- Classes next week will discuss ICA 2020.



- Summative Assignment 2 has been posted and will be due at 11:59 pm on 27 November 2022.



- Solution to Summative Assignment 1 has been posted.

# Lecture 8: Implementing Risk Forecasts

FM321: Risk Management and Modelling





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# General approaches

- In Lecture 7, we said that in order to compute VaR and expected shortfall one of the key inputs we need is a distribution of P/L, which is unobserved.



- Two ways to estimate the distribution:
- Non-parametric approach



  - Apply information from historical data to existing portfolios to compute risk measures.
  - No models are assumed, and no parameters need to be estimated



- Parametric approach
  - Require analyst to obtain risk forecasts from a model for the distribution of returns for the portfolio or securities in question.
  - Rely on a framework for understanding the process that determines the distribution of common and idiosyncratic risks.

# Non-parametric approach

- The procedure for computing $VaR_p$ for an asset is:



  - Choose a historical sample length, and gather data for the returns on that asset for each day in that sample (that is, $r_1$, $r_2$, ..., $r_{T-1}$).



  - Compute the p-th quantile of the distribution of returns during that sample, and construct the $VaR_p$ accordingly.



  - Scale the measure up by the size of the holdings to obtain a monetary measure, if necessary (that is, multiply the measure by $P_{T-1}$).

- At a portfolio level, one can:
  - Choose a historical sample length, and gather data for the returns on each of the assets in the portfolio for each day in that sample (that is, $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, . . . , $\boldsymbol{r}_{T-1}$).
  - For each day in the sample, compute the hypothetical returns on the portfolio by using current holdings and the individual asset returns for that day (that is, $\boldsymbol{w}_T'\boldsymbol{r}_1$, $\boldsymbol{w}_T'\boldsymbol{r}_2$, . . . , $\boldsymbol{w}_T'\boldsymbol{r}_{T-1}$ - this will give us one data point of hypothetical portfolio returns, so we'll have $T-1$ hypothetical returns).
  - The $VaR_p$ of the portfolio is based on the p-th quantile of this distribution of portfolio returns.
- In both cases, expected shortfall is computed in a similar manner, by using the average of the payoffs in all of the $VaR_p$ breach events.

# Historical simulation considerations

- Historical simulations can be attractive in situations where it is difficult to estimate models that replicate the distribution of payoffs in the given portfolio.



- There is a trade-off involved in choosing the period length: a longer sample improves statistical accuracy, but only if the additional past data is relevant for current payoffs, and data in the more distant past is typically less relevant than recent data.



- As a general practical rule, one needs to have a sample size in which at least 3 violations would be expected for an accurate model (that is, a sample size of $\frac{3}{p}$).



- Given that it is difficult to obtain long enough samples for accurate estimates using this model, parametric approaches are generally preferred.

# Parametric approach

- Consider a portfolio with a single asset whose return is given by

$$\frac{P_T}{P_{T-1}} - 1 = r_T = \sigma_T z_T$$



where $z_T$ has a cumulative density function $F(z)$.

- P/L is given by



$$P/L_T = P_T - P_{T-1}$$

- Therefore,



$$\begin{aligned} p &= Prob\left(P_T - P_{T-1} \le -VaR_p\right) \\ &= Prob\left(r_T P_{T-1} \le -VaR_p\right) \\ &= Prob\left(\frac{r_T}{\sigma_T} \le \frac{-VaR_p}{P_{T-\mathbf{1}}\sigma_T}\right) \\ &= F\left(\frac{-VaR_p}{P_{T-\mathbf{1}}\sigma_T}\right) \end{aligned}$$

- Inverting the equation above yields,

$$F^{-1}(p) = -\frac{VaR_p}{P_{T-1}\sigma_T}$$



Therefore,

$$VaR_p = -P_{T-1}\sigma_T F^{-1}(p)$$



- This implies that to compute VaR, one needs:
  - $\sigma_T$: an estimate of conditional volatility for the portfolio returns.
  - $P_{T-1}$: previous portfolio value.
  - $F^{-1}(p)$: knowledge regarding the distribution of standardized returns.

- If the distribution of standardized conditional returns is normal,

$$VaR_p = -P_{T-1}\sigma_T\Phi^{-1}(p)$$

where $\Phi(p)$ is the c.d.f of $N(0,1)$.



- Given $\Phi^{-1}(5\%) \approx -1.645$, we have



$$VaR_{5\%} \approx 1.645P_{T-1}\sigma_T$$

# Parametric approach

- The general process in applying a parametric approach to implementing VaR in a univariate context (single asset) is as follows.



  - Choose a model (univariate or multivariate) to estimate conditional variance.



  - Using historical data for asset returns (that is, $r_1$, $r_2$, ..., $r_{T-1}$), estimate model parameters, and use estimates for determining the model's current estimate of conditional variance (that is, $\sigma_T$).



  - Compute VaR from the distribution obtained using the conditional volatility estimate $\sigma_T$ and the assumed distribution of standardized returns.

  - Scale the measure obtained this way by portfolio value (that is, multiply by $P_{T-1}$) to obtain a monetary measure if appropriate.

- If a portfolio consists of multiple assets, there are two ways of implementing VaR:



  - Univariate approach:



    - compute a hypothetical series of portfolio returns using current weights and historical returns for all of the assets in each date in the historical sample.



    - use this series of portfolio returns as a basis for univariate modelling.
    - from the estimated univariate model, construct the current estimate of the conditional variance of the portfolio return $\sigma_T$.

# Parametric approach

- Multivariate approach:



  - use the series of historical returns ($\boldsymbol{r}_1$, $\boldsymbol{r}_2$, ..., $\boldsymbol{r}_{T-1}$) to estimate a multivariate volatility model for the assets in the portfolio.



  - from the estimated model above, construct the current estimate of the conditional variance matrix $\boldsymbol{\Sigma}_T$ of the assets in this model.



  - derive the distribution of conditional returns for the portfolio given the conditional variance and portfolio weights (that is, using $\boldsymbol{w}_T$, $\boldsymbol{\Sigma}_T$ and information regarding the assumptions made regarding conditional standardized returns).

# Parametric approach

- The univariate approach has the advantage that it usually involves estimating fewer parameters, so there is less statistical error involved.
- But it reduces the analyst's ability to understand sources of shocks and how likely they are to persist.
- Example:
  - Suppose that we have ten equally weighted securities in a portfolio.
  - Scenario A: returns are zero for all of the securities, except that one of them loses half of its value.
  - Scenario B: returns in each security are equal to -5%.
  - In both scenarios, portfolio returns are the same (-5%).
  - But the implications for future volatility can be different.
  - The multivariate approach can distinguish these two cases, but the univariate one cannot.

- In the parametric case, samples need to be long enough for estimates of conditional volatility or variance to be reliable – appropriate sample sizes depend on the characteristics of the problem, but typically no less than one year.



- With the same idea, we can compute expected shortfall based on the expectation of all possible outcomes of this distribution conditional on a breach of VaR.

- To derive an expression for expected shortfall, note that

$$
\begin{aligned}
ES_p &= -E\left(P_T - P_{T-1} | P_T - P_{T-1} \le -VaR_p\right) \\
&= -E\left(r_T P_{T-1} | r_T P_{T-1} \le -VaR_p\right) \\
&= -\sigma_T P_{T-1} E\left(\frac{r_T}{\sigma_T} | \frac{r_T}{\sigma_T} \le \frac{-VaR_p}{P_{T-1}\sigma_T}\right)
\end{aligned}
$$



- The last expectation can be written as

$$
E\left(\frac{r_T}{\sigma_T} | \frac{r_T}{\sigma_T} \le \frac{-VaR_p}{P_{T-1}\sigma_T}\right) = \frac{\int_{-\infty}^{F^{-1}(p)} x f(x) dx}{p}
$$



- To compute the last integral, we need the conditional distribution of standardized returns.

- With normally distributed conditional returns, we have

$$
\begin{aligned}
\int_{-\infty}^{F^{-1}(p)} x f(x) dx &= \int_{-\infty}^{\Phi^{-1}(p)} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \\
&= \int_{-\infty}^{\Phi^{-1}(p)} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) d\left(\frac{x^2}{2}\right) \\
&= -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\Big|_{-\infty}^{\Phi^{-1}(p)} \\
&= -\phi(x)\Big|_{-\infty}^{\Phi^{-1}(p)} \\
&= -\phi(\Phi^{-1}(p))
\end{aligned}
$$

where $\phi()$ is the p.d.f of $N(0,1)$.

- Thus,

$$ES_p = \sigma_T P_{T-1} \frac{\phi(\Phi^{-1}(p))}{p}$$

- Examples:

| $p$ | $\frac{\phi(\Phi^{-1}(p))}{p}$ | $\frac{ES}{VaR_p}$ |
|---|---|---|
| 0.001 | 3.367 | 1.090 |
| 0.01 | 2.665 | 1.146 |
| 0.05 | 2.063 | 1.254 |

- With $p = 5\%$, we have

$$ES_p \approx 2.063 \sigma_T P_{T-1}$$

# Examples: $z_T$ follows a t-distribution

- For a Student t distribution with four degrees of freedom, we have:

| $p$ | $-F^{-1}(p)$ | $\frac{\int_{-\infty}^{F^{-1}(p)} xf(x)dx}{p}$ | $\frac{ES}{VaR_p}$ |
|---|---|---|---|
| 0.001 | 5.072 | 6.849 | 1.350 |
| 0.01 | 2.649 | 3.692 | 1.393 |
| 0.05 | 1.507 | 2.265 | 1.502 |

- For example, with $p = 5\%$,

$$VaR_p \approx 1.507 \sigma_T P_{T-1}$$

$$ES_p \approx 2.265 \sigma_T P_{T-1}$$

- Important note: the standard t(4) distribution has variance equal to 2, but our assumptions are that the standardized returns have unit variance. Thus, we are really working with a scaled version of a t(4) that satisfies this assumption.