Lecture 7: Risk Measures

FM321: Risk Management and Modelling

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LSE Finance

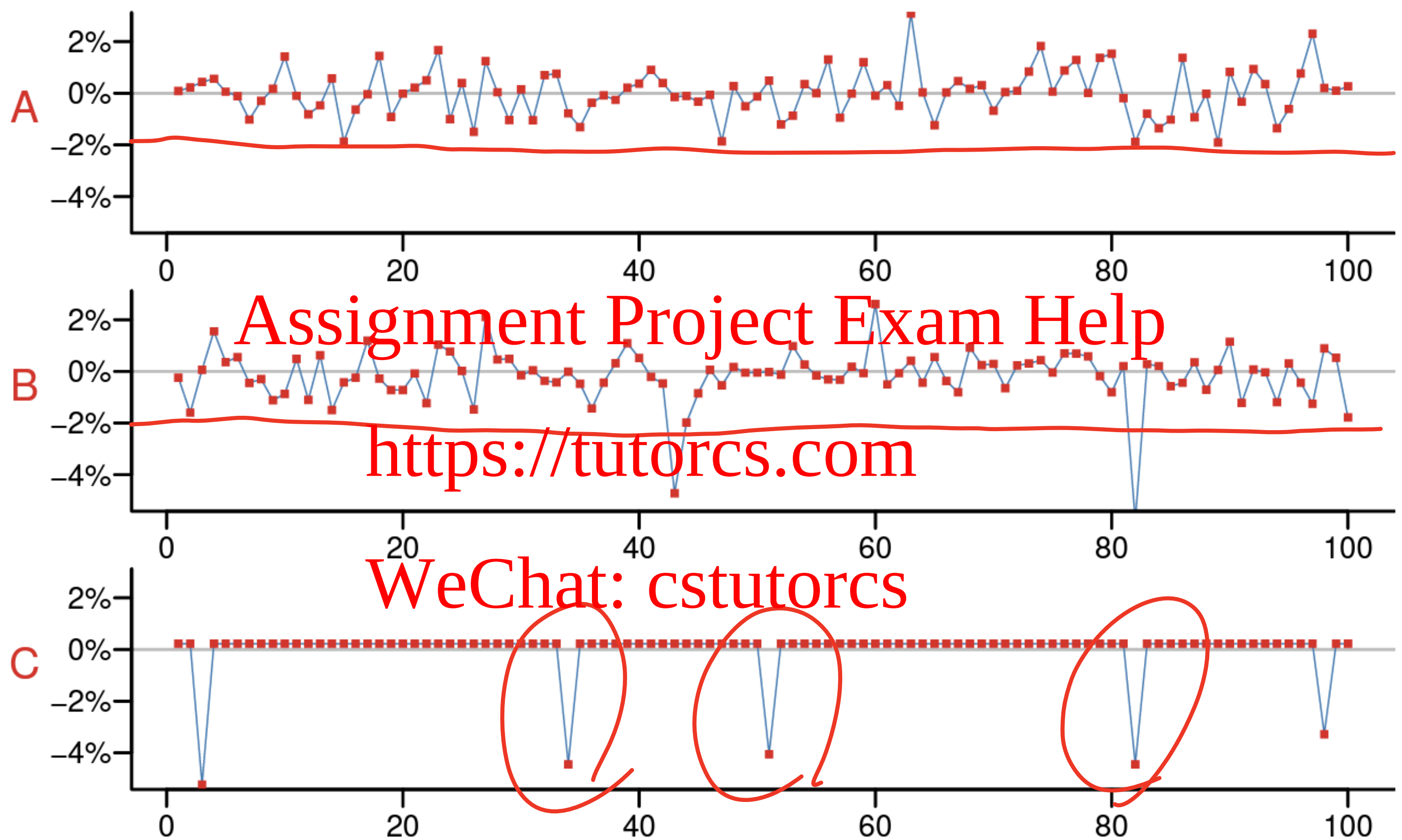
What is risk?

- There is no universal definition of what constitutes risk.
- There are many identifiable types of risk (market, credit, legal, operational, reputational, and so on), and each can be defined individually.

- Financial risk can be described as the chance of losing part or all of an investment.

- There is also no single measure that captures all useful information regarding financial risk.
- Note that risk is not observable (called "latent"), so it can only be inferred from other observable information.

What is risk?



Note: all assets above have zero mean and standard deviation equal to 1

What is risk?

- There is no obvious way to discriminate between the assets.
- One can try to model the underlying distribution of market prices and returns of assets, but it is generally unknown.

- Practically, it is impossible to accurately identify the distribution of financial returns.

- As a result, financial risk is latent. It has to be inferred from the behavior of observed market prices.
- Risk measure is a mathematical concept that captures some relevant aspect of the risk associated with a distribution.

Volatility

- Volatility is the standard deviation of returns.
- Main measure of risk in most financial analysis.
- It is a sufficient measure of risk when returns are normally distributed.

  - For this reason, in mean-variance analysis the efficient frontier shows the best investment decision.

  - If returns are not normally distributed, solutions on the efficient frontier may be inefficient.

- The assumption of normality of return is violated for most if not all financial returns.
- For most applications in financial risk, volatility is likely to systematically underestimate risk

Risk measures

- We are going to discuss two risk measures.

- The one-day Value-at-Risk (at 1% level) of John Smith's trading book is \$1 million.

- The one-day expected shortfall (at 1% level) of John Smith's trading book is \$1.4 million.

- Definition: VaR is the loss on a trading portfolio or asset such that there is a probability p of losses equaling or exceeding VaR in a given trading period.

- VaR is a quantile on the distribution of P/L.

$$Prob\left(P/L \leq -VaR_p\right) = p$$

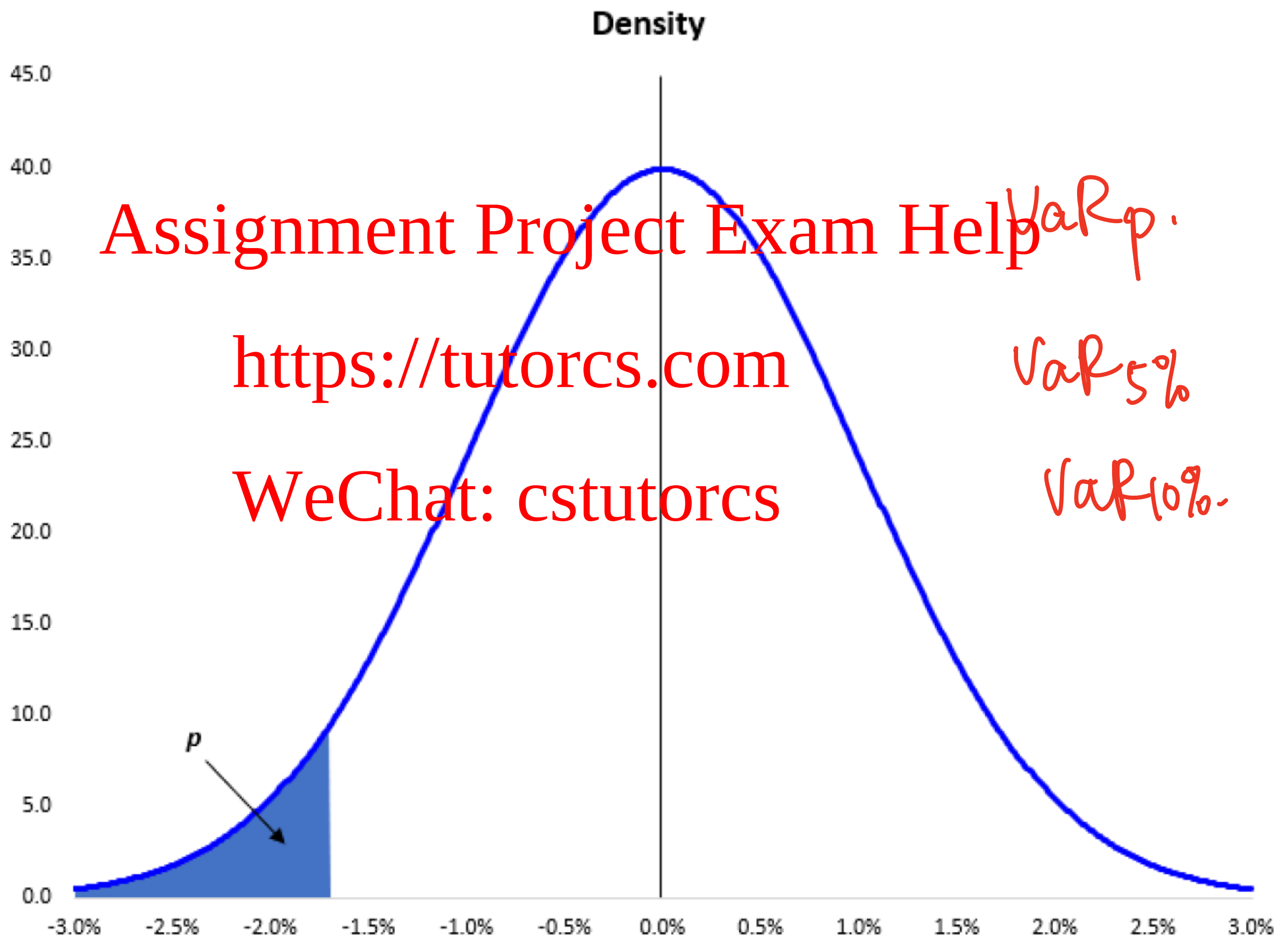
- Equivalently,

$$p = F(-VaRp) = \int_{-\infty}^{-VaR_p} f(x)dx$$

where $F(.)$ is c.d.f of P/L and $f(.)$ is its p.d.f.

- Thus, the VaR that corresponds to level p is (the negative of) the p-th quantile of the distribution in question.

- The following picture illustrates the concept of VaR relative to a probability density function.



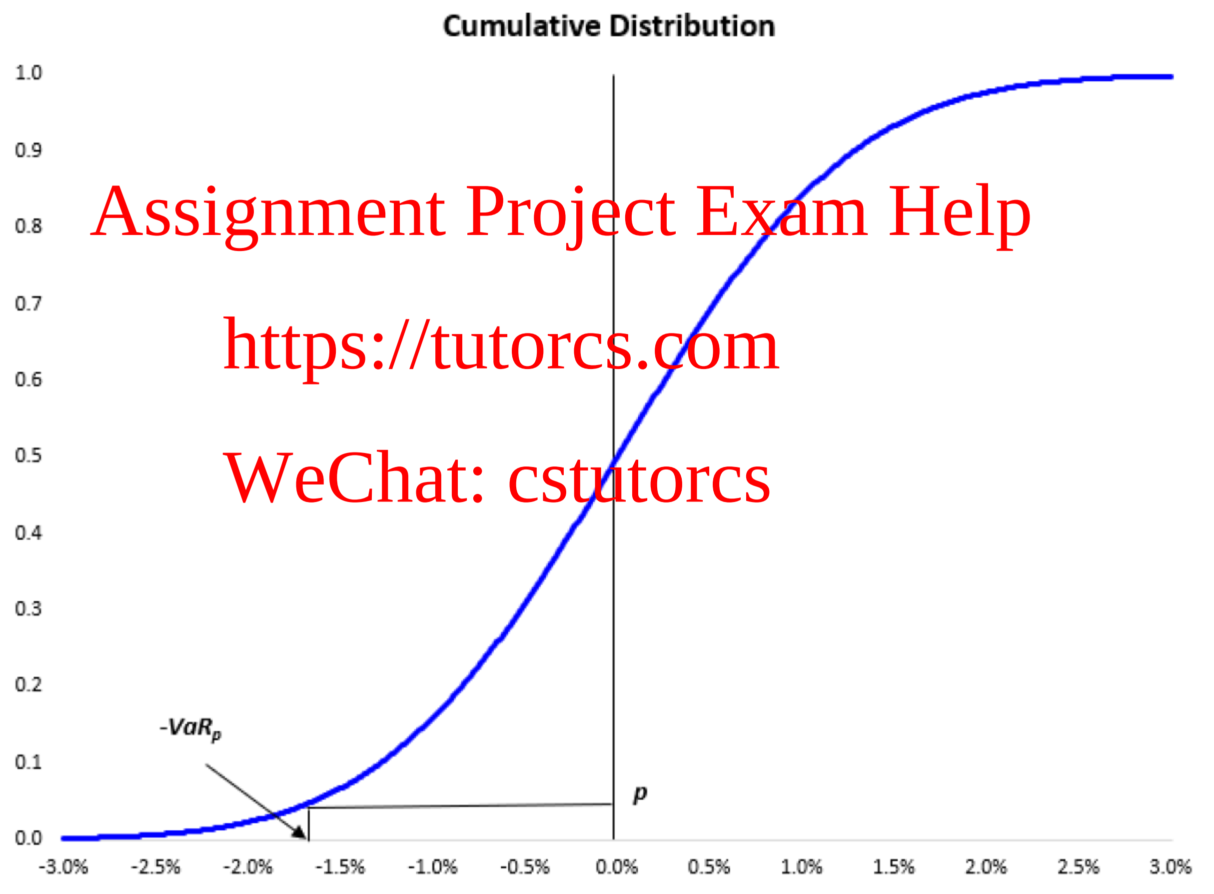
VaR_p.

$VaR_{5\%}$

$VaR_{10\%}$.

VaR on cdf

- The following picture illustrates the concept of VaR relative to a cumulative density function.



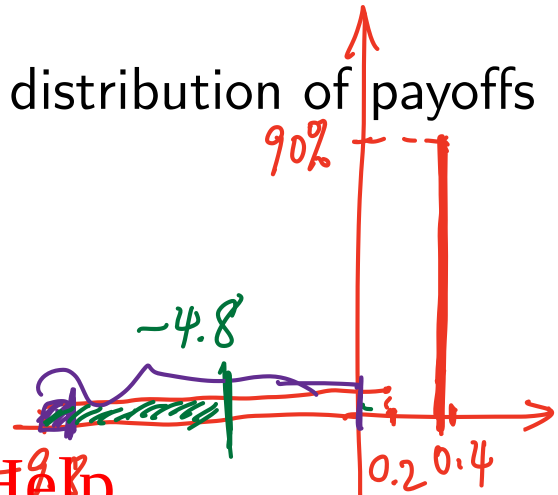
VaR: an example

- Consider a portfolio with two bonds, A and B.
- Each has a face value of $10 million (mln).
- Each has a 5% probability of default.

- In the event of no default, the investor makes a profit of $0.2mln on a bond.

- In the event of default, the recovery rate on the bond is uniformly distributed in $[0, 1]$.
- For simplicity, assume that only one of the two bonds may default (so that if one of them defaults, the other does not).
- What is $VaR_{5\%}$ and $VaR_{1\%}$ of the portfolio?

- To compute VaR, we need to characterize the distribution of payoffs on this portfolio.



Case 1: Bond A defaults (with prob 0.05)

- payoff consists of two components:
  - a profit of \$0.2mln from B.
  - a loss in A, uniformly distributed between 0 and \$10mln.
- payoff is uniformly dist. between -\$9.8mln and \$0.2mln.

Case 2: Bond B defaults (with prob 0.05)

- payoff is uniformly dist. between -\$9.8mln and \$0.2mln.

Case 3: neither bond defaults (with prob 0.9) $= 1 - 0.05 - 0.05$

- payoff $=$ \$0.2 mln $+$ \$0.2 mln $=$ \$0.4 mln

- It follows that $VaR_{5\%} =$ \$4.8mln and $VaR_{1\%} =$ \$8.8mln.

$VaR_{10\%} = -0.2$

VaR

- The risk measure is the negative of the relevant quantile, which will make VaR a positive number in typical situations.

- The notation for VaR makes the relevant level explicit, usually in one of the following formats: VaR_p, VaR^p or $VaR(p)$.

- Sometimes the confidence level is indicated, rather than the usual VaR level (so that what I call $VaR_{1\%}$ might be referred to as $VaR_{99\%}$)

VaR

- If VaR is defined in terms of the profit or loss in a given portfolio, it is a monetary measure.
- This property makes it easier to understand for a broad general audience than a statistical measure (such as standard deviation), and was crucial in popularizing the measure.

- The fact that VaR can be defined in monetary terms makes it easier to compare:

  - investments in different asset classes
  - investments of different horizons
  - different types of potential losses (e.g. VaR can be used to analyze credit risk)

- In certain situations, VaR can also be defined for a distribution of returns, in which case it will be given as a number.

VaR

- Three pieces of information are necessary for computing and interpreting a VaR:

  - The level p.
  - A time horizon (holding period over which one would like to measure risk).
  - The distribution of the returns or profits/losses of interest over that horizon.

VaR: level

- The most common level is 1%, although 5% is also used widely.
- Capital requirements have often been computed based on 1% VaR.
- Less extreme levels (say, 10%) are sometimes used in trading floors.

- More extreme levels (say, 0.1%) can be used for survival analysis.

- Example: 1% daily VaR means that one can expect losses less than VaR in 99 out of 100 days; with 252 trading days per year, expected number of violations is 2.5 per year.
- An event in which losses exceed VaR is called a breach.

VaR: holding period

- The holding period is the time period over which losses are computed.
- In order for it to be useful, it must be long enough for corrective measures to be taken to deal with violations.
- For this reason, it needs to reflect the liquidity of assets in portfolio:
  - Most common period is one day.
  - In high-frequency trading, it can be minutes or hours.
  - Institutional investors often use longer periods (weeks).

VaR: distribution

- As we mentioned, the probability distribution of P/L or returns cannot be observed.
- In practice, the most common procedure is to estimate a conditional variance model and to use the resulting distribution of conditional returns.
- Note that the concept of VaR can be applied to any distribution, so it does not require normality.
- As seen previously, one can use conditionally normal returns in modeling series which have unconditional heavy tails.

Pitfalls of VaR

- VaR is only a quantile.

- VaR is not a coherent risk measure.

Pitfalls of VaR

- VaR does not take into account the shape of the tail of the distribution into account, so it cannot provide any information regarding potential losses when breaches happen.

- For this reason, VaR tends to create the impression that potential losses are less severe than really is the case.

Pitfalls of VaR

- Consider two scenarios for the distribution of returns of a portfolio:

  Scenario 1: returns are normally distributed, with mean 5% and standard deviation 10%.

  Scenario 2: the distribution of returns depend on the state of markets:

  - If markets are calm (which happens with probability 0.9), returns are normally distributed with mean 14.16% and standard deviation 10%.

  - If markets are turbulent (which happens with probability 0.1), returns are normally distributed with mean -25% and standard deviation 15%.

- The second scenario involves a distribution known as a mixture of normals.

Pitfalls of VaR

- Consider two scenarios for the distribution of returns of a portfolio:

  Scenario 1: returns are normally distributed, with mean 5% and standard deviation 10%.

$$VaR_{10\%} = -(\mu - z_{0.9}\sigma) = -(5\% - 1.282 \times 10\%) = 7.82\%$$

where $z_{0.9}$ is the 90-th percentile of $N(0,1)$.

Pitfalls of VaR

- Consider two scenarios for the distribution of returns of a portfolio:

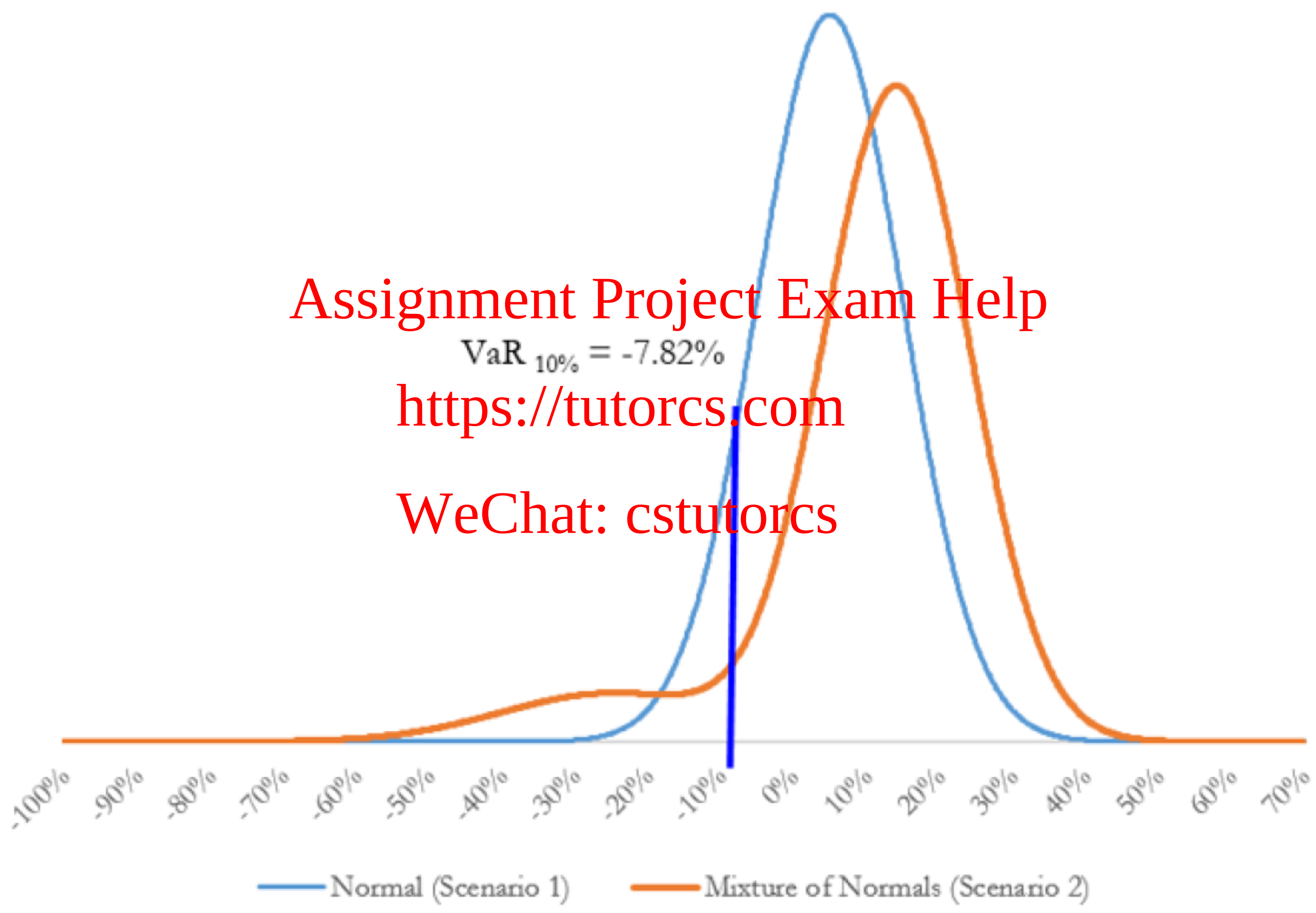
  Scenario 2: the distribution of returns depend on the state of markets:

  - If markets are calm (which happens with probability 0.9), returns are normally distributed with mean 14.16% and standard deviation 10%.
  - If markets are turbulent (which happens with probability 0.1), returns are normally distributed with mean -25% and standard deviation 15%.

A simulation of the mixed normal distribution shows that in this scenario we also have:

$$VaR_{10\%} = 7.82\%$$

Pitfalls of VaR



Pitfalls of VaR

- The two scenarios yield the same $VaR_{10\%}$.
- However, in the event of a breach of this VaR:
  - in scenario 1 losses are concentrated near the VaR.
  - in scenario 2 losses are typically much higher than the VaR.
- Therefore, VaR does not provide enough information to quantify potential losses.
- Expected shortfall, another risk measure we will discuss later in this lecture, will be able to distinguish these two distributions.

Pitfalls of VaR

- VaR is not a coherent risk measure. Specifically, in some circumstances, VaR can be manipulated by breaking up portfolios.
- Consider a portfolio with two bonds, A and B.
  - Each bond has a probability of default equal to 3%.
  - The events of default in the two bonds are independent.
  - Losses for each bond are 0 in the event of no default, and 100 in the event of default.
- Two questions:
  - What is $VaR_{5\%}$ of the portfolio consisting of A and B?
  - What is $VaR_{5\%}$ of the portfolio consisting of only A?

Pitfalls of VaR

- What is $VaR_{5\%}$ of the portfolio consisting of only A?
- Distribution of payoffs:

  - with probability 3%, the payoff is -100.

  - with probability 97%, the payoff is 0.
- $VaR_{5\%} = 0$

Pitfalls of VaR

- What is $VaR_{5\%}$ of the portfolio consisting of A and B?
- To compute the distribution of payoffs in this portfolio, note that
  - the probability of both bonds defaulting is $3\% \times 3\% = 0.09\%$, in which case the payoff is -200.
  - the probability of neither bond defaulting is $(1 - 3\%)^2 = 94.09\%$, and the payoff is 0.
  - the probability of a single bond defaulting is $1 - 0.09\% - 94.09\% = 5.82\%$, in which case the payoff is -100.
- $VaR_{5\%} = 100$

Pitfalls of VaR

- Is it an issue and why?

- If agents would like to reduce total reported risk, the use of VaR produces an incentive to separate the two bonds, compute risk measurements for each individually, then aggregate the resulting measurements.

- In the language of Artzner et. al (1999), VaR is not a coherent risk measure because it is not sub-additive.

- A risk measure $\psi(.)$, which maps random variables X into a risk measurement $\psi(X)$, is said to be coherent if it satisfies the following four properties:
  - Monotonicity: $\psi(X) \geq \psi(Y)$ if $X \leq Y$ in every state of the world.
    A portfolio with greater future returns has less risk.
  - Sub-additivity: $\psi(X+Y) \leq \psi(X) + \psi(Y)$ for any X and Y.
    Diversification is beneficial.
  - Homogeneity: $\psi(cX) = c\psi(X)$ for any X and any constant $c > 0$.
    Risk of a position is proportional to its size. (But may be violated in markets where large quantities cannot be traded without price impact.)
  - Translation invariance: $\psi(X + c) = \psi(X) - c$ for any X and any constant c.
    Addition of a sure amount of capital reduces the risk by the same amount.
- Coherence was proposed by Artzner et. al in 1999, in a paper entitled "Coherent Measures of Risk".

Coherence of VaR for normal outcomes

- VaR is subadditive in the special case of normally distributed returns.

- Suppose that X and Y are joint-normally distributed, so that their sum $X + Y$ is also normally distributed.

- In this case, sub-additivity is satisfied for $p < 0.5$, i.e.

$$VaR_p(X) + VaR_p(Y) \geq VaR_p(X + Y)$$

Coherence of VaR for normal outcomes

- Denote means, standard deviations and correlations of each of these variables by μ, σ and ρ (properly subscripted).

- For normal outcomes X and Y, we have:

$$VaR_p(X) = -[\mu_X + \sigma_X \Phi^{-1}(p)]$$

$$VaR_p(Y) = -[\mu_Y + \sigma_Y \Phi^{-1}(p)]$$

$$VaR_p(X+Y) = -[\mu_{X+Y} + \sigma_{X+Y} \Phi^{-1}(p)]$$

- Therefore,

$$VaR_p(X) + VaR_p(Y) - VaR_p(X+Y) = -\Phi^{-1}(p)[\sigma_X + \sigma_Y - \sigma_{X+Y}]$$

Coherence of VaR for normal outcomes

- $\sigma_X + \sigma_Y - \sigma_{X+Y} \geq 0$ because

$$\begin{aligned}(\sigma_X + \sigma_Y)^2 &= \sigma_X^2 + \sigma_Y^2 + 2\sigma_X\sigma_Y \\ &\geq \sigma_X^2 + \sigma_Y^2 + 2\rho_{X,Y}\sigma_X\sigma_Y \\ &= \sigma_{X+Y}^2\end{aligned}$$

- So as long as $\Phi^{-1}(p) \leq 0$ (that is $p \leq 0.5$), we would have

$$VaR_p(X) + VaR_p(Y) \geq VaR_p(X+Y)$$

i.e. VaR is sub-additive in this case.

- Note that the argument above shows that volatility also satisfies sub-additivity.

VaR and coherence

- Why is VaR sometimes sub-additive and sometimes not?
- Subadditivity for the VaR is violated when the tails are super fat (first example on Slide 25 - 28).
- Examples of fat-tailed assets for which VaR is not a good (coherent) risk measure.
    - Exchange rates in countries that peg their currency but are subject to occasional devaluations.
    - Electricity prices subject to very extreme price swings.
    - Junk bonds where most of the time the bonds deliver a steady positive return.
    - Short deep out of the money options.
- Many assets do not have extremely fat tails, including most equities, exchange rates and commodities, so the range of situations in which VaR can be applied is reasonably broad.
- The conceptual simplicity and relative ease of computation of VaR means that it remains in use in those circumstances.

Expected shortfall

- Expected shortfall is defined as the expected loss conditional on a breach of VaR.

$$ES_p = -E\left(P/L \mid P/L \leq -VaR_p\right) = -\frac{\int_{-\infty}^{-VaR_p} xf(x)dx}{p}$$

- Expected shortfall is also known as expected tail loss or tail VaR.

- Once again, note the negative sign to make the measure positive.

Expected shortfall: an example

- Consider the example of the two-bond portfolio Slide 11.
  - With prob 0.9, payoff is \$0.4 mln.
  - With prob 0.1, payoff is uniformly dist. between -\$9.8mln and \$0.2mln.

- What is $ES_{5\%}$?

$ES_{5\%} = \$7.3\text{mln}.$

- What is $ES_{1\%}$?

$ES_{1\%} = \$9.3\text{mln}.$

ES captures the entire shape of the tail of a distribution

- This example is based on the computation of VaR in the two scenarios in slide 20.

- In both scenarios we have $VaR_{10\%} = 7.82\%$ of portfolio value.

- However, in scenario 1 (normal distribution), we have $ES_{10\%} = 12.55\%$ of portfolio value.

- In scenario 2 (mixture of normals), we have $ES_{10\%} = 26.38\%$ of portfolio value.

- Hence, expected shortfall can clearly distinguish the riskiness of the two distributions in tail events.

ES and coherence

- It is easy to show that Expected Shortfall satisfies Monotonicity, Homogeneity and Translation Invariance.

- We will show that Expected Shortfall satisfies Sub-Additivity, i.e.

$$ES_p(X+Y) \leq ES_p(X) + ES_p(Y)$$

for any probability $p \in (0, 1)$, random payoffs X and Y.

- For simplicity, define $Z = X + Y$ and

$$q_X = -VaR_p(X) \quad q_Y = -VaR_p(Y) \quad q_Z = -VaRp(Z)$$

$$ES_p(X+Y) = -E(X+Y|Z \leq q_Z)$$

Sub-additivity of ES

- With the notations above, we have

$$
\begin{aligned}
& ES_p(X) + ES_p(Y) - ES_p(X+Y) \\
= & -E(X|X \leq q_X) + (-E(Y|Y \leq q_Y)) - [-E(X+Y|Z \leq q_Z)] \\
= & [E(X|Z \leq q_Z) - E(X|X \leq q_X)] \\
& [E(Y|Z \leq q_Z) - E(Y|Y \leq q_Y)]
\end{aligned}
$$

- Each of these two difference terms in brackets turns out to be non-negative. We will show that for the first one, as the proof for the second one is analogous.

Sub-additivity of ES

- We need to show that $E[X|Z \leq q_Z] \geq E[X|X \leq q_X]$, that is, that the expected payoff in X is worse in the case of a breach in X than in the case of a breach in $Z = X + Y$.
- Consider three events:
  - $A = \{X \leq q_X \text{ and } Z \leq q_Z\}$ with probability p_A.
  - $B = \{X > q_X \text{ and } Z \leq q_Z\}$ with probability p_B.
  - $C = \{X \leq q_X \text{ and } Z > q_Z\}$ with probability p_C.
- We have

$$p = Prob(X \leq q_X) = p_A + p_C$$
$$p = Prob(Z \leq q_Z) = p_A + p_B$$

Thus, $p_B = p_C$.

Sub-additivity of ES

- $$\begin{aligned} E(X|Z \le q_Z) &= E(X|X \le q_X \text{ and } Z \le q_Z) Prob(X \le q_X | Z \le q_Z) \\ &+ E(X|X > q_X \text{ and } Z \le q_Z) Prob(X > q_X | Z \le q_Z) \\ &= E(X|A)\frac{p_A}{p} + E(X|B)\frac{p_B}{p} \end{aligned}$$

$$\begin{aligned} E(X|X \le q_X) &= E(X|X \le q_X \text{ and } Z \le q_Z) Prob(Z \le q_Z | X \le q_X) \\ &+ E(X|X \le q_X \text{ and } Z > q_Z) Prob(Z > q_Z | X \le q_X) \\ &= E(X|A)\frac{p_A}{p} + E(X|C)\frac{p_C}{p} \end{aligned}$$

Sub-additivity of ES

- Taking the difference between the two terms,

$$
\begin{aligned}
E(X|Z \leq q_Z) - E(X|X \leq q_X) &= E(X|B)\frac{p_B}{p} - E(X|C)\frac{p_C}{p} \\
&= \frac{p_B}{p}[E(X|B) - E(X|C)]
\end{aligned}
$$

- In event B $X > q_X$ and in event C $X \leq q_X$, so

$$E(X|B) \geq q_X \geq E(X|C)$$

- Thus,

$$E(X|Z \leq q_Z) - E(X|X \leq q_X) = \frac{p_B}{p}[E(X|B) - E(X|C)] \geq 0$$

- This completes the proof.

Comparing VaR with Expected shortfall

- Advantages of Expected Shortfall:
  - Expected Shortfall takes into account the entire shape of losses in tail events, and therefore provides a more accurate picture of tail risk.
  - Expected Shortfall is coherent, while VaR is not, so it behaves more in accordance with the intuition of how risk measures should behave.

- Advantages of VaR:

  - VaR is much easier to estimate than Expected Shortfall, because one needs a lot more information to estimate average tail losses than to estimate quantiles.

  - VaR is much easier to backtest than Expected Shortfall.
- In practice, both measures remain in use, although in data-rich environments the adoption of Expected Shortfall is increasing over time.