Task 1. **[5+5 points]**

Your group has a special pdf π_g for generating a discrete random variable X with values in $\{1, 2, 3\}$. This $\pi_g = (\pi(1), \pi(2), \pi(3))$ with $\pi(i) = \Pr(X = i)$, $i = 1, 2, 3$, is obtained as follows: Subtract your group number g from 559 and express the result as 3-digit decimal number $a_1 \cdot 10^2 + a_2 \cdot 10 + a_3$ with $a_1, a_2, a_3 \in \{0, \ldots, 9\}$. Then,

$$\pi(i) := a_i/(a_1 + a_2 + a_3), \quad i = 1, 2, 3.$$

(a) Give a Matlab implementation of a function `GenerateRandomNumbers(n,g)` that takes a number `n` and your group number g as input and returns `n` random numbers from $\{1, 2, 3\}$ following your pdf π_g. Your function is not allowed to use any built-in number generators except for `rand`.

(b) Plot a histogram of `nTrials=10^6` random numbers generated by your function `GenerateRandomNumbers` (using your π_g). The bins should be centered at $1, 2$, and 3.

Task 2. **[5+5 points]**

Consider the following problem (a version of the so-called Monty Hall problem). A car is randomly placed behind one of the three doors #1, #2, #3 following your special pdf π_g from Task 1 (i.e., the probability that the car is placed behind door #i is $\pi(i)$, $i = 1, 2, 3$). The contestant chooses door #1. If the car is behind door #1, the host chooses door #2 with probability q and door #3 with probability $1 - q$. If the car is not behind door #1, then the host opens whichever of doors #2 or #3 that the car is NOT behind. The contestant makes now a final choice, sticking to door #1 or switching to the unopened door. The contestant wins if the car is behind the final chosen door.

Several strategies are available to the contestant. Relevant are here the following three strategies:

- Strategy 1: Stick always to the original choice, door #1.
- Strategy 2: Switch always to the unopened door.
- Strategy 3: Switch always to door #3 if door #2 is opened, otherwise stick to door #1.

(a) Implement a Matlab function that performs simulations to estimate the probability of winning for each of the three strategies.

(b) Provide a single plot that shows these estimated probabilities of winning, for each of the three strategies, as a function of q.

Task 3. **[8+8+6+8 points]**

Alice wants to solve six computational tasks ($T_1, \ldots, T_6$) on her computer, which is equipped with five Central Processing Units ($\text{CPU}_1, \ldots, \text{CPU}_5$). Each task can be split into a certain number of subtasks that can be processed in parallel (i.e., on different CPUs). However, each CPU can process only a specific number of subtasks (otherwise it would overheat).

Taking all these constraints into account, Alice came up with the computation plan shown below. It is given in form of a binary matrix A with entries a_{ij}, $i = 1, \dots, 5$, $j = 1, \dots, 6$, where $a_{ij} = 1$ if, and only if, a subtask of T_j is processed on CPU$_i$.

$$A = \begin{array}{c c} \begin{matrix} T_1 & T_2 & T_3 & T_4 & T_5 & T_6 \end{matrix} & \\ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \end{pmatrix} & \begin{matrix} \text{CPU}_1 \\ \text{CPU}_2 \\ \text{CPU}_3 \\ \text{CPU}_4 \\ \text{CPU}_5 \end{matrix} \end{array} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Alice wonders whether there are other, 'equivalent', computation plans. A computation plan, i.e., a binary matrix B, is called *equivalent* to A if in each row and each column B has the same number of 1s as A. A matrix A' obtained from A by replacing a 2×2 submatrix of the type

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

by the other is said to be obtained from A by an *interchange*. It is a classical result (and you can use it without proof) that B is equivalent to A, if, and only if, B can be obtained from A by a sequence of interchanges.

(a) Write a Matlab function `EnumerateAllSolutions(A)`, which takes Alice's A as input and returns, by checking systematically all 5×6 binary matrices, the number of different matrices that are equivalent to A. How many different matrices are equivalent to A?

(b) Write a Matlab function `RunMarkovChain(A)`, which takes Alice's A as input and generates a Markov Chain, based on interchanges, that samples the matrices equivalent to A uniformly at random. Provide a histogram that demonstrates that your `RunMarkovChain(A)` samples the matrices equivalent to A uniformly at random.

(c) Prove that your Markov Chain from (b) is aperiodic, irreducible, and has the uniform distribution as stationary distribution.

(d) Take your group number g and compute i_0, j_0 in Matlab as follows:

```
i0=ceil(g/5), j0=rem(g,5)+1;
```

Write a Matlab function `RunMarkovChainNonUniform(A)`, which takes Alice's A as input and generates a Markov Chain, based on interchanges, that randomly samples the matrices equivalent to A. The stationary distribution π of this Markov Chain should satisfy

$$\pi(A') = 2\pi(A'') \tag{1}$$

for any matrix A' with entry $a'_{i_0 j_0} = 1$ and any matrix A'' with $a''_{i_0 j_0} = 0$. Provide a histogram that demonstrates that your `RunMarkovChainNonUniform(A)` randomly samples the matrices equivalent to A satisfying (1).

Submit your answers —including all plots, commented (!) matlab code, and calculations— as one single PDF file via Canvas.