Repeat_exam_WiSe18

Machine learning (Technische Universität München)

Machine Learning — Repeat Exam

1	2	3	4	5	6	7	8	9	10	11	$\sum$
4	5	[illegible]	[illegible]	[illegible]	5	4	7	4	2	7	54

[illegible] anything above this line

Name:

Student ID: Signature:

- Only write on the sheets given to you by supervisors. If you need more paper, ask the supervisors.
- Pages 16-18 can be used as scratch paper.
- All sheets (including scratch paper) have to be returned at the end.
- **Do not unstaple the sheets!**
- Wherever answer boxes are provided, please write your answers in them.
- Please write your student ID (*Matrikelnummer*) on every sheet you hand in.
- **Only use a black or a blue pen (no pencils, red or green pens!).**
- You are allowed to use your A4 sheet of handwritten notes (two sides). **No other materials (e.g. books, cell phones, calculators) are allowed!**
- Exam duration - 120 minutes.
- This exam consists of 18 pages, 11 problems. You can earn 54 points.

Probability distributions

For your reference, we provide the following probability distribution.

- Univariate normal distribution

$$\mathcal{N}(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

- Bernoulli distribution

$$\mathrm{Bern}(x|\theta) = \theta^x (1-\theta)^{(1-x)}$$

Decision Trees

Problem 1 [(2+2)=4 points] Assume you want to build a decision tree. Your data set consists of N samples, each with k [illegible]

a) If the features are [illegible] maximum possible number of leaf nodes and the maximum depth of your dec[illegible]

b) If the features are continuous, what is the maximum possible number of leaf nodes and the maximum depth of your decision tree?

Regression

Problem 2 [(1+4)=5 points] We want to perform regression on a dataset consisting of N samples $\boldsymbol{x}_i \in \mathbb{R}^D$ with corresponding targets $y_i \in \mathbb{R}$ (represented compactly as $\boldsymbol{X} \in \mathbb{R}^{N \times D}$ and $\boldsymbol{y} \in \mathbb{R}^N$).

Assume that we have fitted an L_2-regularized linear regression model and obtained the optimal weight vector $\boldsymbol{w}^* \in \mathbb{R}^D$ as

$$\boldsymbol{w}^* = \arg\min_{\boldsymbol{w}} \frac{1}{2}\sum_{i=1}^{N}(\boldsymbol{w}^T\boldsymbol{x}_i - y_i)^2 + \frac{\lambda}{2}\boldsymbol{w}^T\boldsymbol{w}$$

Note that there is no bias term.

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Now, assume that we obtained a new data matrix $\boldsymbol{X}_{new}$ by scaling all samples by the same positive factor $a \in (0, \infty)$. That is, $\boldsymbol{X}_{new} = a\boldsymbol{X}$ (and respectively $\boldsymbol{x}_i^{new} = a\boldsymbol{x}_i$).

a) Find the weight vector $\boldsymbol{w}_{new}$ that will produce the same predictions on $\boldsymbol{X}_{new}$ as $\boldsymbol{w}^*$ produces on $\boldsymbol{X}$.

b) Find the regularization factor $\lambda_{new} \in \mathbb{R}$, such that the solution $\boldsymbol{w}^*_{new}$ of the new L_2-regularized linear regression problem

$$\boldsymbol{w}^*_{new} = \arg\min_{\boldsymbol{w}} \frac{1}{2}\sum_{i=1}^{N}(\boldsymbol{w}^T\boldsymbol{x}_i^{new} - y_i)^2 + \frac{\lambda_{new}}{2}\boldsymbol{w}^T\boldsymbol{w}$$

will produce the same predictions on $\boldsymbol{X}_{new}$ as $\boldsymbol{w}^*$ produces on $\boldsymbol{X}$.

Provide a mathematical justification for your answer.

Classification

Problem 3 [(1+2+3)=6 points] We would like to perform binary classification on multivariate binary data. That is, t[illegible] $\{0,1\}^D$ are binary vectors of length D, and each sample belongs to one of two c[illegible]

Consider the following [illegible]on model. We place a categorical prior on y

$$[illegible] = \pi_1 \qquad p(y = 2) = \pi_2.$$

The class-conditional d[illegible]ucts of independent Bernoulli distributions

$$p(\boldsymbol{x} \mid y = 1, \boldsymbol{\alpha}) = \prod_{j=1}^{D} \text{Bern}(x_j \mid \alpha_j),$$

$$p(\boldsymbol{x} \mid y = 2, \boldsymbol{\beta}) = \prod_{j=1}^{D} \text{Bern}(x_j \mid \beta_j),$$

where $\boldsymbol{\alpha} \in [0,1]^D$ and $\boldsymbol{\beta} \in [0,1]^D$ are the respective parameter vectors for both classes. That is, each component x_j is distributed as $x_j \sim \text{Bern}(\alpha_j)$ if $y = 1$ or $x_j \sim \text{Bern}(\beta_j)$ if $y = 2$.

a) Write down the expression for the posterior distribution $p(y \mid \boldsymbol{x}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\pi})$.

b) Assume that $D = 3$, $\boldsymbol{\alpha} = [1/3, 1/3, 3/4]$, $\boldsymbol{\beta} = [2/3, 1/2, 1/2]$, $\pi_1 = 1/3$ and $\pi_2 = 2/3$.

Write down a data point $\boldsymbol{x}_1 \in \{0,1\}^3$ that will be classified as class 1 by our model. Additionally, compute the posterior probability $p(y = 1 \mid \boldsymbol{x}_1, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\pi})$.

c) Consider the case when $D = 2$, $\pi_1 = \pi_2 = 1/2$, and $\boldsymbol{\alpha} \in [0,1]^2$ and $\boldsymbol{\beta} \in [0,1]^2$ are known and fixed. Show that the resulting classification rule can be represented as a linear function of $\boldsymbol{x}$. That is, find $\boldsymbol{w} \in \mathbb{R}^2$ and $b \in \mathbb{R}$, such that

$\{\boldsymbol{x} \in \{$ [illegible] $\} = \{\boldsymbol{x} \in \{0,1\}^2 : p(y = 1 \mid \boldsymbol{x}) > p(y = 2 \mid \boldsymbol{x})\}$

Kernels

Problem 4 [(4)=4 points] Prove or disprove whether the following operations on sets $A, B \subseteq \mathcal{X}$, where $\mathcal{X}$ is a finite set, define a valid kernel.

a) $k(A, B) = |A \times B|$, where $A \times B = \{(a, b) : a \in A, b \in B\}$ denotes the cartesian product and $|S|$ denotes the cardinality of set S, i.e. the number of elements in S.

b) $k(A, B) = |A \cap B|$

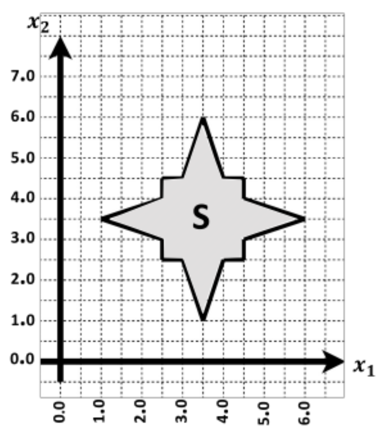
c) $k(A, B) = |A \cup B|$

Optimization

Problem 5 [(1+3+2)=6 points] Let f be the following convex function on $\mathbb{R}^2$:

$$f(x_1, x_2) = e^{x_1+x_2} - 5 \cdot \log(x_2)$$

a) Consider the following shaded region $\boldsymbol{S}$ in $\mathbb{R}^2$. Is this region convex? Why?



b) Find the maximiz[illegible] the shaded region $\boldsymbol{S}$. For your computations, you can pick values from t[illegible] justify your answer.

$e^{4.5} = 90.017$	$e^{5.0} = 148.41$	$e^{5.5} = 244.69$	$e^{6.5} = 665.14$
$e^{7.0} = 1096.63$	$e^{7.5} = 1808.04$	$e^{8.0} = 2980.95$	$e^{8.5} = 4914.76$
$e^{9.0} = 8103.08$	$e^{9.5} = 13359.726$	$e^{10.0} = 22026.46$	$e^{10.5} = 36315.50$
$\log(1.0) = 0$	$\log(2.5) = 0.9162$	$\log(3.0) = 1.0986$	$\log(3.5) = 1.2527$
$\log(4.0) = 1.3862$	$\log(4.5) = 1.5040$	$\log(5.0) = 1.6094$	$\log(6.0) = 1.7917$

c) Assume that we are given an algorithm $\texttt{ConvOpt}(f, \mathcal{X})$ that takes as input a convex function f and any convex region $\mathcal{X}$, and returns the minimum of f over $\mathcal{X}$.

Using the $\texttt{ConvOpt}$ algorithm, how would you find the global minimum of f over the shaded region $\boldsymbol{S}$?

SVM

Problem 6 [(5)=5 points] Given the data points

$$\boldsymbol{x}_1 = (1, 1, 0, \text{[illegible]} 0)^T \qquad \boldsymbol{x}_3 = (0, 1, 1, 1)^T \qquad \boldsymbol{x}_4 = (0, 0, 1, 1)^T$$

Prove or disprove wheth[illegible]binations of labels $\boldsymbol{y}$ and dual variables $\boldsymbol{\alpha}$ are the optimal solutions of a soft-marg[illegible]

a) $\boldsymbol{y} = (-1, -1, 1, 1)^T$ [illegible] $)^T$

b) $\boldsymbol{y} = (-1, -1, 1, 1)^T$, $\boldsymbol{\alpha} = (\frac{\text{[illegible]}}{3}, \frac{\text{[illegible]}}{3}, \frac{\text{[illegible]}}{3}, 0)$

c) $\boldsymbol{y} = (-1, 1, -1, 1)^T$, $\boldsymbol{\alpha} = (1, 1, 1, 1)^T$

Deep Learning

Problem 7 [(2+2)=4 points] You are trying to solve a regression task and you want to choose between two approache

1. A simple linear re
2. A feed forward ne vith L hidden layers, where each hidden layer $l \in \{1, ..., L\}$ has a weight matr ReLU activation function. The output layer has a weight matrix $\boldsymbol{W}_{L+1} \in \mathbb{R}$ n function.

In both models, there a

Your dataset $\mathcal{D}$ contains data points with nonnegative features $\boldsymbol{x}_i$ and the target y_i is continuous:

$$\mathcal{D} = \{\boldsymbol{x}_i, y_i\}_{i=1}^{N}, \quad \boldsymbol{x}_i \in \mathbb{R}^D_{\geq 0}, \quad y_i \in \mathbb{R}$$

Let $\boldsymbol{w}^*_{LS} \in \mathbb{R}^D$ be the optimal weights for the linear regression model corresponding to a global minimum of the following least squares optimization problem:

$$\boldsymbol{w}^*_{LS} = \arg\min_{\boldsymbol{w} \in \mathbb{R}^D} \mathcal{L}_{LS}(\boldsymbol{w}) = \arg\min_{\boldsymbol{w} \in \mathbb{R}^D} \frac{1}{2} \sum_{i=1}^{N} (\boldsymbol{w}^T \boldsymbol{x}_i - y_i)^2$$

Let $\boldsymbol{W}^*_{NN} = \{\boldsymbol{W}^*_1, \ldots, \boldsymbol{W}^*_{L+1}\}$ be the optimal weights for the neural network corresponding to a global minimum of the following optimization problem:

$$\boldsymbol{W}^*_{NN} = \arg\min_{\boldsymbol{W}} \mathcal{L}_{NN}(\boldsymbol{W}) = \arg\min_{\boldsymbol{W}} \frac{1}{2} \sum_{i=1}^{N} (f_{\boldsymbol{W}}(\boldsymbol{x}_i) - y_i)^2$$

a) Assume that the optimal $\boldsymbol{W}^*_{NN}$ you obtain are non-negative.
What will be the relation ($<, \leq, =, \geq, >$) between the neural network loss $\mathcal{L}_{NN}(\boldsymbol{W}^*_{NN})$ and the linear regression loss $\mathcal{L}_{LS}(\boldsymbol{w}^*_{LS})$? Provide a mathematical argument to justify your answer.

b) In contrast to (a), now assume that the optimal weights $\boldsymbol{w}^*_{LS}$ you obtain are non-negative. What will be the relation $(<, \leq, =, \geq, >)$ between the linear regression loss $\mathcal{L}_{LS}(\boldsymbol{w}^*_{LS})$ and the neural network loss $\mathcal{L}_{NN}(\boldsymbol{W}^*_{NN})$? Provide a mathematical argument to justify your answer.

Dimensionality Reduction

Problem 8 [(3+2+2)=7 points] You are given $N = 4$ data points: $\{\boldsymbol{x}_i\}_{i=1}^4, \boldsymbol{x}_i \in \mathbb{R}^3$, represented with the matrix $\boldsymbol{X} \in \mathbb{R}^{4\times 3}$.

$$\boldsymbol{X} = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 1 & -2 \\ 4 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix}$$

Hint: In this task the results of all (final and intermediate) computations happen to be integers.

a) Perform principal component analysis (PCA) of the data $\boldsymbol{X}$, i.e. find the principal components and their associated variances in the transformed coordinate system. Show your work.

b) Project the data to two dimensions, i.e. write down the transformed data matrix $\boldsymbol{Y} \in \mathbb{R}^{4\times 2}$ using the top-2 principal components you computed in (a). What fraction of variance of $\boldsymbol{X}$ is preserved by $\boldsymbol{Y}$?

c) Let $\boldsymbol{x}_5 \in \mathbb{R}^3$ be a new data point. Specify the vector $\boldsymbol{x}_5$ such that performing PCA on the data including the new data point $\{\boldsymbol{x}_i\}_{i=1}^5$ leads to exactly the same principal components as in (a).

Clustering

Problem 9 [(4)=4 points] Let $\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_K$ be the centroids computed by the K-means algorithm. Prove that the set $\mathcal{X}_j$ o[illegible]signed during inference to the cluster j is a convex set.

$$\mathcal{X}_j := \text{[illegible]e assigned to centroid } \boldsymbol{\mu}_j \text{ by } K\text{-means}\}$$

Hint: start by thinking [illegible] $K = 2$.

Problem 10 [(2)=2 points] Given three 1-dimensional Gaussian distributions $\mathcal{N}(\mu_i, \sigma_i^2)$ with parameters

$$\mu_1 = 1, \qquad \mu_2 = -1, \qquad \mu_3 = 0,$$
$$\sigma_1 = \text{[illegible]} \qquad \sigma_2 = 0.5, \qquad \sigma_3 = 2.5$$

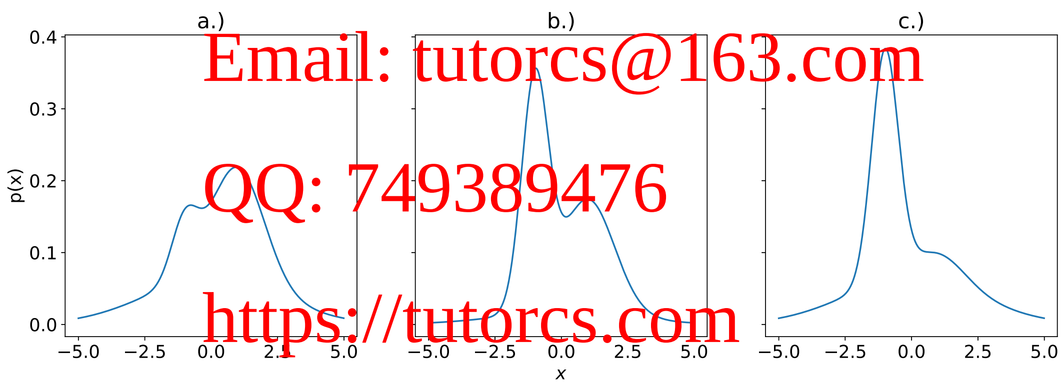
and three different vect[illegible]ents π defining categorical cluster priors.

Match the value of π in [illegible]owing table with one of the probability density functions

$$\text{[illegible]} \sum_{i=1}^{3} \pi_i \mathcal{N}(x \mid \mu_i, \Sigma_i)$$

of the resulting GMM showed below. Fill in the last column of the table, no argumentation required.

	π_1	π_2	π_3	PDF (a, b or c)
case 1	0.111...	0.444...	0.444...	
case 2	0.444...	0.111...	0.444...	
case 3	0.444...	0.444...	0.111...	



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Variational Inference

Problem 11 [(3+1+1+2)=7 points] Consider the following latent variable probabilistic model

$$p(z) = \mathcal{N}(z \mid 0, 1)$$
$$p(x \mid z) = \mathcal{N}(x \mid z, 1)$$

We want to approximate the posterior distribution $p(z \mid x)$ using the following variational family

$$\mathcal{Q} = \{q(z) = \mathcal{N}(z \mid \mu, 1) \text{ for } \mu \in \mathbb{R}\}$$

that includes all normal distributions with unit variance.

Questions (a), (b), (c) and (d) are all concerning this setup.

Hint: Variance of $p(z \mid x)$ is equal to 0.5.

a) Write down the closed-form expression for ELBO $\mathcal{L}(q)$ and simplify it. You can ignore all the terms constant in μ.

b) Find the optimal variational distribution $q^* \in \mathcal{Q}$ that maximizes the ELBO

$$q^* = \arg\max_{q \in \mathcal{Q}} \mathcal{L}(q)$$

i.e. find the mean μ^* of the optimal variational distribution q^*.

c) Assume that the optimal q^* (i.e. the optimal μ^*) from question (b) is given. Which of the following statements is true?

(1) $\mathbb{KL}(q(z \mid \mu^*) \parallel p(z \mid x)) < 0$

(2) $\mathbb{KL}(q(z \mid \mu^*) \parallel p(z \mid x)) = 0$

(3) $\mathbb{KL}(q(z \mid \mu^*) \parallel p(z \mid x)) > 0$

Justify your answer.

d) For each of the conditions (1), (2), (3) from question (c) above, provide a parametric variational family $\mathcal{Q}_i$, such that the optimal q_i^* from each family would fulfill the respective condition, or explain why it's impossible.

That is, provide $\mathcal{Q}_1$, such that for $q_1^* = \arg\max_{q \in \mathcal{Q}_1} \mathcal{L}(q)$ we have $\mathbb{KL}(q_1^*(z) \parallel p(z \mid x)) < 0$, for $q_2^* = \arg\max_{q \in \mathcal{Q}_2} \mathcal{L}(q)$ we have $\mathbb{KL}(q_2^*(z) \parallel p(z \mid x)) = 0$, and for $q_3^* = \arg\max_{q \in \mathcal{Q}_3} \mathcal{L}(q)$ we have $\mathbb{KL}(q_3^*(z) \parallel p(z \mid x)) > 0$.

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