}) \quad\land\quad (\neg\text{Voldemort} \lor \neg\text{Dolores}) \quad\land\quad (\neg\text{Bellatrix} \lor \neg\text{Dolores})
\end{aligned}
$$

This is now in CNF.

Challenge: how long would an equivalent DNF expression be?

Reading

See Sipser, pp. 14–15, and top paragraph of p. 302.