Online Examination–V1–Fundamentals of Real Analysis

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MATH 2027

1. Justify every assertion with a proof or a valid quote, and show your work and rationale in all questions. I like to give partial credit, so try to make it easy for me to give it to you, by clearly explaining what you are doing. In your responses, feel free to quote any result from the book to support your claims. A valid quote used to justify a claim is as good as its proof, and will grant you full marks for that particular claim. So use facts from the book as much as you can.
2. Notation is the same as that used in the textbook.
3. This exam has 4 pages.
4. Total Marks: 100

Recall (see Definition 2.2.4) that the *ε-neighbourhood of* $a \in \mathbb{R}$ is the set $V_\varepsilon(a) = \{x \in \mathbb{R} : |x-a| < \varepsilon\}$, and that (see Definition 3.2.11) the closure of a set $A \subset \mathbb{R}$ is defined as $\overline{A} = A \cup L(A)$, where $L(A) := \{x \in \mathbb{R} \;:\; \forall\, \varepsilon > 0,\;\; A \cap V_\varepsilon(x) \text{ contains points different than } x\}$ is the set of *limit points* of A (see Definition 3.2.4). Recall (see Exercise 3.2.14) that $A^\circ := \{x \in A : \exists\, \varepsilon > 0 \text{ such that } V_\varepsilon(x) \subset A\}$ is the *interior* of A. Denote by $\operatorname{iso}(A)$ the *isolated points* of the set A, defined as those points $a \in A$ for which there exists $\varepsilon > 0$ such that $V_\varepsilon(a) \cap A = \{a\}$. Given a set $S \subset \mathbb{R}$, recall that $S^c := \{x \in \mathbb{R} : x \notin S\}$.

Question 1. (Topology of $\mathbb{R}$, open and closed sets, closure of a set)

(a) [3 POINTS] Show that $a \in \overline{A}$ if and only if for every $\varepsilon > 0$ we have $A \cap V_\varepsilon(a) \neq \emptyset$.

(b) [3 POINTS] Let I be an arbitrary nonempty set. Show that

$$\bigcup_{i\in I} \overline{A_i} \subset \overline{\bigcup_{i\in I} A_i}. \tag{1}$$

Hint: use the characterization given in part (a).

(c) [2 POINTS] Given $n \in \mathbb{N}$, define $A_n := \left(\dfrac{\sqrt{2}}{3n}, \sqrt{2}\right)$. Show that $\bigcup_{n\in\mathbb{N}} A_n = (0, \sqrt{2})$.

(d) [2 POINTS] Use the family of sets in part (c) to show that the opposite inclusion in (1) is false.

(e) [1+2=3 POINTS] Consider now a single point $x \in \mathbb{R}$. Show that a set $S_x := \{x\}$ consisting of a single point is a closed set (hint: check its complement). Use the family given by $S_n := \left\{\dfrac{\sqrt{2}}{3n}\right\}$ to contradict again the opposite inclusion to (1). In other words, show that

$$\bigcup_{n\in\mathbb{N}} \overline{S_n} \subsetneq \overline{\bigcup_{n\in\mathbb{N}} S_n}.$$

Hint: Note that $\bigcup_{n\in\mathbb{N}} S_n = \{\dfrac{\sqrt{2}}{3n} : n \in \mathbb{N}\}$ and compute the closure of this union (check Example 3.2.9(i)).

[3+3+2+2+3=13 POINTS]

Please turn over

Question 2. (Supremum and Infimum, Sequences and series in $\mathbb{R}$, Topology of $\mathbb{R}$)
Let $y_1 = 3$, and for each $n \in \mathbb{N}$ define $y_{n+1} := (2\,y_n - 3)/3$.

(a) [3 POINTS] Use induction to prove that the sequence (y_n) satisfies $y_n > -3$ for all $n \in \mathbb{N}$.

(b) [3 POINTS] Use induction to prove the sequence (y_n) is strictly decreasing.

(c) [2 POINTS] Explain why (a) and (b) imply that the sequence (y_n) converges.

(d) [2 POINTS] Compute $\lim\limits_{n\to\infty} y_n$.

(e) [5 POINTS] Let $T := \{y_n \ : \ n \in \mathbb{N}\}$. Compute $\overline{T}$, $T^\circ, L(T)$ and $\mathrm{iso}(T)$. Is the set T compact? Justify your answers.

(f) [2 POINTS] Find $\sup(T)$ and $\inf(T)$. Is $\sup(T) = \max(T)$? Is $\inf(T) = \min(T)$? Justify your answers.

(g) [3 POINTS] Show by induction that the sequence (y_n) defined above verifies that

$$y_n = -3 + \frac{2^n}{3^{n-2}}, \quad \text{for all } n \in \mathbb{N}.$$

(h) [4 POINTS] Define $a_n = \dfrac{(y_n + 3)}{9}$. Use part (g) to show that the two series given by $\sum a_n$ and $\sum(-1)^n a_n$ converge.

(i) [2 POINTS] Show that the series $\sum y_n$ diverges.

[3+3+2+2+5+2+3+4+2=26 POINTS]

Question 3. (Supremum and Infimum)
Decide if the following statements about suprema and infima are true or false. In all cases, the sets A and B are nonempty subsets of the real line $\mathbb{R}$. Give a short proof for those statements that are true. For any that are false, supply an example where the statement does not hold.

(a) [3 POINTS] If $A \subsetneq B$, and B is bounded below, then $\inf B > \inf A$.

(b) [3 POINTS] If $\sup B < \inf A$, then $A \cap B = \emptyset$.

(c) [6 POINTS] Assume that $B \subset A$ and that A is bounded above and verifying the following property:

$$\forall\, x \in A \text{ there exists } y \in B \text{ such that } y \geq x.$$

In this situation, we must have $\sup(B) = \sup(A)$.

[3+3+6=12 POINTS]

Please turn over

Question 4. (Cardinality, countable and uncountable sets)
Denote by H the set of positive numbers which are multiples of 11, namely $H := \{t \in \mathbb{N} : \exists k \in \mathbb{N} \text{ such that } t = 11k\}$. Use results from the book or a direct proof to explain why the following claims hold.

(a) [4 POINTS] There is no injective function $f : (-1, 1) \to H$.

(b) [4 POINTS] There is a bijective function $f : H \to \mathbb{Q}$.

[4+4=8 POINTS]

Question 5. (Supremum and Infimum, Topology of $\mathbb{R}$, Continuity of functions)
Give an example of the situation described or state that there is no such example. When there is no such example, provide a compelling argument for why this is the case. In all cases, the sets are subsets of the real line $\mathbb{R}$.

(a) [4 POINTS] Two nonempty sets A and B with $A \cap B = \emptyset$, $\inf A = \inf B$, $\inf A \notin A$, and $\inf B \notin B$.

(b) [4 POINTS] Two functions f and g, neither of which is continuous at $x = 2$, but $f(x)(g(x))^5$ and $f(x) + 2g(x)$ are continuous at $x = 2$.

(c) [2 POINTS] A function f discontinuous at $x = 2$, but $\sin(f(x) + \pi)$ is continuous at $x = 2$.

(d) [6 POINTS] Two nonempty sets C and D, with C bounded above, $\overline{C} = D$, and $\sup C < \sup D$.

[4+4+2+6=16 POINTS]

Question 6. (Sequences, Cauchy sequences)
Let (x_n) and (y_n) be two real sequences. In parts (a)–(c), assume that (x_n) is a Cauchy sequence and that (y_n) is bounded. Decide whether the following sequences are or not Cauchy. To justify your answers, you may quote a result from the book or provide a direct proof. Provide a counterexample if you decide the sequence is not Cauchy.

(a) [4 POINTS] The sequence (a_n) defined by $a_n := (3x_n + (-1)^n)$.

(b) [4 POINTS] The sequence (b_n) defined by $b_n := ((x_n)^2\, y_n)$.

(c) [4 POINTS] The sequence (z_n) defined as $z_n := x_n - \dfrac{(-1)^n}{2^n} y_n$.

[4+4+4=12 POINTS]

Question 7. (Series)
Give an example of each or explain why the request is impossible referencing the proper results.

(a) [3 POINTS] The series $\sum x_n$ converges but $\sum (x_n)^2$ diverges.

(b) [3 POINTS] A divergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n\,(y_n)^2$ converges.

(c) [3 POINTS] Two divergent series $\sum x_n$ and $\sum y_n$ such that $\sum (x_n - y_n)$ converge.

[3+3+3=9 POINTS]

Question 8. [4 POINTS, 0.5 PER ITEM]
Circle the correct answer. No justification is needed.

(i) An open set $C \subset \mathbb{R}$ that contains every rational number must be equal to $\mathbb{R}$.
TRUE FALSE

(ii) Every open set $B \subseteq \mathbb{R}$ contains an infinite number of irrationals.
TRUE FALSE

(iii) If $f : \mathbb{R} \to \mathbb{R}$ is continuous everywhere and $f(x) = \pi$ for every $x \in \mathbb{Q} \cap (-1, 1)$, then $f(1) = \pi$.
TRUE FALSE

(iv) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = \sin x$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \notin \mathbb{Q}$. Then f is discontinuous at $x = 0$.
TRUE FALSE

(v) If K compact and F closed, then $K \cap F^c$ compact.
TRUE FALSE

(vi) The set $T := \{x \in \mathbb{Q} \,:\, e^x \leq 1\}$ is compact.
TRUE FALSE

(vii) If K is compact and not empty, then $\sup(K) \in K$.
TRUE FALSE

(viii) A compact set K cannot have an infinite number of isolated points.
TRUE FALSE

End of Examination