Predicate Logic

Part 2

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Recall Syntax of a grammatically correct sentence (wff) in predicate logic

- **$p(t_1,\ldots, t_n)$ is a wff if p is an n-ary predicate symbol and the t_i are terms.**
- **If W, W1, and W2 are wffs then so are the following:**

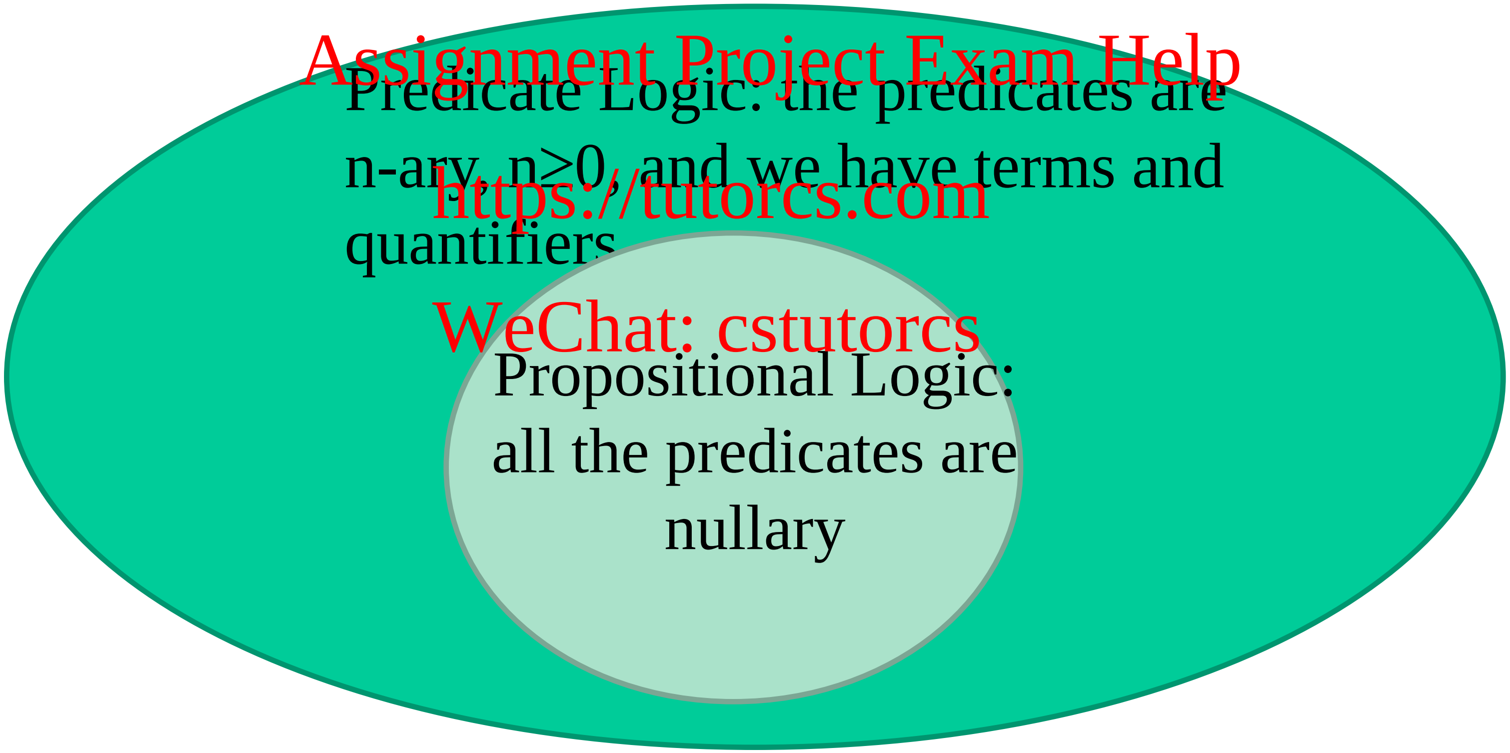
  $\neg$ W   W1 $\wedge$ W2   W1 $\vee$ W2

  W1 $\rightarrow$ W2   W1 $\leftrightarrow$ W2

  $\forall$X(W) $\exists$X(W)

  where X is a variable symbol.
- **There are no other wffs.**

From the description above you can see that propositional logic is a special case of predicate logic.



Some useful equivalences

All propositional logic equivalences hold for predicate logic wffs.

E.g. $\mathbf{A \rightarrow B \equiv \neg A \lor B}$

So

able_to_work(john) → employed(john) ≡

¬ able_to_work(john) ∨ employed(john)

∀X (able_to_work(X) → employed(X)) ≡

∀X (¬ able_to_work(X) ∨ employed(X))

Some useful equivalences cntd.

E.g. $\neg(A \wedge B) \equiv \neg A \vee \neg B$

So

$\neg$ (academic(john) $\wedge$ rich(john)) $\equiv$

$\neg$ academic(john) $\vee$ $\neg$ rich(john)

Another instance of the same equivalence:

$$\neg(A \wedge B) \equiv \neg A \vee \neg B$$

$$\neg(\overbrace{\forall X\ (\text{able\_to\_work}(X) \rightarrow \text{employed}(X))}^{A} \wedge \underbrace{\text{inflation}(\text{low})}_{B}) \equiv$$

$$\neg\ (\forall X\ (\text{able\_to\_work}(X) \rightarrow \text{employed}(X))) \vee \neg\ \text{inflation}(\text{low})$$

Some other equivalences in predicate logic

- $\forall X p(X) \equiv \neg \exists X \neg p(X)$

  all true, none false

- $\forall X \neg p(X) \equiv \neg \exists X\, p(X)$

  all false - none true

- $\exists X p(X) \equiv \neg \forall X \neg p(X)$

  at least one true - not all false

- $\exists X \neg p(X) \equiv \neg \forall X\, p(X)$

  at least one false - not all true

Equivalence exercises

$\forall X\ (\text{cautious}(X) \lor \text{normal}(X) \rightarrow$

$\exists Y\ \text{shelter}(Y,X)) \equiv$

$\neg\exists X\ ((\text{cautious}(X) \lor \text{normal}(X)) \land$

$\neg\exists Y\ \text{shelter}(Y,X))$

$\forall X\ \forall Y\ (\text{aggresive}(X) \land \text{sees}(X, Y) \rightarrow$

$\text{fights}(X, Y)) \equiv$

$\forall X\ \neg\exists Y\ (\text{aggresive}(X) \land \text{sees}\ (X, Y) \land$

$\neg\text{fights}\ (X, Y))$

Some other equivalences in predicate logic

Suppose W1, W2 are wffs.

If W1 can be transformed to W2 by a consistent renaming of variables, then W1 and W2 are equivalent.

E.g.

$\forall X\ p(X) \equiv \forall Y\ p(Y)$

$\forall X\ \exists Y\ (p(X,Y) \rightarrow q(Y, X)) \equiv$

$\forall Z\ \exists W\ (p(Z,W) \rightarrow q(W, Z))$

Some other equivalences in predicate logic

If two wffs differ only in the order of two adjacent quantifiers of the same kind, then they are equivalent. E.g.

$$\forall X\ \forall Y\ p(X,Y) \equiv \forall Y\ \forall X\ p(X,Y)$$

$$\exists X\ \exists Y\ p(X,Y) \equiv \exists Y\ \exists X\ p(X,Y)$$

But

$\forall X\ \exists Y\ p(X,Y)$ **is not equivalent to**

$\exists Y\ \forall X\ p(X,Y)$

More Equivalences

$\exists X(A \lor B) \equiv \exists XA \lor \exists XB$

E.g.

$\exists X(\text{male}(X) \lor \text{female}(X)) \equiv$

$\exists X\ \text{male}(X) \lor \exists X\ \text{female}(X)$

More Equivalences

$$\forall X\,(A \wedge B) \equiv \forall X\, A \wedge \forall X\, B$$

E.g.

$$\forall X\,(\ \text{mscDegree}(X) \rightarrow \text{duration}(X,\ 12\text{months}) \wedge$$
$$\text{phdDegree}(X) \rightarrow \text{duration}(X,\ 42\text{months})) \equiv$$

$$\forall X\,(\text{mscDegree}(X) \rightarrow \text{duration}(X,\ 12\text{months})) \wedge$$
$$\forall X\,(\text{phdDegree}(X) \rightarrow \text{duration}(X,\ 42\text{months}))$$

Some notes on quantifiers

1. Free and Bound variables:

   An occurrence of a variable in a wff is bound if it is within the scope of a quantifier in that wff. It is free if it is not within the scope of any quantifier in that wff.

Examples

$$\forall X\ (p(X) \rightarrow q(Y,X))$$

- Both occurrences of X in the above wff are bound (they are both within the scope of the $\forall$.)

- The occurrence of Y is free (it is not within the scope of any quantifier.)

$$(\forall X\ p(X)) \wedge (\exists X q(X))$$

In the wff above, both occurrences of X are bound, the first by the $\forall$, the second by the $\exists$.

$$(\forall X\ p(X)) \wedge (\exists Y q(X,Y))$$

In the wff above, the first occurrence of X is bound, the second is free. The occurrence of Y is bound.

Definition.

If a wff contains no free occurrences of variables it is said to be **closed**, otherwise it is said to be **open**.

A wff with no free occurrences of variables is also called a **sentence**.

E.g.

$\text{Bird}(X) \rightarrow \text{has\_beak}(X)$

is a wff but not a sentence.

$\forall X\ (\text{Bird}(X) \rightarrow \text{has\_beak}(X))$

is a wff and a sentence.

Back to equivalences

2. A particular occurrence of a variable is bound by the closest quantifier which can bind it.

E.g.

$$\forall X\ (p(X) \rightarrow \forall X\ q(X)) \equiv$$

$$\forall X\ (p(X) \rightarrow \forall Y\ q(Y))$$

3. Law of vacuous quantification

$\forall X\ W \equiv W$ if W (a wff) contains no free occurrences of X.

$\forall X\ (p(a) \rightarrow q(a)) \equiv p(a) \rightarrow q(a)$

$\forall X\ \exists X\ p(X) \equiv \exists X\ p(X)$

$\forall X\ \exists Y(p(X) \rightarrow \exists Y\ q(X,Y))$

which quantification can we drop?

More Equivalences

If X does not occur free in A then

$\forall X(A \rightarrow B) \equiv A \rightarrow \forall XB$, and

$\exists X(A \rightarrow B) \equiv A \rightarrow \exists XB$.

E.g.

$\forall X(\text{funny}(\text{john}) \rightarrow \text{loves}(X, \text{john})) \equiv$

$\text{funny}(\text{john}) \rightarrow \forall X\ \text{loves}(X, \text{john})$

More Equivalences

If X doesn't occur free in A, then

$\exists X(A \wedge B) \equiv A \wedge \exists X B$, and

$\forall X(A \vee B) \equiv A \vee \forall X B$.

E.g.

$\exists X(\text{station}(\text{victoria}) \wedge \text{tubeLine}(X, \text{victoria}))$

$\equiv \text{station}(\text{victoria}) \wedge \exists X \ \text{tubeLine}(X, \text{victoria})$

More Equivalences

If X does not occur free in B then

$\forall X(A \rightarrow B) \equiv \exists XA \rightarrow B$, and

$\exists X(A \rightarrow B) \equiv \forall XA \rightarrow B$.

Be careful:

The quantifier changes.

$\forall X(A \rightarrow B)$ is equivalent to $\exists X A \rightarrow B$, and

$\exists X(A \rightarrow B)$ is equivalent to $\forall X A \rightarrow B$

E.g.

$\forall X(\text{loves}(X, \text{john}) \rightarrow \text{happy}(\text{john})) \equiv$

$(\exists X\ \text{loves}(X, \text{john})) \rightarrow \text{happy}(\text{john})$

Warning: non-equivalences

The following are *NOT* logically equivalent (though always, the first $\models$ the second):

$\forall X(A \rightarrow B)$ and $\forall XA \rightarrow \forall XB$

$\exists X(A \wedge B)$ and $\exists XA \wedge \exists XB$

$\forall XA \vee \forall XB$ and $\forall X (A \vee B)$

Can you find a 'counter-example' for each one?

Counter-example for

$\forall X(p(X) \to q(X))$ and $\forall X p(X) \to \forall X q(X)$

Take

$p(a)$	$p(b)$	$\neg p(c)$
$q(a)$	$\neg q(b)$	

Then RHS is true, but LHS is not.

Examples for slide 26

$\exists X(A \wedge B)$ and $\exists XA \wedge \exists XB$

Not equivalent

$\exists X(\text{male}(X) \wedge \text{female}(X))$ and

$\exists X\ \text{male}(X) \wedge \exists X\ \text{female}(X)$

$\forall XA \vee \forall XB$ and $\forall X\ (A \vee B)$

$\forall X\ \text{msc}(X) \vee \forall X\ \text{meng}(X)$ and

$\forall X\ (\text{msc}(X) \vee \text{meng}(X))$