School of Mathematics and Statistics

MAST30031 Methods of Mathematical Physics, Semester 2 202[illegible]

Written assignment 3 and Cover Sheet

Student Name	Student Number

Submit your assig[illegible]er with this coversheet via the MAST30031 Gradescope website before [illegible]am AET) sharp. **No extensions will be granted! Only [illegible]ly justified reasons can be granted.**

- This assignment is worth 10% of your final MAST30031 mark.
- Assignments must be either neatly **handwritten** or can be written with LaTeX.
- Full working must be shown in your solutions.
- Marks will be deducted in every question for incomplete working, insufficient justification of steps and incorrect mathematical notation.
- You must use methods taught in MAST30031 Methods of Mathematical Physics to solve the assignment questions.
- **All tasks are mandatory for everyone!**
- There are in total 40 points to achieve.
- Begin your answer for each question on a new page!

Please, turn the page for the other questions!

1. **Summary 10 points**

Write a summary of the third part of the lectures called "Differential Forms". **The summary should be between two and three A4 pages!** To make it simple, pick **ten out of the sixteen topics below** and briefly describe those in a concise way. Use the space on the following three pages.

- differentia[illegible]orms as vector spaces
- differentia[illegible]r maps
- exact and [illegible]
- wedge product as multilinear maps
- exterior derivative
- closed and exact p-forms
- wedge product and grad, div, curl,
- integrating p-forms (higher dimensional contour integrals)
- orientations and differential p-forms
- boundaries and their orientations
- generalised Stoke's theorem
- Levi-Cevita symbol and the determinant
- Hodge star operator
- the Laplacian for differential forms
- de Rahm cohomologies
- Maxwell's equations

Please, turn the page for the other questions!

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2. **Simple Question 10 points.**

Compute the following contour integrals

(a) Compute the 1-dim contour integral

$$I_1 = \int_\gamma \omega$$

for the differ

$$\omega = ydx + xdy - xydz$$

and the 1-dir

$$\gamma(t) = (x, y, z) = (e^t, t, t^2).$$

with $t \in]-\infty$

(b) Compute the 2-dim contour integral

$$I_2 = \int_S \sigma$$

for the differential 2-form

$$\sigma = ydx \wedge dy + xdy \wedge dz - yxdx \wedge dz$$

and the 2-dim contour

$$S(s, t) = (x, y, z) = (e^t, t + s, t^2 + s^2)$$

with $t \in [-\infty, 0]$ and $s \in [0, 1]$ and the orientation $ds \wedge dt = +dsdt$.

3. **Moderate Question 10 points.**

Consider the differential forms

$$\omega_1 = \frac{zx}{\sqrt{x^2+y^2}[(1-\sqrt{x^2+y^2})^2+z^2]}dx + \frac{zy}{\sqrt{x^2+y^2}[(1-\sqrt{x^2+y^2})^2+z^2]}dy + \frac{(1-\sqrt{x^2+y^2})}{(1-\sqrt{x^2+y^2})^2+z^2}dz,$$

$$\omega_2 = \frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx.$$

(a) Change the coordinates to those on the torus

$$T = \left\{(x, y, z) \in \mathbb{R}^3 : \left(1 - \sqrt{x^2+y^2}\right)^2 + z^2 = \frac{1}{4}\right\},$$

which are

$$(x, y, z) = \left(\left(1 - \frac{1}{2}\cos(\vartheta)\right)\cos(\varphi), \left(1 - \frac{1}{2}\cos(\vartheta)\right)\sin(\varphi), \frac{1}{2}\sin(\vartheta)\right)$$

with $\vartheta, \varphi \in \mathbb{R}$, and show that the two differential forms become

$$\omega_1 = d\vartheta \qquad \text{and} \qquad \omega_2 = d\varphi.$$

Hint: this computation can be messy if you do not do it in the proper order. First, simplify all coefficient functions and then compute the differentials. Once this is done put everything together.

(b) Compute the wedge product $\sigma = \omega_1 \wedge \omega_2$ in these new coordinates.

(c) Show that ω_1, ω_2, and σ are closed on the torus T.
Hint: you can use the drastically simplified form.

(d) Explain why the integrals

$$I_1 = \frac{1}{2\pi}\oint_\gamma \omega_1, \qquad I_2 = \frac{1}{2\pi}\oint_\gamma \omega_2$$

are integers for every closed curve $\gamma : [0, 1] \to T$. Deduce from this that ω_1 and ω_2 are inexact on the torus T.

Please, turn the page for the other questions!

4. **Challenge Question 10 points.**

Let $U \subset \mathbb{R}^N$ and we choose the orientation $dx^1 \wedge \cdots \wedge dx^N = +dx^1 \cdots dx^N$. Consider real valued differential forms $\omega, \sigma \in \Omega^k(U)$ with $k \leq N$ real valued differential k-form and a general Riemannian metric following from the length element

$$ds^2 = \sum_{a,b=1}^{N} dx^a g_{ab} dx^b.$$

Then, we define

$$\langle \omega, \sigma \rangle = \int_U \omega \wedge (*\sigma).$$

(a) Explain why

$$dx^{b_{s(1)}} \wedge \ldots \wedge dx^{b_{s(k)}} = \epsilon_{s(1)\ldots s(k)} dx^{b_1} \wedge \ldots \wedge dx^{b_k},$$

for any permutation $s : \{1, \ldots, k\} \to \{1, \ldots, k\}$ (meaning $s \in \mathbb{S}_k$ the symmetric group), where $\epsilon_{s(1)\ldots s(k)}$ is the Levi-Cevita symbol. Why is then

$$dx^{b_1} \wedge \ldots \ldots \wedge dx^{b_N} = \epsilon_{b_1 \ldots b_N} dx^1 \wedge \ldots \wedge dx^N$$

true for all $b_1, \ldots, b_N \in \{1, \ldots, N\}$?

(b) Show that

$$\sum_{b_{k+1}, \ldots, b_N = 1}^{N} (\epsilon_{b_1 \ldots b_N})^2 = (N-k)!$$

for all fixed $b_1, \ldots, b_k \in \{1, \ldots, N\}$ satisfying $b_i \neq b_j$ for all $i \neq j$ for all $i, j = 1, \ldots k$.
Hint: the number of permutations in the symmetric group $\mathbb{S}_l$ is $l!$.

(c) With the help of (a) and (b), show that

$$\langle \omega, \sigma \rangle = k! \int_U \left(\sum_{a_1, \ldots, a_k, b_1, \ldots, b_k = 1}^{N} \omega_{a_1 \ldots a_k} g^{a_1 b_1} \cdots g^{a_k b_k} \sigma_{b_1 \ldots b_k} \right) \sqrt{\det(g)} dx^1 \cdots dx^N$$

for any two square integrable differential forms

$$\omega = \sum_{a_1, \ldots, a_k = 1}^{N} \omega_{a_1 \ldots a_k} dx^{a_1} \wedge \cdots \wedge dx^{a_k} \quad \text{and} \quad \sigma = \sum_{b_1, \ldots, b_k = 1}^{N} \sigma_{b_1 \ldots b_k} dx^{b_1} \wedge \cdots \wedge dx^{b_k}.$$

Recall that the coefficient functions are skew-symmetric in their indices!