Monash University
Faculty of Information Technology



# FIT2014 Theory of Computation

## Lecture 12

## Context-Free Grammars



slides by Graham Farr

based in part on previous slides by David Albrecht

# Overview

- Inductive Definitions
- Context Free Grammars
- Parse Trees
- Derivations

# Arithmetic Expressions



1. All integers are Arithmetic Expressions
2. If $A$ and $B$ are Arithmetic Expressions, so are:
   (i) $A + B$
   (ii) $A - B$
   (iii) $A * B$
   (iv) $A/B$
   (v) $(A)$

# Production Rules



$$
\begin{aligned}
AE &\rightarrow \text{integer} \\
AE &\rightarrow AE + AE \\
AE &\rightarrow AE - AE \\
AE &\rightarrow AE * AE \\
AE &\rightarrow AE / AE \\
AE &\rightarrow (AE)
\end{aligned}
$$

$$
\begin{aligned}
S &\rightarrow A \\
A &\rightarrow \text{integer} \\
A &\rightarrow A + A \\
A &\rightarrow A - A \\
A &\rightarrow A * A \\
A &\rightarrow A / A \\
A &\rightarrow (A)
\end{aligned}
$$

# Backus-Naur Form (a.k.a. Backus Normal Form)

$$
\begin{aligned}
S &\rightarrow A \\
A &\rightarrow \text{integer} \mid A + A \mid A - \;\cdots\; A \mid A/A \mid (A)
\end{aligned}
$$

John Backus (1924–2007)
`https://mathshistory.st-andrews.ac.uk/Biographies/Backus/`

Peter Naur (1928–2016)
`https://datamuseum.dk/`

Historical example: fragment of the BNF of ALGOL 60

### 4.1. Compound Statements and Blocks

#### 4.1.1. Syntax



⟨unlabelled basic statement⟩ ::= ⟨assignment statement⟩|
  ⟨go to statement⟩|⟨dummy statement⟩|⟨procedure statement⟩
⟨basic statement⟩ ::= ⟨unlabelled basic statement⟩|⟨label⟩:
  ⟨basic statement⟩
⟨unconditional statement⟩ ::= ⟨basic statement⟩|⟨for statement⟩|
  ⟨compound statement⟩|⟨block⟩
⟨statement⟩ ::= ⟨unconditional statement⟩|
  ⟨conditional statement⟩
⟨compound tail⟩ ::= ⟨statement⟩ **end**|⟨statement⟩ ;
  ⟨compound tail⟩
⟨block head⟩ ::= **begin**⟨declaration⟩|⟨block head⟩ ;
  ⟨declaration⟩
⟨unlabelled compound⟩ ::= **begin** ⟨compound tail⟩
⟨unlabelled block⟩ ::= ⟨block head⟩ ; ⟨compound tail⟩
⟨compound statement⟩ ::= ⟨unlabelled compound⟩|
  ⟨label⟩:⟨compound statement⟩
⟨block⟩::=⟨unlabelled block⟩|⟨label⟩:⟨block⟩



from: J. W. Backus *et al.*, *Comm. ACM* **3** (5) (May 1960) 299–314.

# EQUAL

A string is in EQUAL if it has an equal number of a's and b's.

$$\{\varepsilon, \mathtt{ab}, \mathtt{ba}, \mathtt{aabb}, \mathtt{abab}, \mathtt{abba}, \mathtt{baba}, \ldots\}$$

An a-type string has one more `a` than `b`.
A b-type string has one more `b` than `a`.

# EQUAL

A string is in EQUAL if it is

- $\varepsilon$, *or*
- a followed by a string of b-type, *or*
- b followed by a string of a-type.

$$S \longrightarrow \varepsilon$$
$$S \longrightarrow \mathtt{a}B$$
$$S \longrightarrow \mathtt{b}A$$

A string is of a-type if it is

- just a, *or*
- a followed by a string in EQUAL, *or*
- b followed by two strings of a-type.

$$A \longrightarrow \mathtt{a}$$
$$A \longrightarrow \mathtt{a}S$$
$$A \longrightarrow \mathtt{b}AA$$

A string is of b-type if it is

- just b, *or*
- b followed by a string in EQUAL, *or*
- a followed by two strings of b-type.

$$B \longrightarrow \mathtt{b}$$
$$B \longrightarrow \mathtt{b}S$$
$$B \longrightarrow \mathtt{a}BB$$

# Context Free Grammar (CFG)



A Context Free Grammar consists of:

1. An alphabet
   - The letters are called **terminals**.
2. Another set of symbols
   - We call these symbols **nonterminals**.
   - often represented by upper-case letters.
   - One of these symbols is the **start symbol** .
   - $S$ is often used as the start symbol.
3. A finite set of **production rules** of the form:

$$\text{One nonterminal} \longrightarrow \text{finite string of terminals and/or nonterminals}$$

# Context Free Grammar (CFG)



**Definition**

The **language generated** by a Context Free Grammar (CFG) consists of those *strings of terminals* which can be produced from the start symbol using the production rules.



A language generated by a CFG is called a **Context Free Language (CFL)**.

# EQUAL

Terminals: $\texttt{a}$, $\texttt{b}$

Nonterminals: $S$, $A$, $B$

Production rules:

$$
\begin{aligned}
S &\longrightarrow \varepsilon \\
S &\longrightarrow \texttt{b}A \\
S &\longrightarrow \texttt{a}B \\
A &\longrightarrow \texttt{a} \\
A &\longrightarrow \texttt{a}S \\
A &\longrightarrow \texttt{b}AA \\
B &\longrightarrow \texttt{b} \\
B &\longrightarrow \texttt{b}S \\
B &\longrightarrow \texttt{a}BB
\end{aligned}
$$

This CFG generates the language EQUAL.

# History

Pāṇini (c520BC–c460BC)

- studied *Sanskrit*

Noam Chomsky (b. 1928)

- studied *natural languages*

John Backus

- studied *programming languages*




https://mathshistory.st-andrews.ac.uk/Biographies/Panini/

Noam Chomsky, during visit to Australia in 2011 to accept Sydney Peace Prize.
http://www.abc.net.au/news/2011-06-02/noam-chomsky/2741826

## $S \to aS \mid Sa \mid \varepsilon$

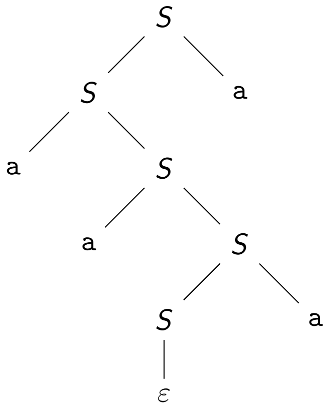


| | | | |
|---|---|---|---|
| 1. | $S$ | $\to$ | $Sa$ |
| 2. | $S$ | $\to$ | $aS$ |
| 3. | $S$ | $\to$ | $\varepsilon$ |

…on of aaaa

$$
\begin{aligned}
S &\Rightarrow Sa && \text{(Rule 1)} \\
&\Rightarrow aSa && \text{(Rule 2)} \\
&\Rightarrow aaSa && \text{(Rule 2)} \\
&\Rightarrow aaSaa && \text{(Rule 1)} \\
&\Rightarrow aaaa && \text{(Rule 3)} \\
&= aaaa
\end{aligned}
$$

# Parse Tree






on of aaaa

$$
\begin{aligned}
S &\Rightarrow Sa && \text{(Rule 1)}\\
&\Rightarrow aSa && \text{(Rule 2)}\\
&\Rightarrow aaSa && \text{(Rule 2)}\\
&\Rightarrow aaSaa && \text{(Rule 1)}\\
&\Rightarrow aa\varepsilon aa && \text{(Rule 3)}\\
&= aaaa
\end{aligned}
$$

# EQUAL

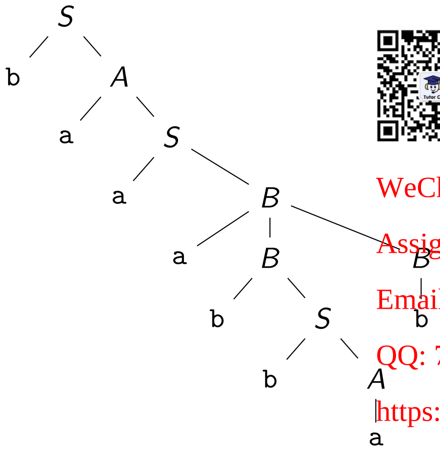
| | | | |
|---|---|---|---|
| 1. | $S$ | $\rightarrow$ | $\varepsilon$ |
| 2. | $S$ | $\rightarrow$ | $\mathtt{b}A$ |
| 3. | $S$ | $\rightarrow$ | $\mathtt{a}B$ |
| 4. | $A$ | $\rightarrow$ | $\mathtt{a}$ |
| 5. | $A$ | $\rightarrow$ | $\mathtt{a}S$ |
| 6. | $A$ | $\rightarrow$ | $\mathtt{b}AA$ |
| 7. | $B$ | $\rightarrow$ | $\mathtt{b}$ |
| 8. | $B$ | $\rightarrow$ | $\mathtt{b}S$ |
| 9. | $B$ | $\rightarrow$ | $\mathtt{a}BB$ |



Derivation of baaabbab

| | | | |
|---|---|---|---|
| $S$ | $\Rightarrow$ | $\mathtt{b}A$ | (Rule 2) |
| | $\Rightarrow$ | $\mathtt{ba}S$ | (Rule 5) |
| | $\Rightarrow$ | $\mathtt{baa}B$ | (Rule 3) |
| | $\Rightarrow$ | $\mathtt{baaa}BB$ | (Rule 9) |
| | $\Rightarrow$ | $\mathtt{baaa}B\mathtt{b}$ | (Rule 7) |
| | $\Rightarrow$ | $\mathtt{baaab}S\mathtt{b}$ | (Rule 8) |
| | $\Rightarrow$ | $\mathtt{baaabb}A\mathtt{b}$ | (Rule 2) |
| | $\Rightarrow$ | $\mathtt{baaabbab}$ | (Rule 4) |

# Parse Tree






Derivation of baaabbab

$$
\begin{aligned}
S &\Rightarrow \mathtt{b}A && \text{(Rule 2)}\\
&\Rightarrow \mathtt{ba}S && \text{(Rule 5)}\\
&\Rightarrow \mathtt{baa}B && \text{(Rule 3)}\\
&\Rightarrow \mathtt{baaa}BB && \text{(Rule 9)}\\
&\Rightarrow \mathtt{baaa}B\mathtt{b} && \text{(Rule 7)}\\
&\Rightarrow \mathtt{baaab}S\mathtt{b} && \text{(Rule 8)}\\
&\Rightarrow \mathtt{baaabb}A\mathtt{b} && \text{(Rule 2)}\\
&\Rightarrow \mathtt{baaabbab} && \text{(Rule 4)}
\end{aligned}
$$

# PARENTHESES: the Dyck Language

PARENTHESES is the language over the two-letter alphabet $\{\,(,)\,\}$ consisting of all strings of correctly matched parentheses.



PARENTHESES $= \{\varepsilon, (), ()($ … $()()(), ()(()), (())(), (()()), ((())), \ldots\}$.

Non-members: $())$ $((((\,$…

Expressing PARENTHESES strings in terms of smaller PARENTHESES strings:



Any non-empty string of parentheses must start with $($. Where is its matching $)$ ?



It could be at the other end: $(\underbrace{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots}_{\text{smaller PARENTHESES string}})$

It could be before the other end: $\underbrace{(\cdots\cdots\cdots\cdots\cdots)}_{\text{smaller PARENTHESES string}}\underbrace{(\cdots\cdots\cdots)}_{\text{smaller PARENTHESES string}}$

# PARENTHESES: the Dyck Language



Inductive Definition → Context-Free Grammar

A **string of parentheses** $S$ is c
following:

- the empty string, $\varepsilon$
- $(S')$, where $S'$ is a string of parentheses
- $S_1 S_2$, where $S_1, S_2$ are strings of parentheses.



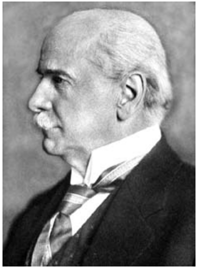
| | | | |
|---|---|---|---|
| 1. | $S$ | $\rightarrow$ | $\varepsilon$ |
| 2. | $S$ | $\rightarrow$ | $(S)$ |
| 3. | $S$ | $\rightarrow$ | $SS$ |

$$
\begin{array}{lll}
1. & S & \to \varepsilon \\
2. & S & \to \\
3. & S & \to
\end{array}
$$



Derivation of ()(())

$$
\begin{array}{lll}
S & \Rightarrow SS & \text{(Rule 3)} \\
& \Rightarrow (S)S & \text{(Rule 2)} \\
& \Rightarrow (S)(S) & \text{(Rule 2)} \\
& \Rightarrow ()(S) & \text{(Rule 1)} \\
& \Rightarrow ()((S)) & \text{(Rule 2)} \\
& \Rightarrow ()(()) & \text{(Rule 1)}
\end{array}
$$

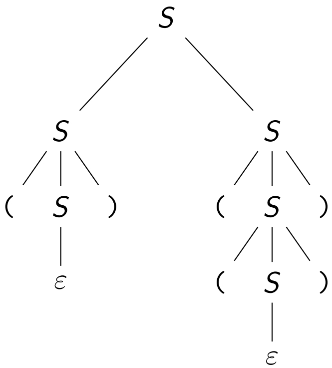


Walther von Dyck (1856–1934)
`https://mathshistory.st-andrews.ac.uk/Biographies/Von_Dyck/`

# PARENTHESES: the Dyck Language



Parse tree:




Derivation of $()(())$

$$
\begin{array}{rcll}
S & \Rightarrow & SS & \text{(Rule 3)} \\
 & \Rightarrow & (S)S & \text{(Rule 2)} \\
 & \Rightarrow & (S)(S) & \text{(Rule 2)} \\
 & \Rightarrow & ()(S) & \text{(Rule 1)} \\
 & \Rightarrow & ()((S)) & \text{(Rule 2)} \\
 & \Rightarrow & ()(()) & \text{(Rule 1)}
\end{array}
$$

# Exercises



- Suppose we have two types of brackets, such as round and square: ( ) and [ ] . Find a context-free language for the set of all valid strings of such brackets.



- Find a context-free grammar for PALINDROMES



- For other languages we have met:
  - find context-free grammars for them, OR
  - if you think they don't have one, think about why.

# A simple property of derivations



At any stage, the string to the left of the first nonterminal must be a prefix of the final (derived) string.

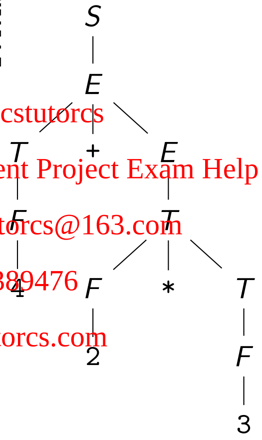


$$
\begin{array}{rcll}
S & \Longrightarrow & \cdots & \\
\vdots & & \vdots & \\
& \Longrightarrow & x_1 \cdots x_k AB \cdots & \\
& \Longrightarrow & x_1 \cdots x_k aXYB \cdots & (\text{using} \quad A \longrightarrow aXY) \\
\vdots & & \vdots & \\
& \Longrightarrow & x_1 \cdots x_k a \cdots \cdots & (\text{derived string})
\end{array}
$$

# 4 + 2*3

$$
\begin{array}{lcl}
S & \longrightarrow & E \\
E & \longrightarrow & T + E \mid T - E \mid T \\
T & \longrightarrow & F * T \mid F / T \mid F \\
F & \longrightarrow & \text{integer} \mid (E)
\end{array}
$$





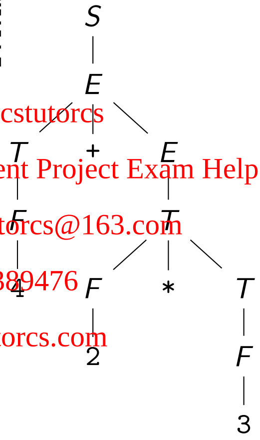

**Leftmost** derivation:

$$
\begin{array}{lll}
S & \Rightarrow & \mathbf{E} \\
 & \Rightarrow & \mathbf{T} + E \\
 & \Rightarrow & \mathbf{F} + E \\
 & \Rightarrow & 4 + \mathbf{E} \\
 & \Rightarrow & 4 + \mathbf{T} \\
 & \Rightarrow & 4 + \mathbf{F} * T \\
 & \Rightarrow & 4 + 2 * \mathbf{T} \\
 & \Rightarrow & 4 + 2 * \mathbf{F} \\
 & \Rightarrow & 4 + 2 * 3
\end{array}
$$

# 4 + 2*3

$$
\begin{aligned}
S &\longrightarrow E \\
E &\longrightarrow T + E \mid T - E \mid T \\
T &\longrightarrow F * T \mid F / T \mid F \\
F &\longrightarrow \text{integer} \mid (E)
\end{aligned}
$$






**Rightmost** derivation:

$$
\begin{aligned}
S &\Rightarrow \mathbf{E} \\
&\Rightarrow T + \mathbf{E} \\
&\Rightarrow T + \mathbf{T} \\
&\Rightarrow T + F * \mathbf{T} \\
&\Rightarrow T + F * \mathbf{F} \\
&\Rightarrow T + \mathbf{F} * 3 \\
&\Rightarrow \mathbf{T} + 2 * 3 \\
&\Rightarrow \mathbf{F} + 2 * 3 \\
&\Rightarrow 4 + 2 * 3
\end{aligned}
$$

# Leftmost and rightmost derivations



In a **Leftmost derivation**, the leftmost non-terminal is always replaced first.
In a **Rightmost derivation**, the rightmost non-terminal is always replaced first.

**Theorem.**

Whenever a string has a derivation, it also has a leftmost derivation of the same length.


**Proof.** See Tute 4.


Does the same hold for rightmost derivations?

# A simple property of leftmost derivations

Whenever a production

$$X \longrightarrow \text{terminals}\ \text{Non-terminal}\ \textit{theRest}$$

is applied, the terminal letters of terminals that are appended to the current prefix to give a larger prefix of the derived string.

$$\begin{aligned}
S &\Longrightarrow \cdots \\
\vdots &\quad \vdots \\
&\Longrightarrow x_1 \cdots x_k \underbrace{A} B \cdots \\
&\Longrightarrow x_1 \cdots x_k \underbrace{aXY} B \cdots \qquad (\text{using} \quad A \longrightarrow aXY) \\
\vdots &\quad \vdots \\
&\Longrightarrow x_1 \cdots x_k a \cdots\cdots \qquad (\text{derived string})
\end{aligned}$$

# Revision



- Context Free Grammars
  - Definition. How to use
- Parse Trees
  - Definition. How to make them.



- The Dyck language
- Leftmost and rightmost derivations.

Read:
Sipser, Ch. 2, pp. 101–108.