# Machine Learning Exercise Sheet 06

## Optimization

Exercise sheets consist o[...]k and in-class exercises. You solve the homework exercises on your own or with you[...]d upload it to Moodle for a possible grade bonus. The in-class exercises will be sol[...]ring the tutorial. You do not have to upload any solutions of the in-class exercises.

## In-class Exercises



**Problem 1:** Prove or disprove whether the following functions $f : D \to \mathbb{R}$ are convex

a) $D = (1, \infty)$ and $f(x) = \log(x) - x^3$,



b) $D = \mathbb{R}^+$ and $f(x) = -\min(\log(3x + 1), -x^4 - 3x^2 + 8x - 42)$,

c) $D = (-10, 10) \times (-10, 10)$ and $f(x, y) = y \cdot x^3 - y \cdot x^2 + y^2 + y + 4$.



a) The second derivative of $f$ is $\frac{d^2 f(x)}{dx^2} = \frac{d}{dx}\left(\frac{1}{x} - 3x^2\right) = -\frac{1}{x^2} - 6x$, which is negative on the given set $D$ and therefore $f$ is not convex.



b) Transform min to max

$$-\min\{\log(3x+1), -x^4 - 3x^2 + 8x - 42\} = \max\{-\log(3x+1), x^4 + 3x^2 - 8x + 42\}\,.$$

max($g_1(x), g_2(x)$) is convex if both $g_1$ and $g_2$ are convex on $D = \mathbb{R}^+$ (see Exercise Sheet 6, Problem 1c). $g_1(x) = -\log(3x + 1)$ is convex since the second derivative is positive on $\mathbb{R}^+$:



$$\frac{d^2}{dx^2}\left(-\log(3x+1)\right) = \frac{d}{dx}\left(-\frac{3}{3x+1}\right) = \frac{9}{(3x+1)^2} > 0$$

$g_2(x) = x^4 + 3x^2 - 8x + 42$ is also convex:

$$\frac{d^2}{dx^2}\left(x^4 + 3x^2 - 8x + 42\right) = \frac{d}{dx}\left(4x^3 + 6x - 8\right) = 12x^2 + 6 > 0$$

Thus $f$ is convex.

c) For the function $f(x, y)$ to be convex (on $D$) it has to hold for all $x_1, x_2, y \in D$ and $\lambda \in (0, 1)$ that

$$\lambda f(x_1, y) + (1 - \lambda) f(x_2, y) \geq f(\lambda x_1 + (1 - \lambda) x_2, y)\,.$$

It does not hold in our case, consider $y = 1, x_1 = -4, x_2 = 0$ and $\lambda = 0.5$:

$$0.5 f(-4, 1) + 0.5 f(0, 1) = 0.5 \cdot (-74) + 0.5 \cdot 6 = -34$$
$$f\big(0.5 \cdot (-4) + 0.5 \cdot 0, 0.5 \cdot 1 + 0.5 \cdot 1)\big) = f(-2, 1) = -6 \;>\; -34$$

Thus $f(x, y)$ is not convex.

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**Problem 2:** Prove that the following function (the loss function of logistic regression) $f : \mathbb{R}^d \to \mathbb{R}$ is convex:

$$f(\boldsymbol{w}) = -\ln p(\boldsymbol{y} \mid \boldsymbol{w}, \boldsymbol{X}) = -\sum_{i=1}^{N} \left( y_i \ln \sigma(\boldsymbol{w}^T \boldsymbol{x}_i) + (1 - y_i) \ln(1 - \sigma(\boldsymbol{w}^T \boldsymbol{x}_i)) \right) .$$

First, let's simplify the loss function. For this we will need the following two facts

$$\sigma(z) = \frac{1}{1 + e^{-z}} \qquad \text{and} \qquad 1 - \sigma(z) = \sigma(-z) = \frac{1}{1 + e^{z}},$$

which implies that

$$\ln \sigma(z) = \ln \left( \frac{e^z}{1 + e^z} \right) = z - \ln(1 + e^z) \qquad \text{and} \qquad \ln(1 - \sigma(z)) = -\ln(1 + e^z).$$

Plugging this into the definition of the loss function we obtain

$$\begin{aligned} f(\boldsymbol{w}) &= -\sum_{i=1}^{N} \left( y_i \ln \sigma(\boldsymbol{w}^T \boldsymbol{x}_i) + (1 - y_i) \ln(1 - \sigma(\boldsymbol{w}^T \boldsymbol{x}_i)) \right) \\ &= -\sum_{i=1}^{N} \left( y_i \left( \boldsymbol{w}^T \boldsymbol{x}_i - \ln(1 + e^{\boldsymbol{w}^T \boldsymbol{x}_i}) \right) - (1 - y_i) \ln(1 + e^{\boldsymbol{w}^T \boldsymbol{x}_i}) \right) \\ &= \sum_{i=1}^{N} \left( -y_i (\boldsymbol{w}^T \boldsymbol{x}_i) + \ln(1 + e^{\boldsymbol{w}^T \boldsymbol{x}_i}) \right) \end{aligned}$$

We know that $\boldsymbol{w}^T \boldsymbol{x}_i$ is a convex (and concave) function of $\boldsymbol{w}$. Therefore, the first term $-y_i(\boldsymbol{w}^T \boldsymbol{x}_i)$ is also convex.

Now, if we show that $\ln(1 + e^z)$ is a nondecreasing and convex function of $z$ on $\mathbb{R}$, we will be able to use the convexity preserving operations to prove that $f(\boldsymbol{w})$ is convex.

The first derivative of $\ln(1 + e^z)$ is

$$\frac{d}{dz} \ln(1 + e^z) = \frac{e^z}{1 + e^z} = \sigma(z),$$

which is positive for all $z \in \mathbb{R}$, which means that $\ln(1 + e^z)$ is an nondecreasing function.

The second derivative is

$$\frac{d^2}{dz^2} \ln(1 + e^z) = \frac{d}{dz} \sigma(z) = \sigma(z)\sigma(-z),$$

which is also positive for all $z \in \mathbb{R}$, which means that $\ln(1 + e^z)$ is a convex function.

Using the following two facts

1. Sum of convex functions is convex
2. Composition of a convex function with a convex nondecreasing function is convex

we can verify that $f(\boldsymbol{w})$ is indeed convex in $\boldsymbol{w}$ on $\mathbb{R}^d$.

**Problem 3:** Prove that for differentiable convex functions each local minimum is a global minimum. More specifically, given a differentiable convex function $f : \mathbb{R}^d \to \mathbb{R}$, prove that



a) if $\boldsymbol{x}^*$ is a local minimum, then $\nabla f(\boldsymbol{x}^*) = \boldsymbol{0}$.

b) if $\nabla f(\boldsymbol{x}^*) = \boldsymbol{0}$, the[illegible]mum.

We will show that if t[illegible]oint $\boldsymbol{x}^*$ is not equal to zero, then this point cannot be a local optimum — we c[illegible] direction of the negative gradient and end up in a point with a lower value of t[illegible]

More formally, suppos[illegible]ne $\boldsymbol{x}^*$. Then by Taylor's theorem for a sufficiently small $\varepsilon > 0$ we get

$$
\begin{aligned}
f(\boldsymbol{x}^* - \varepsilon \nabla f(\boldsymbol{x}^*)) &= f(\boldsymbol{x}^*) - (\varepsilon \nabla f(\boldsymbol{x}^*))^T \nabla f(\boldsymbol{x}^*) + O(\varepsilon^2 \|\nabla^2 f(\boldsymbol{x}^*)\|_2^2) \\
&= f(\boldsymbol{x}^*) - \varepsilon \|\nabla f(\boldsymbol{x}^*)\|_2^2 + O(\varepsilon^2 \|\nabla f(\boldsymbol{x}^*)\|_2^2) \\
&< f(\boldsymbol{x}^*)
\end{aligned}
$$



Which means that $\boldsymbol{x}^*$ is not a local optimum. Therefore, the gradient must be equal to zero for any local optimum $\boldsymbol{x}^*$.



We will prove (b) using the first-order criterion for convexity:

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + (\boldsymbol{y} - \boldsymbol{x})^T \nabla f(\boldsymbol{x}).$$



If we plug in $\boldsymbol{x}^*$ and use the fact that $\nabla f(\boldsymbol{x}^*) = 0$ we get: $f(\boldsymbol{y}) \geq f(\boldsymbol{x}^*)$ for all $\boldsymbol{y}$, meaning $\boldsymbol{x}^*$ is a global minimum.



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## Homework

## 1 Convexity of fu[illegible]

**Problem 4:** Assume th[illegible] $\to \mathbb{R}$ are convex functions. Prove or disprove the following statements:

a) The function $h(x)$ [illegible]

b) The function $h(x)$ [illegible]f $g$ is non-decreasing.

*Note: For this exercise you are not allowed to use the convexity preserving operations from the lecture.*

a) Statement is false. Proof by counterexample: Suppose $f(x) = x^2$ and $g(z) = -x$. Since the function $h(x) = g(f(x)) = -x^2$ is twice differentiable, we can inspect its second derivative:



$$\frac{d^2}{dx^2}h(x) = -1.$$

Since the second derivative is negative for all $x$, we conclude that the function $h$ is not convex.



*(Note: It would actually be sufficient to show that the second derivative is negative for a single value of $x$)*



b) Statement is true. Suppose $x_0, x_1 \in \mathbb{R}$ and $\lambda \in (0, 1)$. We will use a shorthand notation $x_\lambda = \lambda x_1 + (1-\lambda)x_0$.

We will prove the convexity of $h$ using the definition of convexity and the properties of $f$ and $g$:



$$\begin{aligned}
&f \text{ convex} && \Rightarrow && f(x_\lambda) \le \lambda f(x_1) + (1-\lambda)f(x_0) \\
&g \text{ non-decresing} && \Rightarrow && g(f(x_\lambda)) \le g(\lambda f(x_1) + (1-\lambda)f(x_0)) && (1)\\
&g \text{ convex} && \Rightarrow && g(\lambda f(x_1) + (1-\lambda)f(x_0)) \le \lambda g(f(x_1)) + (1-\lambda)g(f(x_0)) && (2)\\
&(1) \text{ and } (2) && \Rightarrow && g(f(x_\lambda)) \le \lambda g(f(x_1)) + (1-\lambda)g(f(x_0)) \\
& && \Leftrightarrow && h(x_\lambda) \le \lambda h(x_1) + (1-\lambda)h(x_0).
\end{aligned}$$



Therefore $h$ is convex.

## 2 Optimization / Gradient descent

**Problem 5:** You are given the following objective function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x_1, x_2) = 0.5x_1^2 + x_2^2 + 2x_1 + x_2 + \cos(\sin(\sqrt{\pi})).$$

a) Compute the minimizer $\boldsymbol{x}^*$ of $f$ analytically.

As $f$ is a sum of convex functions, it is convex. To find the global minimizer, we compute the gradient and set it to zero

$$[illegible] \begin{pmatrix} \cdots + 2 \\ \cdots + 1 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} -2 \\ -\frac{1}{2} \end{pmatrix}.$$



b) Perform 2 steps of [illegible] $f$ starting from the point $\boldsymbol{x}^{(0)} = (0, 0)$ with a constant learning rate $\tau = 1$[illegible]

We already know how [illegible] ent from a).

first step $$\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix} - \tau \begin{pmatrix} x_1^{(0)} + 2 \\ 2x_2^{(0)} + 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - 1 \begin{pmatrix} 0 + 2 \\ 0 + 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$



second step $$\begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \end{pmatrix} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} - \tau \begin{pmatrix} x_1^{(1)} + 2 \\ 2x_2^{(1)} + 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} - 1 \begin{pmatrix} -2 + 2 \\ -2 + 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$



c) Will the gradient descent procedure from Problem b) ever converge to the true minimizer $\boldsymbol{x}^*$? Why or why not? If the answer is no, how can we fix it?



By performing one more iteration of gradient descent we observe that

$$\begin{pmatrix} x_1^{(3)} \\ x_2^{(3)} \end{pmatrix} = \begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \end{pmatrix} - \tau \begin{pmatrix} x_1^{(2)} + 2 \\ 2x_2^{(2)} + 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} - 1 \begin{pmatrix} -2 + 2 \\ 0 + 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix}.$$



That is, we are stuck iterating between $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ forever. We can fix this by decreasing the learning rate (adaptive stepsize, etc.).



**Problem 6:** Load the notebook `exercise_06_notebook.ipynb` from Moodle. Fill in the missing code and run the notebook. Export (download) the evaluated notebook as PDF and add it to your submission.

*Note: We suggest that you use Anaconda for installing Python and Jupyter, as well as for managing packages. We recommend that you use Python 3.*

*For more information on Jupyter notebooks, consult the Jupyter documentation. Instructions for converting the Jupyter notebooks to PDF are provided on Piazza.*

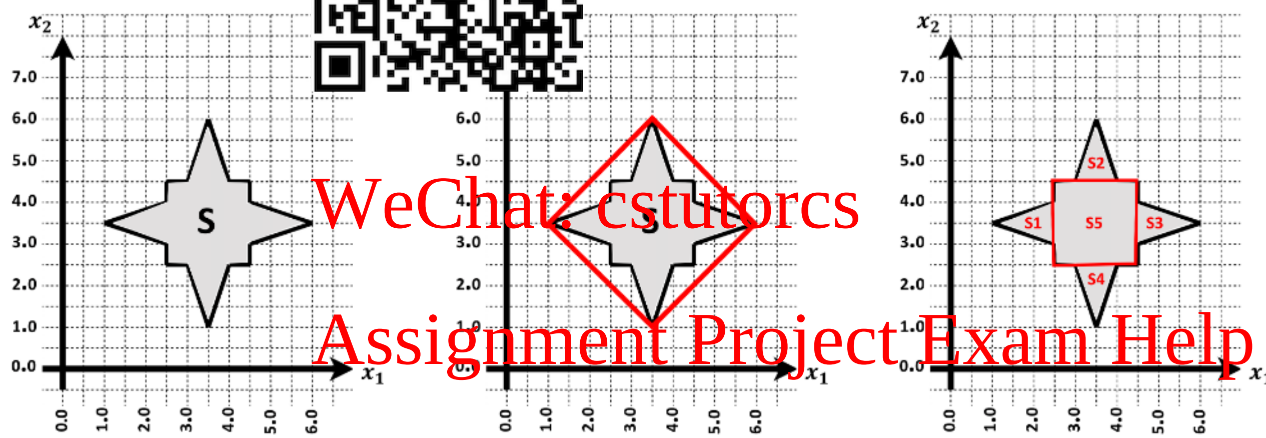
The solution notebook is uploaded to Moodle.

**Problem 7:** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the following convex function:

$$f(x_1, x_2) = e^{x_1 + x_2} - 5 \cdot \log(x_2)$$

a) Consider the follow[illegible] $\subset \mathbb{R}^2$. Is this region convex? Why?

b) Assume that we ar[illegible] $\texttt{ConvOpt}(f, D)$ that takes as input a convex function $f$ and convex region $D$, a[illegible]um of $f$ over $D$. Using the $\texttt{ConvOpt}$ algorithm, how would you find the global [illegible] the shaded region $S$?




(a) initial non-convex set (b) convex hull (c) union of convex sets

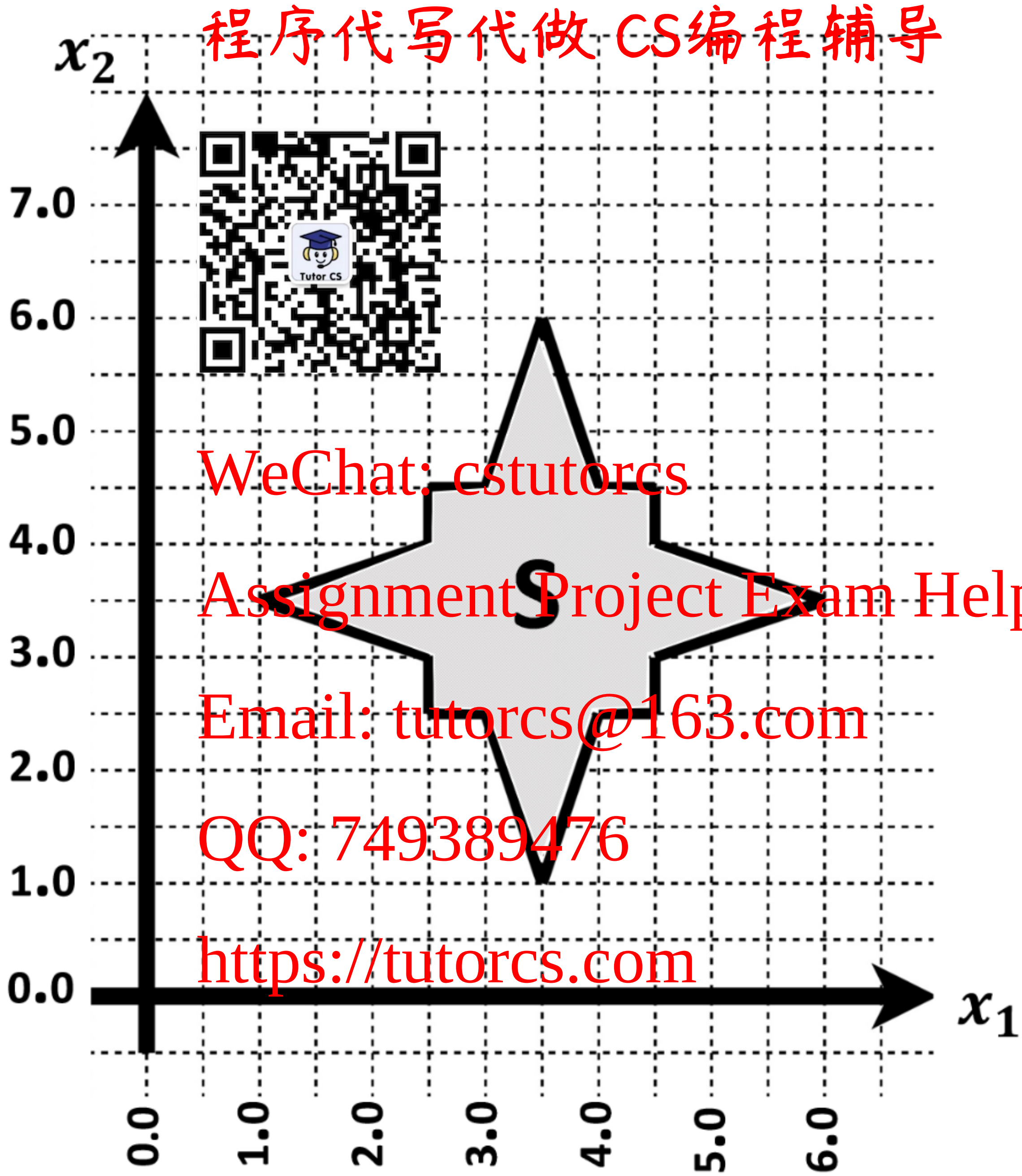


a) It is not because we can choose two points in $S$ such that the line connecting the points does not completely reside in $S$, for example $(3.0, 2.5)^T$ and $(3.5, 6.0)^T$ (see Figure 1b).

b) We can partition the shaded region $S$ to the following five convex regions $S_1, \dots, S_5$ (see Figure 1c). Afterwards, we run the $\texttt{ConvOpt}$ algorithm separately for the 5 regions and obtain

$$m_i = \min_{x \in S_i} f(x) = \texttt{ConvOpt}(f, S_i) .$$

Finally, the minimum over the whole $S$ can be computed as the smallest of these values, that is $\min_{x \in S} f(x) = \min(m_1, \dots, m_5)$.


$x_2$
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0.0
S
$x_1$
0.0
1.0
2.0
3.0
4.0
5.0
6.0