Math 191 Sample Midterm 2B
Spring 2023, Wilkening

Your Name: Solutions

1a. (4 points) Let $A =$ D of an arbitrary $m \times n$ real matrix A. Show that the singular values $\sigma_i = S_{ii}$ which orthogonal matrices U and V are used.

$A^T A = V S^T$

$\det(A^TA - \lambda I) = \det\left(V(S^2 - \lambda I)V^T\right)$

$= \det(V)\det(S^2 - \lambda I)\det(V^T)$

$= \det(\underbrace{VV^T}_{I})\det(S^2 - \lambda I) = \det(S^2 - \lambda I)$

the σ_i^2 are the roots of $\det(A^TA - \lambda I)$, counting multiplicity, which don't depend on U or V. (We take the positive square roots to obtain the σ_i from σ_i^2)

1b. (4 points) Suppose A is an $m \times n$ matrix with real entries. Prove that $\|A\|_2 = \sigma_1$, where σ_1 is the largest singular value of A and we recall the definition

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

for $x \neq 0$, $\quad y = V^T x \Rightarrow \|y\| = \|x\|$

$$\|Ax\| = \|\underbrace{USV^T}_{\text{full SVD}}x\| = \|SV^Tx\| = \|Sy\| = \sqrt{\sum_{j=1}^n \underbrace{\sigma_j^2 y_j^2}_{\leq \sigma_1^2 y_j^2}}$$

$$\leq \sigma_1 \sqrt{\sum_{j=1}^n y_j^2} = \sigma_1 \|y\| = \sigma_1 \|x\|$$

so $\frac{\|Ax\|}{\|x\|} \leq \sigma_1$ for all $x \neq 0$.

It's maximized with $x = v_1$ since $\|Av_1\| = \|\sigma_1 u_1\| = \sigma_1 \overbrace{\|u_1\|}^{1} = \sigma_1 \overbrace{\|v_1\|}^{1}$

2. (8 points) Compute the LDL^T factorization of $A = \begin{pmatrix} 3 & 3 & 1 \\ 3 & a & 2 \\ 1 & 2 & b \end{pmatrix}$, assuming it exists. Which values of a and b cause (i) A to be singular? (ii) A to be positive definite?

$$\begin{pmatrix} 3 & 3 & \\ 1 & a- & \\ 1/3 & 1 & \end{pmatrix}$$

$$\begin{pmatrix} 3 & 3 & 1 \\ 1 & a-3 & 1 \\ 1/3 & \frac{1}{a-3} & \underbrace{b-\frac{1}{3}-\frac{1}{a-3}}_{b-\left(\frac{1}{3}+\frac{1}{a-3}\right)} \end{pmatrix}$$

$$b-\frac{a}{3(a-3)}$$

$$L = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 1/3 & \frac{1}{a-3} & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & & \\ & a-3 & \\ & & b-\frac{a/3}{a-3} \end{pmatrix}$$

(i) $a \neq 3$, $b \neq \frac{a/3}{a-3}$

(ii) $a > 3$, $b > \frac{a/3}{a-3}$

3. A real 3×3 matrix A, not necessarily symmetric, has eigenvalues 5, 3 and 0 with corresponding eigenvectors u, v and w, all unit vectors. Suppose z is a unit vector orthogonal to u and to v.

(a) (3 points) Find any solution of $Ax = b$ if $b = 2u + 3v$.

(b) (5 points) Find the minimum norm least squares solution of $Ax = b$ if $b = 2u + 3v + 4z$. (The answer will contain ww^T somewhere in the formula. Justify your answer.)

Au ... $Aw = 0$

Av ... $z \perp \operatorname{span}(u, v)$

(a) $Ax = b = 2u + 3v$

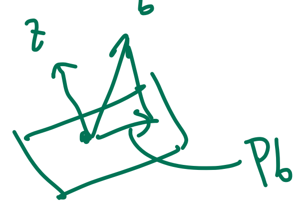
$\boxed{x = \frac{2}{5}u + v} \Rightarrow Ax = \frac{2}{5}(5u) + (3v) = 2u + 3v$ ✓

(b) $b = 2u + 3v + 4z$

$R(A)^\perp = \operatorname{span}(z)$

$Ax = Pb = 2u + 3v$

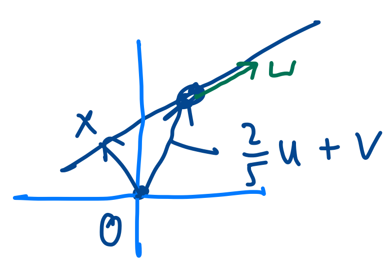
LS solution criterion



$P = I - zz^T$

all solutions: $x = \frac{2}{5}u + v - \alpha w$

best: $x = \underbrace{(I - ww^T)}_{\text{projection onto } w^\perp}\left(\frac{2}{5}u + v\right)$



$\Leftrightarrow w^T x = 0$ so $\|x + \beta w\|^2 = \|x\|^2 + \beta^2 \|w\|^2 \geq \|x\|^2$

4. (8 points) Compute the polar decompositon $A = Q|A|$ of

$$A = \begin{pmatrix} 9 & 6 & 6 \\ 6 & 0 & 12 \\ 6 & 6 & 0 \end{pmatrix} = \underset{U}{\begin{pmatrix} 2/3 & 1/3 \\ 2/3 & -2/3 \\ 1/3 & 2/3 \end{pmatrix}} \underset{S}{\begin{pmatrix} 18 & \\ & 9 \end{pmatrix}} \underset{V^T}{\begin{pmatrix} 2/3 & 1/3 & 2/3 \\ 1/3 & 2/3 & -2/3 \end{pmatrix}},$$

i.e., compute the matrix entries of the partial isometry Q with the same nullspace as A and the symmetric, positive sem ... $|A|$.

$$|A| = VS \ldots$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 2 & -2 \end{pmatrix} \underbrace{\begin{pmatrix} 1/3 & \\ & 1/3 \end{pmatrix}\begin{pmatrix} 18 & \\ & 9 \end{pmatrix}\begin{pmatrix} 1/3 & \\ & 1/3 \end{pmatrix}}_{\begin{pmatrix} 2 & \\ & 1 \end{pmatrix}} \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 & 4 \\ 1 & 2 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 6 & 6 \\ 6 & 6 & 0 \\ 6 & 0 & 12 \end{pmatrix}$$

$$Q = UV^T = \frac{1}{9} \begin{pmatrix} 2 & 1 \\ 2 & -2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 5 & 4 & 2 \\ 2 & -2 & 8 \\ 4 & 5 & -2 \end{pmatrix}$$

5. (8 points) Draw the tilted ellipse $(5/2)x^2 + 2xy + y^2 = 1$ and find the half-lengths of its major and minor axes from the eigenvalues of the corresponding matrix S for which $(5/2)x_1^2 + 2x_1x_2 + x_2^2 = x^T Sx$.

$$S = \begin{pmatrix} 5/2 & 1 \\ 1 & 1 \end{pmatrix} \qquad (x_1 \;\; x_2)\, S \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =$$

$$\det(S - \lambda I) = \left(\tfrac{5}{2} - \lambda\right)(1 - \lambda) - 1$$

$$= \lambda^2 - \tfrac{7}{2}\lambda + \tfrac{3}{2} = 0$$

$$\lambda = \frac{\frac{7}{2} \pm \sqrt{\frac{49}{4} - 6}}{2} = \frac{7}{4} \pm \frac{1}{2}\sqrt{\frac{25}{4}} = \frac{12}{4}, \frac{2}{4}$$

$$\lambda_1 = 3, \quad \lambda_2 = \tfrac{1}{2}$$

$$q_1 \in N(S - 3I) = N\begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & -2 \end{pmatrix} \qquad q_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$q_2 \in N\left(S - \tfrac{1}{2}I\right) = N\begin{pmatrix} 2 & 1 \\ 1 & \frac{1}{2} \end{pmatrix} \qquad q_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$Q = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 3 & \\ & 1/2 \end{pmatrix}, \quad S = Q \Lambda Q^T$$

$$x^T S x = (Q^T x)^T \Lambda\, Q^T x = u^T \Lambda u = 3u_1^2 + \tfrac{1}{2}u_2^2$$

$$u = Q^T x$$

$$u_1 = \frac{1}{\sqrt{5}}(2x_1 + x_2)$$

$$u_2 = \frac{1}{\sqrt{5}}(-x_1 + 2x_2)$$