Leture 4: Multivariate Volatility Modelling – Part I

FM321: Risk Management and Modelling

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LSE Finance

Motivation: a hedge fund's problem

- Imagine you run a quantitative equity fund.
- 6,000+ US stocks to choose from (around 1,000 considered very liquid, Russell 1000)
- If the fund trades international equity, 60,000+ stocks to choose from, though some not accessible from abroad.
- A global macro strategy reviews securities in other asset classes (fixed income, currency, commodities etc.) although most likely you will trade index futures rather than individual securities.
- How would you decide which securities to invest in and how much?

- Security return vector: $r_t = (r_{1,t}, \ldots, r_{N,t})'$.

- Conditional variance-covariance matrix of r_t (given information up to $t-1$): Σ_t.

- Analogous to the univariate case, we can write our model as

$$r_t = \Sigma_t^{1/2} z_t$$

where $z_t = (z_{1,t}, \ldots, z_{N,t})$, $z_{i,t} \sim$ iid $(0,1)$ and $z_{i,t}$'s are independent across i.

- Portfolio weight vector: $w_t = (w_{1,t}, \ldots, w_{N,t})'$.

- Portfolio return: $r_{P,t} = w_t' r_t = \sum_{i=1}^{N} w_{i,t} r_{i,t}$

- Portfolio variance: $\sigma_{P,t}^2 = w_t' \Sigma_t w_t$

- A simplest way to frame a hedge fund's problem at $t-1$ is given below.

$$
\begin{aligned}
\max_{w_t} \quad & E_{t-1}[r_{P,t}] - \frac{\lambda}{2}\sigma^2_{P,t} \\
s.t. \quad & r_{P,t} = w_t' r_t \\
& \sigma^2_{P,t} = w_t' \Sigma_t w_t
\end{aligned}
$$

- This is called the quadratic optimization problem and is often used as the benchmark problem for portfolio construction. The fund chooses λ depending on its risk preference.

- To solve the problem, we need an estimate of Σ_t.

- First, we must have $\sigma^2_{P,t} \geq 0$ for any vector w_t, which implies that the matrix Σ_t must be positive semidefinite.

What do we need for a good estimate of Σ_t?

- Second, all elements of Σ_t, including covariance terms, should be easily estimated.

- As the number N of securities grows, the number of distinct elements of their covariance matrix grows quadratically.

- For $N = 3$, we need to estimate 6 distinct elements of the covariance matrix:

$$\Sigma_t = \begin{bmatrix} \sigma^2_{1,t} & & \\ \sigma_{12,t} & \sigma^2_{2,t} & \\ \sigma_{13,t} & \sigma_{23,t} & \sigma^2_{3,t} \end{bmatrix}$$

(Note that, by symmetry, $\sigma_{ij} = \sigma_{ji}$)

- With N assets, the number of distinct elements is $\frac{N(N+1)}{2}$.

The curse of dimensionality

- Depending on what model we're considering, the number of parameters to be estimated can grow at even higher rates than quadratically in N.

- Recall that ideally we'd work with many more observations than parameters to be estimated.

- Financial institutions often work with hundreds or thousands of securities in their universe (if not more).

- Models that suffer from the Curse of Dimensionality will suffer from a great degree of statistical uncertainty for portfolios with typical number of securities, and won't be useful in practice.

- Challenge 1: $\hat{\Sigma}_t$ must be positive semidefinite.

- Challenge 2: need to deal with the curse of dimensionality

Multivariate moving averages: EWMA

- The exponentially-weighted version is given by

$$\Sigma_t = (1-\lambda) r_{t-1} r'_{t-1} + \lambda \Sigma_{t-1}$$

- Individual element:

$$\sigma_{ij,t} = (1-\lambda) r_{i,t-1} r_{j,t-1} + \lambda \sigma_{ij,t-1} \qquad i,j = 1,\ldots,K$$

- EWMA ensures that $\hat{\Sigma}_t$ is positive semidefinite.
- Suffers from the same problems as in the univariate case.

- A single λ would lead to excessive movements of covariance terms.
- Multiple λ's would lead to problem of the curse of dimensionality. The number of parameters is quadratic in N.

BEKK model

- Recall the univariate GARCH model:

$$\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2$$

- A generalization to the multivariate case (where r_{t-1} becomes a vector and σ_t become a matrix Σ_t):

$$\Sigma_t = \Omega\Omega' + A r_{t-1} r_{t-1}' A' + B \Sigma_{t-1} B'$$

- Guarantees positive definite estimates.

- The number of parameters is still quadratic in N.

- Developed by Baba, Engle, Kraft and Kroner (1990).

- The specification for $N = 2$ and 1 lag is:

$$
\begin{aligned}
\Sigma_t = {} & \underbrace{\begin{bmatrix} \omega_{11} & 0 \\ \omega_{12} & \omega_{22} \end{bmatrix}}_{\Omega} \underbrace{\begin{bmatrix} \omega_{11} & \omega_{12} \\ 0 & \omega_{22} \end{bmatrix}}_{\Omega'} \\
& + \underbrace{\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} r_{1,t-1}^2 & r_{1,t-1} r_{2,t-1} \\ r_{1,t-1} r_{2,t-1} & r_{2,t-1}^2 \end{bmatrix}}_{r_{t-\mathbf{1}} r'_{t-\mathbf{1}}} \underbrace{\begin{bmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \end{bmatrix}}_{A'} \\
& + \underbrace{\begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \sigma_{1,t-1}^2 & \sigma_{12,t-1} \\ \sigma_{12,t-1} & \sigma_{2,t-1}^2 \end{bmatrix}}_{\Sigma_{t-\mathbf{1}}} \underbrace{\begin{bmatrix} \beta_{11} & \beta_{21} \\ \beta_{12} & \beta_{22} \end{bmatrix}}_{B'}
\end{aligned}
$$

Appendix (will not be tested in ICA)

- Usually expressed in compact form to avoid repeated elements
- For example, with $N = 2$ we denote

$$\Sigma_t = \begin{bmatrix} \sigma^2_{1,t} & \sigma_{12,t} \\ \sigma_{12,t} & \sigma^2_{2,t} \end{bmatrix} \qquad r_t = \begin{bmatrix} r_{1,t} \\ r_{2,t} \end{bmatrix}$$

- The model formulation for a V-GARCH(1,1) in this case is

$$\begin{bmatrix} \sigma^2_{1,t} \\ \sigma_{12,t} \\ \sigma^2_{2,t} \end{bmatrix} = \begin{bmatrix} \omega_{11} \\ \omega_{22} \\ \omega_{33} \end{bmatrix} + \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} r^2_{1,t-1} \\ r_{1,t-1} r_{2,t-1} \\ r^2_{2,t-1} \end{bmatrix}$$

$$+ \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} \sigma^2_{1,t-1} \\ \sigma^2_{12,t-1} \\ \sigma^2_{2,t-1} \end{bmatrix}$$

- In the general formulation, every element of Σ_t is allowed to depend on every element of Σ_{t-1} and $r_{t-1}r_{t-1}$.

- The number of parameters in the model is of the order of N^4.

- In addition to severe dimensionality problems, any estimates from V-GARCH are not guaranteed to be positive definite.

- Simplification of V-GARCH that restricts the coefficient matrices to be diagonal.

- The model formulation for a V-GARCH(1,1) in this case is

$$\begin{bmatrix} \sigma^2_{1,t} \\ \sigma_{12,t} \\ \sigma^2_{2,t} \end{bmatrix} = \begin{bmatrix} \omega_{11} \\ \omega_{22} \\ \omega_{33} \end{bmatrix} + \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} r^2_{1,t-1} \\ r_{1,t-1} r_{2,t-1} \\ r^2_{2,t-1} \end{bmatrix} + \begin{bmatrix} B_{11} & 0 & 0 \\ 0 & B_{22} & 0 \\ 0 & 0 & B_{33} \end{bmatrix} \begin{bmatrix} \sigma^2_{1,t-1} \\ \sigma^2_{12,t-1} \\ \sigma^2_{2,t-1} \end{bmatrix}$$

- While this limits the dimensionality problem (the number of parameters is of the order N^2), resulting estimates are still not guaranteed to be positive definite.