Monash University
Faculty of Information Technology

FIT2014 Theory of Computation

Lecture 8

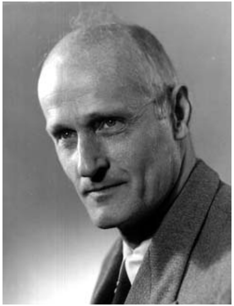
Kleene's Theorem. I.

Regexp $\longrightarrow$ NFA $\longrightarrow$ FA

slides by Graham Farr

Overview

- Questions
- Kleene's Theorem
- Convert Regular Expressions to NFA
- Convert NFA to FA
- *Next lecture:*
  Convert FA to Regular Expression

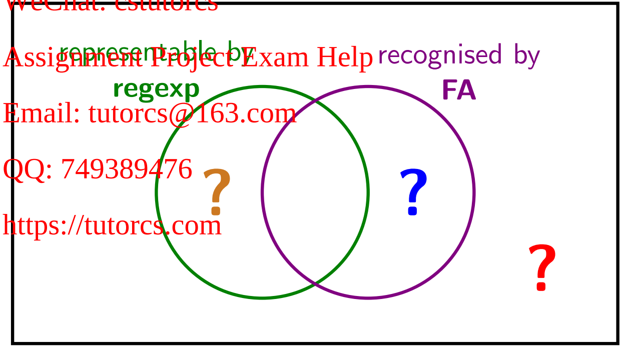


Stephen Cole Kleene (1909–1994)
`https://mathshistory.st-andrews.ac.uk/Biographies/Kleene/`

Questions

- Can every language which is represented by a **regular expression** be described by a **finite automaton**?
- Can every language which is described by a **finite automaton** be represented by a **regular expression**?
- Can every language be represented by a **regular expression** or a **finite automaton**?

{ all languages }



Kleene's Theorem

Theorem.
Any language which can be defi

- Regular Expressions
- Finite Automata
- Nondeterministic Finite Automata (NFA)
- Generalized Nondeterministic Finite Automata (GNFA)

can be defined by any of the other methods.

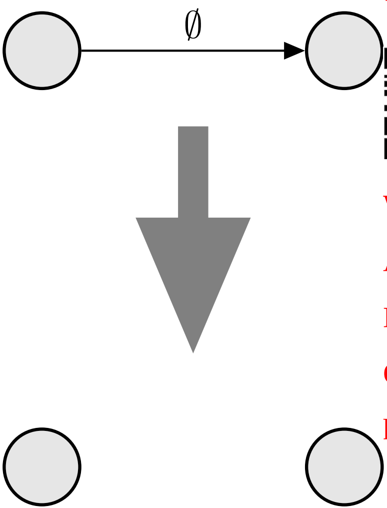
Kleene's Theorem

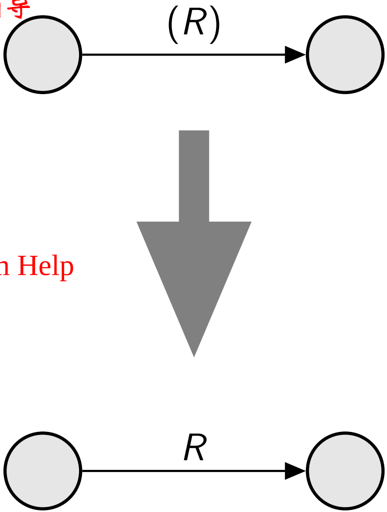
Converting Regular Expression to NFA

Start with:

Apply the following rules until all edges are labelled with a letter or ε:

Converting Regular Expression to NFA





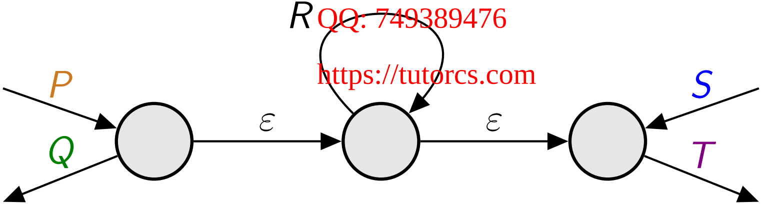
Converting Regular Expression to NFA

Converting Regular Expression to NFA

Converting Regular Expression to NFA

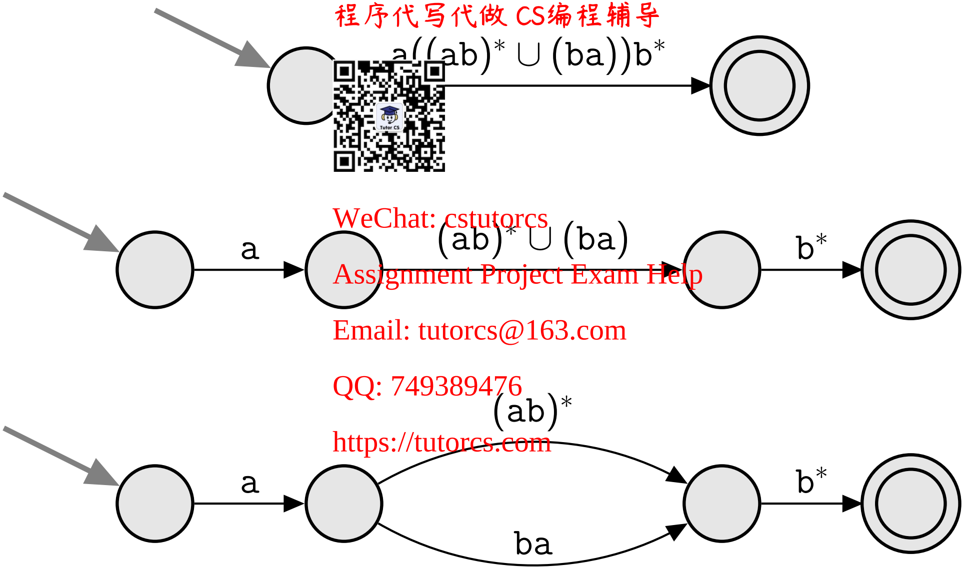
Here, you can't match PR^*Q or SR^*T, in general.

The two ε transitions are necessary, in general.

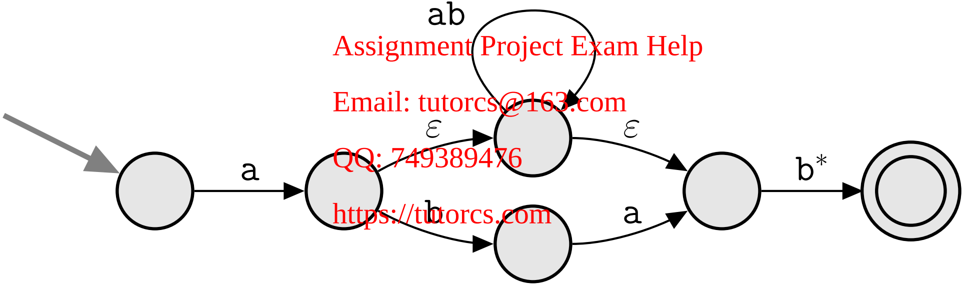


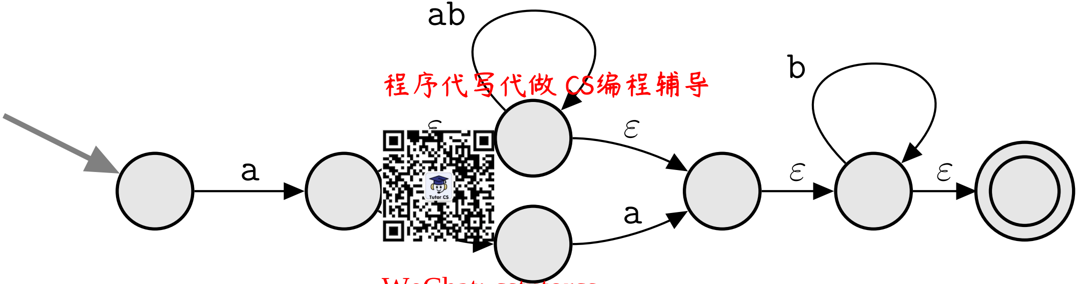
So the R loop cannot be at left node or right node.

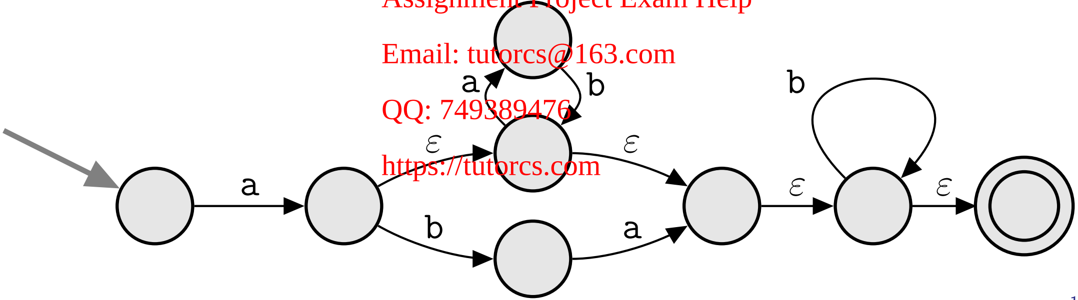
Converting Regular Expression to NFA. Example: $a((ab)^* \cup (ba))b^*$




(ab)*
a
a
b*


ab
ε
ε
a
b
a
b*


ab
b
a
ε
ε
a
ε
ε


a
b
b
a
ε
ε
b
a
ε
ε

Converting a NFA to a FA

Complexity?

How reversible is this construction?

Kleene's Theorem

Converting a NFA to a FA

In a FA:

- Any string w traces a uniq[illegible] starting from the Start State and ending at some unique state, which [illegible]endState(w).
- The string w is accepted i[illegible]e(w) is a Final State, otherwise it is rejected.
- endState(ε) $=$ Start State.

In a NFA:

- Any string w traces a set of paths, starting from the Start State and ending at some set of states, which we'll call endStates(w).

- The set might have zero, one or more members.

- The string w is accepted if endStates(w) contains a Final State, otherwise it is rejected.

- endStates(ε) $=$ $\{$ Start State $\}$ if there are no ε transitions.

Converting a NFA to a FA

$$\begin{aligned} \mathrm{endStates}(\mathtt{ab}) &= \{1,2\} \\ \mathrm{endStates}(\mathtt{aba}) &= \{1,3\} \end{aligned}$$

In general, if w is a string and x is a single letter, then

$$\mathrm{endStates}(wx) = \{q : \text{for some state } p \in \mathrm{endStates}(w), \text{ there is a transition } p \xrightarrow{x} q\}$$

. . . *provided* there are no *empty* transitions.

This suggests part of a method for constructing endStates(w) for all strings w.

Converting a NFA to a FA

Idea:

$$\underline{\textit{sets} \text{ of states}} \text{ in the NFA} \quad \longrightarrow \quad \underline{\text{states}} \text{ in the FA.}$$

Informally (and *assuming no empty transitions*, for the time being):

Start with the one-element set { Start state }.

- This is endStates(ε).
- It's the set of NFA-states we can possibly be in at the very start.

Construct endStates($\mathtt{a}$), the set of all states we could then get to by reading a single $\mathtt{a}$.
Construct endStates($\mathtt{b}$), the set of all states we could then get to by reading a single $\mathtt{b}$.

For each set of states, X, that we construct:

- find the set of states we can get to from X, by reading a single $\mathtt{a}$.
- find the set of states we can get to from X, by reading a single $\mathtt{b}$.

Keep doing this, until we no longer get any new sets of states.

Converting a NFA to a FA

state	a	b
Start {1}		

state	a	b
Start {1}	{1}	{1,2}
{1,2}		

state	a	b
Start {1}	{1}	{1,2}
{1,2}	{1,3}	{1,2,3}
{1,3}		
{1,2,3}		

...

	state	a	b
Start	{1}	{1}	{1,2}
	{1,2}	{1,3}	{1,2,3}
	{1,3}	{1}	{1,2}
	{1,2,3}		

......

Converting a NFA to a FA

. . . →

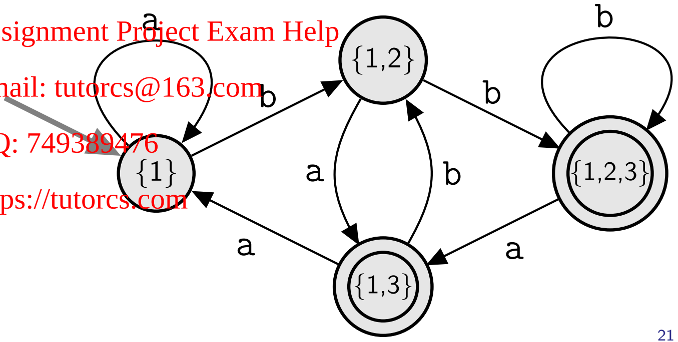
	state	a	b
Start	{1}	{1}	{1,2}
	{1,2}	{1,3}	{1,2,3}
	{1,3}	{1}	{1,2}
	{1,2,3}	{1,3}	{1,2,3}

→

	state	a	b
Start	{1}	{1}	{1,2}
	{1,2}	{1,3}	{1,2,3}
Final	{1,3}	{1}	{1,2}
Final	{1,2,3}	{1,3}	{1,2,3}

Converting a NFA to a FA

	state	a	b
Start	{1}	{1}	{1,2}
	{1,2}	{1,3}	{1,2,3}
Final	{1,3}	{1}	{1,2}
Final	{1,2,3}	{1,3}	{1,2,3}



Algorithm: Conversion of NFA *without empty transitions* to FA

Input: a NFA

NextSetOfStatesOfNFA := { Start State of NFA }.

Create new incomplete row in FA table for Start State called **NextSetOfStatesOfNFA**.

while *the FA table still has at least one incomplete row* **do**

  CurrentStateInFA := the state for the first incomplete row of the FA.

  for *each letter* x *in the alphabet* **do**

    NextSetOfStatesOfNFA :=

      $\{q$: for some NFA-state p in **CurrentStateInFA**, $\exists$ transition $p \xrightarrow{x} q\}$

    Write **NextSetOfStatesOfNFA** in table entry for row **CurrentStateInFA**, column x.

    if **NextSetOfStatesOfNFA** *is new* **then**

      Create new incomplete row in table, using *set* **NextSetOfStatesOfNFA** as state.

Any FA state which (as a *set*) contains an NFA Final State is labelled Final.

Output: the FA

Converting a NFA to a FA

Now suppose that the NFA might have empty transitions, $q_1 \xrightarrow{\varepsilon} q_2$.

These allow change of state without reading any letter of the input string.

Every time we include a new state q in **NextSetOfStatesOfNFA**, we also need to include any state we can reach from it along empty transitions.

Look at all paths from q that just use ε transitions . . .

$$q \xrightarrow{\varepsilon} q_1 \xrightarrow{\varepsilon} q_2 \xrightarrow{\varepsilon} \cdots\cdots \xrightarrow{\varepsilon} q_i$$

. . . and include all states on such paths.

Modify earlier algorithm, for constructing the sets of NFA states, to take account of empty transitions.

Algorithm: Conversion of NFA to FA

Input: a NFA

NextSetOfStatesOfNFA := { Start State of NFA }.

for *each* q $\in$ **NextSetOfStatesOfNFA** **do**

 Add, to **NextSetOfStatesOfNFA**, all states reachable from q along ε-transitions.

Create new incomplete row in FA th[illegible] Start State called **NextSetOfStatesOfNFA**.

while *the FA table still has at least one incomplete row* **do**

 CurrentStateInFA := the state for the first incomplete row of the FA.

 for *each letter* x *in the alphabet* **do**

  NextSetOfStatesOfNFA :=

   $\{q$: for some NFA-state p in **CurrentStateInFA**, $\exists$ transition $p \xrightarrow{x} q\}$

  for *each* q $\in$ **NextSetOfStatesOfNFA** **do**

   Add, to **NextSetOfStatesOfNFA**, all states reachable from q along ε-transitions.

  Write **NextSetOfStatesOfNFA** in table entry for row **CurrentStateInFA**, column x.

  if **NextSetOfStatesOfNFA** *is new* **then**

   Create new incomplete row in table, using *set* **NextSetOfStatesOfNFA** as state.

Any FA state which (as a *set*) contains an NFA Final State is labelled Final.

Output: the FA

Converting a NFA to a FA

state	a	b
Start {1,2,3}		

→

state	a	b
Start {1,2,3}	{1,2,3}	{2,3}
{2,3}		

→

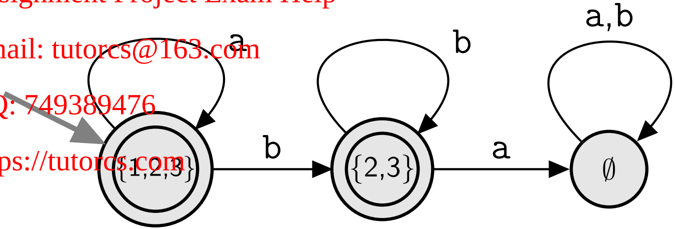
state	a	b
Start {1,2,3}	{1,2,3}	{2,3}
{2,3}	$\emptyset$	{2,3}
$\emptyset$		

→

state	a	b
Start {1,2,3}	{1,2,3}	{2,3}
{2,3}	$\emptyset$	{2,3}
$\emptyset$	$\emptyset$	$\emptyset$

Converting a NFA to a FA

state	a	b
Start {1,2,3}	{1,2,3}	{2,3}
{2,3}	∅	{2,3}
∅	∅	∅



Converting a NFA to a FA

Complexity?

- Think about how many states the FA may have, as a function of the number of states in the NFA.

Revision

Today:

- Understand Kleene's Theorem
- Be able to convert Regular Expression $\longrightarrow$ NFA
- Be able to convert NFA $\longrightarrow$ Finite Automaton

Next lecture:

- Be able to convert FA $\longrightarrow$ Regular Expression

Reading:
Sipser, Ch 1, especially pp. 54–58, 66–69.