Leture 5: Multivariate Volatility Modelling - Part II

FM321: Risk Management and Modelling

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LSE Finance

- In multivariate volatility models, a vector of returns are written as:

$$r_t = \Sigma_t^{1/2} z_t$$

- Previously, we wrote down models for Σ_t by directly generalizing univariate models.

  - EWMA: $\Sigma_t = (1-\lambda) r_{t-1} r'_{t-1} + \lambda \Sigma_{t-1}$
  - BEKK: $\Sigma_t = \Omega'\Omega + A r_{t-1} r'_{t-1} A' + B \Sigma_{t-1} B'$

- Now we will look at alternative approaches particularly suited for multivariate models.
  - Correlation models: $\Sigma_t = D_t C_t D_t$, where $D_t = diag\{\sigma_{1,t}, ..., \sigma_{N,t}\}$ and C_t correlation.
  - Factor models: $r_t = \beta f_t + \epsilon_t$ so that $Var_{t-1}(r_t) = \beta \Sigma_{f,t} \beta' + \Sigma_{\epsilon,t}$.

- If define the matrices D_t and C_t (which contain information about individual asset volatility and correlations, respectively) by:

$$D_t = \begin{bmatrix} \sigma_{1,t} & 0 & \dots & 0 \\ 0 & \sigma_{2,t} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sigma_{N,t} \end{bmatrix} \quad C_t = \begin{bmatrix} 1 & \rho_{12,t} & \dots & \rho_{1N,t} \\ \rho_{21,t} & 1 & \dots & \rho_{2N,t} \\ \dots & \dots & \dots & \dots \\ \rho_{N1,t} & \rho_{N2,t} & \dots & 1 \end{bmatrix}$$

then a simple algebraic verification shows that

$$\Sigma_t = D_t C_t D_t$$

- This allows us to model volatilities (D_t) and correlations (C_t) separately.

- C_t can be modeled as:
  - a constant matrix $\rightarrow$ Constant Conditional Correlation (CCC).
  - a time-varying matrix $\rightarrow$ Dynamic Conditional Correlation (DCC).

- Estimate a univariate volatility model for each asset (e.g., GARCH).

- For each asset, use univariate model to compute estimates for $\hat{\sigma}_t$ for each asset.

- Use conditional volatilities to standardize the returns (that is, compute $\hat{q}_t = \frac{r_t}{\hat{\sigma}_t}$.

- Use the series for $\hat{q}_t$ for all assets to model C_t.

- In CCC, we assume that the true correlation matrix is a constant matrix, $C_t = C$, and estimate it using all available data:

$$\hat{C}_t = \begin{bmatrix} 1 & \hat{\rho}_{12,t} & \cdots & \hat{\rho}_{1N,t} \\ \hat{\rho}_{21,t} & 1 & \cdots & \hat{\rho}_{2N,t} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{\rho}_{N1,t} & \hat{\rho}_{N2,t} & \cdots & 1 \end{bmatrix} \qquad \hat{\rho}_{ij,t} = \frac{\sum_{s=1}^{t-1} \hat{q}_{i,s}\hat{q}_{j,s}}{\sqrt{\left(\sum_{s=1}^{t-1} \hat{q}_{i,s}^2\right)\left(\sum_{s=1}^{t-1} \hat{q}_{j,s}^2\right)}}$$

- The complexity of the model depends on what assumptions we place on C:
  - If no restrictions are placed, the number of correlations to estimate is still of the order N^2.
  - If we say that the pairwise correlation between assets is constant (that is, $\rho_{ij} = \rho$ for every pair of securities), the curse of dimensionality disappears; but should we believe this assumption?

- In DCC, we specify the evolution of C_t.
- The elements of C_t are given by

$$\rho_{ij,t} = \frac{h_{ij,t}}{\sqrt{h_{ii,t} h_{jj,t}}}$$

  $h_{ij,t}$: latent covariance between $\hat{q}_{i,t}$ and $\hat{q}_{j,t}$.

- Two common choices for the evolution of $h_{ij,t}$:
  - EWMA:

$$h_{ij,t} = (1-\lambda) q_{i,t-1} q_{j,t-1} + \lambda h_{ij,t-1}$$

  - GARCH(1,1):

$$h_{ij,t} = (1-\alpha-\beta)\bar{h}_{ij} + \alpha q_{i,t-1} q_{j,t-1} + \beta h_{ij,t-1}$$

- C_t symmetric and positive semidefinite (if we make sure $\bar{h}_{ij} = \bar{h}_{ji}$. Number of parameters fixed regardless of N.

Large Problems

- Managers deal with large universes.

- MSCI World All-Country Index has over 2400 stocks.

- Around 2,800 stocks trade in the NYSE.

- Financial institutions can have tens of thousands of assets, if not more.

- None of the models we have discussed so far can be scaled to that number of securities.

- Basic Idea: find a few sources of risk (factors) that account for a sizeable portion of the common risk across securities, and model risk around those sources.

- Examples:
  - Market risk

  - Country risk
  - Sector risk

  - Macroeconomic risk factors
  - Risk factors derived from anomalies (value, size, momentum)
  - In yield curve contexts: level, slope and convexity
  - In currency contexts: carry trade factor

- In general, the formulation is

$$r_t = \beta f_t + \epsilon_t$$

where

- r is an $N \times 1$ vector of (excess) returns.
- β is an $N \times F$ matrix of factor exposures (containing the exposure of each security to each factor).

- f is an $F \times 1$ vector of factor returns.
- ϵ is an $N \times 1$ vector of residual returns for the securities.

- From these assumptions, we conclude that

$$\Sigma_t = Var_{t-1}(r_t) = \beta \Sigma_{f,t} \beta' + \Sigma_{\epsilon,t}$$

- The number of parameters to estimate is therefore:

  - The $N \times F$ factor exposures in β (can be constant or time-varying);
  - The $\frac{1}{2}F(F+1)$ elements of $\Sigma_{f,t}$; and

  - the N non-zero elements of $\Sigma_{\epsilon,t}$.

- The total number of parameters is $N(F+1) + \frac{1}{2}F(F+1)$, which is linear in N.