ASSIGNMENT 2

MATH3975 Financial Derivatives (Advanced)

Due by 11:59 p.m. on Sunday, 22 October 2023

1. **[10 marks] Path-dependent American claim.** Let $\mathcal{M} = (B, S)$ be the CRR model with $r = 0$ and the stock price S satisfying $S_0 = 4,\ S_1^u = 5.5,\ S_1^d = 3.5$. Consider a path-dependent American claim with maturity $T = 2$ and the reward process g defined as follows: $g_0 = 5.5,\ g_1 = 6$ and the random variable g_2 is given by

$$g_2(S_1^u, S_2^{uu}) = 8,\ g_2(S_1^u, S_2^{ud}) = 4,\ g_2(S_1^d, S_2^{du}) = 5,\ g_2(S_1^d, S_2^{dd}) = 9.$$

   (a) Let $\widetilde{\mathbb{P}}$ be the probability measure under which the process S/B is a martingale. Compute the arbitrage price process $(\pi_t(X^a),\ t = 0, 1)$ for the American claim using the recursive relationship

$$\pi_t(X^a) = \max\left\{ g_t,\ B_t\, \mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{\pi_{t+1}(X^a)}{B_{t+1}} \,\Big|\, \mathcal{F}_t \right) \right\}$$

   with the terminal condition $\pi_2(X^a) = g_2$. Find the rational exercise time τ_0^* of this claim by its holder.

   (b) Find the replicating strategy φ for the claim up to the random time τ_0^* and check that the equality $V_t(\varphi) = \pi_t(X^a)$ is valid for all $t \leq \tau_0^*$.

   (c) Determine whether the arbitrage price process $(\pi_t(X^a);\ t = 0, 1, 2)$ is either a martingale or a supermartingale under $\widetilde{\mathbb{P}}$ with respect to the filtration $\mathbb{F}$.

   (d) Find a probability measure $\mathbb{Q}$ on the space $(\Omega, \mathcal{F}_2)$ such that the arbitrage price process $(\pi_t(X^a);\ t = 0, 1, 2)$ is a martingale under $\mathbb{Q}$ with respect to the filtration $\mathbb{F}$ and compute the Radon-Nikodym density of $\mathbb{Q}$ with respect to $\widetilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}_2)$.

   (e) Let $\widehat{\mathbb{P}}$ be a probability measure under which the process B/S is a martingale. Define the process $(\widetilde{\pi}_t(X^a),\ t = 0, 1)$ through the recursive relationship

$$\widetilde{\pi}_t(X^a) = \max\left\{ g_t,\ S_t\, \mathbb{E}_{\widehat{\mathbb{P}}}\left(\frac{\widetilde{\pi}_{t+1}(X^a)}{S_{t+1}} \,\Big|\, \mathcal{F}_t \right) \right\}$$

   with $\widetilde{\pi}_2(X^a) = g_2$. Is it true that the equality $\widetilde{\pi}_t(X^a) = \pi_t(X^a)$ holds for all $t = 0, 1, 2$? Justify your answer but do not perform any computations with numbers.

2. **[10 marks] Gap option.** We place ourselves with the setup of the Black-Scholes market model $\mathcal{M} = (B, S)$ with a unique martingale measure $\widetilde{\mathbb{P}}$. Let the real numbers α and β satisfy > 0. Consider the *gap option* with the payoff at maturity date T given by the following expression

$$X = h(S_T) = (S_T - \beta)^+ \mathbb{1}_{\{S_T \geq \alpha\}}.$$

   (a) Sketch the graph of the function $g(S_T)$ and show that the inequality $\pi_t(X) < C_t(\beta)$ is valid for every $0 \leq t < T$ where $C_t(\beta)$ is the Black-Scholes price of the standard call option with strike β.

   (b) Show that the payoff of the gap option can be decomposed into the sum of the payoff $C_T(\alpha)$ of the standard call option with the strike price α and $\alpha - \beta$ units of the digital option with the payoff $D_T(\alpha) = \mathbb{1}_{\{S_T \geq \alpha\}}$.

   (c) Compute the arbitrage price $\pi_t(X)$ at time t for the gap option. Take for granted the Black-Scholes formula for the standard call option.

   (d) Assume that $S_0 \neq \alpha$. Find the limit $\lim_{T \to 0} \pi_0(X)$. Explain your result.

   (e) Find the limit $\lim_{\sigma \to \infty} \pi_t(X)$ for a fixed $0 \leq t < T$ and compare with the limits $\lim_{\sigma \to \infty} C_t(\beta)$ and $\lim_{\sigma \to \infty} C_t(\alpha)$. Explain your findings.