Plan for the rest of term

- Tue 22 Nov (today): Lecture 9 Backtesting and Stress Testing
    - Week 10 classes: Class9Lecture.pdf

- Tue 29 Nov: Lecture 10 Risk Forecasts for Bonds and Options
    - Week 11: no classes

- Tue 29 Nov: Summative Assignment 2 due

- Wed 30 Nov: optional Zoom review session at noon

Plan for the rest of the term

- Tue 6 Dec: ICA (75 min)

- Fri 9 Dec: release of course project
  - Due at 4 pm on Fri, 20 Jan 2023

  - Similar to summative assignments and class material
  - For fairness, no questions will be answered about it after the release. So please be well-prepared and make sure you are familiar implementing techniques in R before then.

  - My office hours available for questions: Tue 6 Dec 4-5 pm, Thu 8 Dec 2-3 pm.

Lecture 10: Backtesting and Evaluating Risk Forecasts

FM321: Risk Management and Modelling

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LSE Finance

- Consists of two parts:

  - implementing risk forecasts in the historical sample;

  - evaluating their performance.

- We denote by T the sample size and by W the minimum estimation window; dates are $t = 1, 2, \ldots, T$:

- Steps:
  - Set initial estimation window from $t = 1$ to $t = W$.
  - Estimate model with this window, and produce VaR_p estimate for $t = W + 1$.

  - Change estimation window
    - either by including one more observation in the sample (so that the sample is $t = 1, 2, \ldots, W, W + 1$ i.e., "expanding window")

    - or by shifting start and end points by 1 (so that the sample is $t = 2, 3, \ldots, W + 1$; i.e., "moving window").
  - Repeat procedure from second step to obtain VaR_p estimate for $t = W + 2$.

- The above procedure yields $T - W$ forecasts for VaR_p, which can be compared to realized values for the underlying returns or P/L.

- A violation is the event that VaR is breached.

- We define the violation indicator as

$$V_t = \begin{cases} 1 & \text{if } P_t - P_{t-1} < -VaR_p \\ 0 & \text{otherwise} \end{cases}$$

- We get out of the procedure a sequence of zeros and ones, indicating the periods when the violations occurred:

$$V_{W+1}, V_{W+2}, \ldots, V_T$$

- This is called the hit sequence.

Backtesting: computational tip

- The procedure aims to replicate what one would do in practice by making sure one only uses information that is available in real time.

- Given the need to re-estimate models every period, this is potentially time-consuming and computationally intensive.

- One useful tip: since parameter estimates will typically not change very much by adding one data point to an already large sample, one can start the iteration in each period using the parameter estimates from the previous period as initial values.

- Violation ratios & Unconditional coverage ratio test.

  - If VaR_p forecasts are correct, a breach ($P/L < -VaR_p$) happens with probability p.

- Conditional coverage ratio test.

  - If VaR_p forecasts are correct, the probability of having a breach ($P/L < -VaR_p$) should not depend on whether a breach occurred the day before.

- The violation ratio for the entire simulation by

$$VR = \frac{1}{p} \frac{\sum_{t=W+1}^{T} V_t}{T - W}$$

- Violations should happen with frequency p.

- Violation ratios tell us whether the model overforecasts risk (if $VR < 1$) or underforecasts risk (if $VR > 1$).

Evaluation: unconditional coverage test

- How to test $\mathcal{H}_0$: violations occur with probability $p = p_0$ (say, 1% or 5%)?

- Likelihood-ratio test.

- $\mathcal{L}_U$: likelihood of data $\{V_{W+1}, \ldots, V_T\}$ as a function of parameters p where p is a free parameter.

- $\mathcal{L}_R$: likelihood of data $\{V_{W+1}, \ldots, V_T\}$ as a function of parameters p where $p = p_0$.

- Under $H_0 : p = p_0$, the likelihood ratio test statistic:

$$-2log\left(\mathcal{L}_R - \mathcal{L}_U\right) \sim \chi_1^2$$

- Denote by p the true probability of a violation, so that the distribution of V_t is Bernoulli with parameter p and its density is given by

$$f(V_t|p) = (1-p)^{1-V_t} p^{V_t}$$

- The number of violations V_1 and non-violations V_0 are given by

$$V_1 = \sum_{t=W+1}^{T} V_t \qquad V_0 = (T-W) - V_1$$

- so

$$\mathcal{L} = \Pi_{t=W+1}^{T} f(V_t|p) = (1-p)^{V_0} p^{V_1}$$

Evaluation: unconditional coverage test

- Under the null, the value of the likelihood (p is constrained to be equal to p_0)

$$\mathcal{L}_R = (1 - p_0)^{V_0} p_0^{V_1}$$

- The unconstrained likelihood is given by

$$\mathcal{L}_U = (1 - \hat{p})^{V_0} \hat{p}^{V_1}$$

where $\hat{p}$ is estimated by maximum likelihood $\hat{p} = \frac{\sum_{t=W+1}^{T} V_t}{T-W}$.

- Under the null, the likelihood ratio test statistic:

$$-2log\left(\mathcal{L}_R - \mathcal{L}_U\right) \sim \chi_1^2$$

- This test helps determine whether violations are happening with the expected frequency (either underforecasting or overforecasting risk are generally undesirable).

- We denote by p_{ij} the probability that state $V_t = i$ will be followed by state $V_{t+1} = j$.

$$p_{ij} = Prob(V_{t+1} = j | V_t = i) \qquad i, j \in \{0, 1\}$$

- We can consider the hit sequence $\{V_{W+1}, \dots, V_T\}$ as a realization of a Markov Chain with transition matrix

$$\Pi = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix}$$

- By construction, $p_{00} + p_{01} = p_{10} + p_{11} = 1$.

- So there are only two free parameters in Π to estimate: p_{01} and p_{11}.

- Ideally, the likelihood of a violation at time t would be independent of whether a violation occurred at time $t-1$ or not, so that we should have $p_{01} = p_{11}$ (if $p_{01} < p_{11}$ then violations will cluster).

- How to test whether $p_{01} = p_{11}$?

- Likelihood-ratio test.

- $\mathcal{L}_U$: likelihood of data $\{V_{W+1}, \ldots, V_T\}$ as a function of parameters (p_{01}, p_{11}) when parameters are unconstrained.

- $\mathcal{L}_R$: likelihood of data $\{V_{W+1}, \ldots, V_T\}$ as a function of parameters (p_{01}, p_{11}) when $p_{01} = p_{11}$.

- Under $H_0 : p_{01} = p_{11}$, the likelihood ratio test statistic:

$$-2log\left(\mathcal{L}_R - \mathcal{L}_U\right) \sim \chi^2_1$$

- What is $\mathcal{L}_R$?

- If we constrain $p_{01} = p_{11}$, then our maximum likelihood estimate equals that of the unconstrained estimator for the unconditional coverage test:

$$\hat{p}_{01} = \hat{p}_{11} = \hat{p} = \frac{\sum_{t=W+1}^{T} V_t}{T-W}$$

- Thus, in this case we have

$$\mathcal{L}_R = (1-\hat{p})^{V_0}\hat{p}^{V_1}$$

- What is $\mathcal{L}_U$?
- If p_{01} and p_{11} are unconstrained, the likelihood of V_{t+1} conditional on V_t is

$$f(V_{t+1}|V_t) = (1 - p_{V_{t1}})^{1-V_{t+1}} p_{V_{t1}}^{V_{t+1}}$$

- The likelihood of the hit sequence $\{V_{W+1}, \dots, V_T\}$ is

$$\mathcal{L}_U = f(V_{W+1}) f(V_{W+2}|V_{W+1}) f(V_{W+3}|V_{W+2}) \dots f(V_T|V_{T-1})$$

- Thus, we have

$$log \mathcal{L}_U = \underbrace{(1 - V_{W+1}) log(1 - \hat{p}) + V_{W+1} log(\hat{p})}_{log\ f(V_{W+1})} +$$

$$\sum_{t=W+1}^{T-1} \underbrace{\left[(1 - V_{t+1}) log(1 - p_{V_{t1}}) + V_{t+1} log(p_{V_{t1}})\right]}_{log\ f(V_{t+1}|V_t)}$$

- Maximizing $log\mathcal{L}_U$ with respect to p_{01} and p_{11} yields

$$\hat{p}_{01} = \frac{\sum_{t=W+1}^{T-1}(1-V_t)V_{t+1}}{\sum_{t=W+1}^{T-1}(1-V_t)} = \frac{\sum_{t=W+1}^{T-1}\mathcal{I}_{\{V_t=0 \text{ and } V_{t+1}=1\}}}{\sum_{t=W+1}^{T-1}\mathcal{I}_{\{V_t=0\}}}$$

$$\hat{p}_{11} = \frac{\sum_{t=W+1}^{T-1}V_tV_{t+1}}{\sum_{t=W+1}^{T-1}V_t} = \frac{\sum_{t=W+1}^{T-1}\mathcal{I}_{\{V_t=1 \text{ and } V_{t+1}=1\}}}{\sum_{t=W+1}^{T-1}\mathcal{I}_{\{V_t=1\}}}$$

- so $\mathcal{L}_U$ is given by replacing $p_{V_{t1}}$ in the previous slide by $\hat{p}_{01}$ and $\hat{p}_{11}$.

- Thus, the likelihood ratio test statistic is

$$-2\left(log\mathcal{L}_R - log\mathcal{L}_U\right) \sim \chi^2_1$$

where

$$log\mathcal{L}_U = \underbrace{(1-V_{W+1})log(1-\hat{p}) + V_{W+1}log(\hat{p})}_{log\ f(V_{W+1})} +$$

$$\sum_{t=W+1}^{T-1} \underbrace{\left[(1-V_{t+1})log(1-\hat{p}_{V_{t1}}) + V_{t+1}log(\hat{p}_{V_{t1}})\right]}_{log\ f(V_{t+1}|V_t)}$$

and

$$\mathcal{L}_R = (1-\hat{p})^{V_0}\hat{p}^{V_1}$$

- This test helps determine whether a violation in one period predicts a higher likelihood of a violation in the next.

- Note that coverage tests require a lot of data: if $p = 0.01$, then in order to expect to have three instances of one violation followed by another we would need $30{,}000$ data points (or 120 years with daily data).

Backtesting expected shortfall

- The general process is:

  - Given an assumption for conditional distribution of returns, compute expected shortfall (ES_t) for each day when a violation occurs.

  - For each such that, compute the ratio between actual shortfall (S_t and expected shortfall.

  - The expected value of the ratio $\frac{S_t}{ES_t}$ should be 1 if the model is accurate, so we can carry out tests of this hypothesis.

- Usually, data requirements for testing expected shortfall are much greater than those for backtesting VaR.