

# 1 Phase Estimation

$U$  is a  $2^n \times 2^n$  unitary operator,  $N = 2^n$ .

Recall that if the state of  $n$  qubits is  $|u\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle + \dots + \alpha_{N-1}|N-1\rangle$  and

$$\begin{pmatrix} \beta_0 \\ \vdots \\ \beta_{2^n-1} \end{pmatrix} = U \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{2^n-1} \end{pmatrix},$$

then

$$U|u\rangle = \beta_0|0\rangle + \beta_1|1\rangle + \dots + \beta_{N-1}|N-1\rangle.$$

Let  $|u\rangle$  be an eigenvector of  $U$ , i.e.,

$$U|u\rangle = e^{2\pi i\psi}|u\rangle, \psi \in [0, 1).$$

We want to find  $\psi$ . We will use the following circuit.

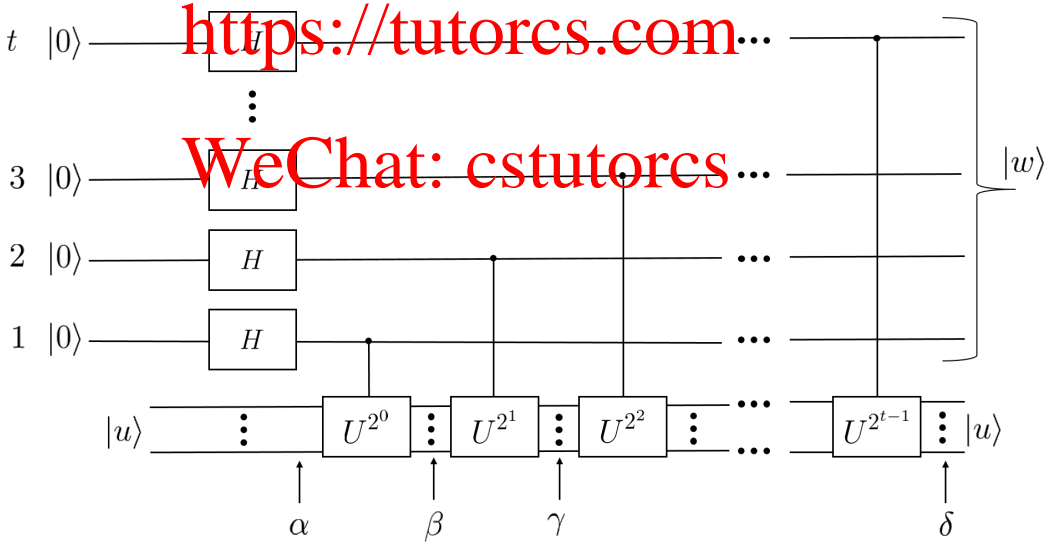


Figure 1: The left part of the quantum circuit for phase estimation

Recall that in a quantum circuit the mostly top qubit is considered to be the first one. So, in the circuit shown in Fig. 2 the joint state of the two input qubits is

$$(\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle).$$

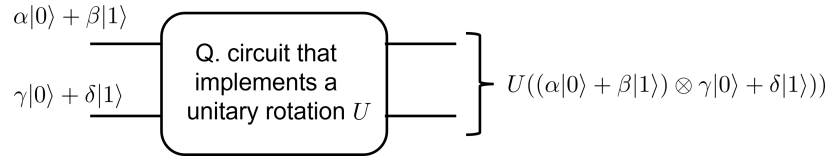


Figure 2: A quantum circuit with two input qubits

However, for the phase estimation circuit shown in Fig. 1 it will be convenient for us to say that the mostly top qubit has "name"  $t$ , the next qubit has "name"  $t - 1$  and so on.

In what follows we will use the following notation for the tensor power

$$a^{\otimes N} = \underbrace{a \otimes a \otimes \dots \otimes a}_{N \text{ times}}.$$

Recall, that it is customary to omit the tensor product notation  $\otimes$ . For instance, instead of  $|0\rangle \otimes |1\rangle$  we often write  $|0\rangle|1\rangle$  or  $|01\rangle$ , or instead of  $|v\rangle \otimes |u\rangle$  we often write  $|v\rangle|u\rangle$ . In what follows we will sometimes drop  $\otimes$  and sometimes use it. I hope the meaning of the expressions will be clear.

Let us consider the evolution of the joint state of the first  $t$  qubits and the last  $n$  qubits in the state  $|u\rangle$ . After the moment  $\alpha$  the qubit 1 interacts with the  $n$  qubits in the state  $|u\rangle$ , so it is convenient to write the joint state of all the  $t + n$  qubits so that to see the state of qubit 1 explicitly:

$$\begin{aligned} \alpha : & \underbrace{\left( \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \right)^{\otimes(t-1)}}_{\text{qubits } t, \dots, 2} \otimes \underbrace{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|u\rangle}_{\text{qubit 1}} \\ &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)^{\otimes(t-1)} \otimes \frac{1}{\sqrt{2}}(|0\rangle|u\rangle + |1\rangle|u\rangle) \end{aligned}$$

Next we have

$$\begin{aligned}
\beta &: \left( \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \right)^{\otimes(t-1)} \otimes \frac{1}{\sqrt{2}}(|0\rangle|u\rangle + |1\rangle U|u\rangle) \\
&= \left( \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \right)^{\otimes(t-1)} \otimes \frac{1}{\sqrt{2}}(|0\rangle|u\rangle + |1\rangle e^{2\pi i\psi}|u\rangle) \\
&= \left( \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \right)^{\otimes(t-1)} \otimes \underbrace{\frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i\psi}|1\rangle)}_{|v_1\rangle} |u\rangle \\
&= \underbrace{\left( \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \right)^{\otimes(t-2)}}_{\text{qubits } t, \dots, 3} \otimes \underbrace{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}_{\text{qubit}} |v_1\rangle |u\rangle
\end{aligned}$$

Using similar arguments, we get

$$\begin{aligned}
\gamma &: \left( \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \right)^{\otimes(t-2)} \otimes \frac{1}{\sqrt{2}}(|0\rangle|v_1\rangle|u\rangle + |1\rangle|v_1\rangle U|u\rangle) \\
&= \left( \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \right)^{\otimes(t-2)} \otimes \frac{1}{\sqrt{2}}(|0\rangle|v_1\rangle|u\rangle + |1\rangle|v_1\rangle e^{2\pi i2\psi}|u\rangle) \\
&= \left( \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \right)^{\otimes(t-2)} \otimes \frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i2\psi}|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i\psi}|1\rangle)|u\rangle
\end{aligned}$$

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Applying similar arguments to all  $t$  qubits, we obtain:

$$\begin{aligned}
\delta &: \left( \frac{1}{\sqrt{2}} \right)^t (|0\rangle + e^{2\pi i2^{t-1}\psi}|1\rangle) \otimes (|0\rangle + e^{2\pi i2^{t-2}\psi}|1\rangle) \otimes \dots \\
&\quad \otimes (|0\rangle + e^{2\pi i\psi}|1\rangle) \otimes |u\rangle
\end{aligned} \tag{1}$$

Let us assume that  $t$  bits is enough to represent  $\psi$  in its binary expansion:

$$\psi = \psi_1 \cdot \frac{1}{2} + \psi_2 \cdot \frac{1}{2^2} + \dots + \psi_t \cdot \frac{1}{2^t}, \quad \psi_j = 0, 1.$$

For example, if  $\psi = \frac{5}{8}$  then it is enough to take  $t = 3$  since

$$\frac{5}{8} = 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2^2} + 1 \cdot \frac{1}{2^3}, \quad \text{so here } \psi_1 = 1, \psi_2 = 0, \psi_3 = 1.$$

Then, taking into account that  $e^{2\pi i \cdot m} = 1$  for any integer  $m$ , we get

$$e^{2\pi i2^{t-1}\psi} = e^{2\pi i2^{t-1}\psi_1 \cdot \frac{1}{2}} \cdot e^{2\pi i2^{t-1}\psi_2 \cdot \frac{1}{2^2}} \cdot \dots \cdot e^{2\pi i2^{t-1}\psi_t \cdot \frac{1}{2^t}} = e^{2\pi i\psi_t/2}.$$

Simplifying other factors of (1) in the same way, we obtain

$$\delta : \frac{1}{\sqrt{2^t}}(|0\rangle + e^{2\pi i \psi_t/2}|1\rangle) \otimes (|0\rangle + e^{2\pi i/\psi_{t-1}/2 + \psi_t/4}|1\rangle) \otimes \dots \\ \otimes (|0\rangle + e^{2\pi i(\psi_1/2 + \dots + \psi_t/2^t)}|1\rangle) \otimes |u\rangle$$

and therefore

$$|w\rangle = \frac{1}{\sqrt{2^t}}(|0\rangle + e^{2\pi i \psi_t/2}|1\rangle) \otimes (|0\rangle + e^{2\pi i/\psi_{t-1}/2 + \psi_t/4}|1\rangle) \otimes \dots \\ \otimes (|0\rangle + e^{2\pi i(\psi_1/2 + \dots + \psi_t/2^t)}|1\rangle)$$

This is the same state as we obtain at the output of Quantum DFT (equation (6) of the lecture notes on Quantum DFT).

**Remark 1.** Note that if we have a quantum circuit that implements a unitary transformation  $U$  (i.e. the circuit does not contain measurement blocks), then it is not difficult to construct an inverse circuit for implementing  $U^\dagger$ . For obtaining the inverse circuit we replace all gates with their Hermitian conjugates and reverse the gates order.

An example is shown in Fig. 3. The quantum circuit consists of Control  $S$  gate ( $U_{CS}$ ), Control NOT gate ( $U_{CNOT}$ ),  $H$  gate applied to the first qubit, and  $T$  gate applied to the second qubit ( $S$  and  $T$  gates are defined in the Lecture Notes on QFT). Thus this circuit implements

$$U = (H \otimes T) \cdot U_{CNOT} \cdot U_{CS}. \quad (2)$$

(Note that the matrices  $(H \otimes T)$ ,  $U_{CNOT}$ , and  $U_{CS}$  appear in (2) in the reverse order compared with the circuit, since in the circuit the input state  $|v\rangle$  is first transformed by the mostly left gates, and in the mathematical expression  $|v\rangle$  is first transformed by the most right matrix in the product.)

The inverse quantum circuit implements Hermitian conjugated rotations in the opposite order (note that  $H^\dagger = H$  and  $U_{CNOT}^\dagger = U_{CNOT}$ , so they do not change). Thus the inverse circuit implements

$$U_{inv} = U_{CS}^\dagger \cdot U_{CNOT} \cdot (H \otimes T^\dagger).$$

It is easy to check that  $UU_{inv} = I_4$ , or equivalently  $U_{inv} = U^\dagger$ .

Taking into account Remark 1, we conclude that using  $|w\rangle$  at the input for the inverse QDFT, we obtain the state  $|\psi_t\rangle \dots |\psi_2\rangle |\psi_1\rangle$  and hence with probability 1 we obtain classical outputs  $\psi_t, \dots, \psi_1$ . This is shown on Fig. 4.

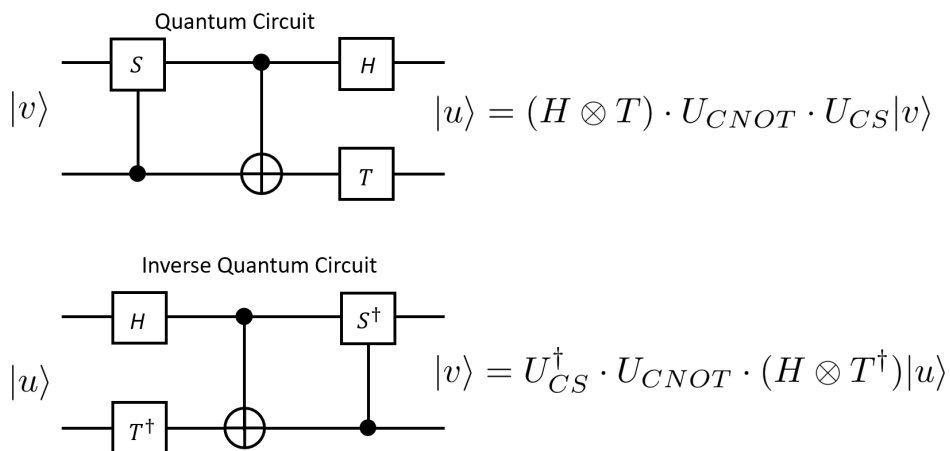


Figure 3: For obtaining a reverse quantum gate, we replace each gate with its Hermitian conjugate and reverse the order of the gates

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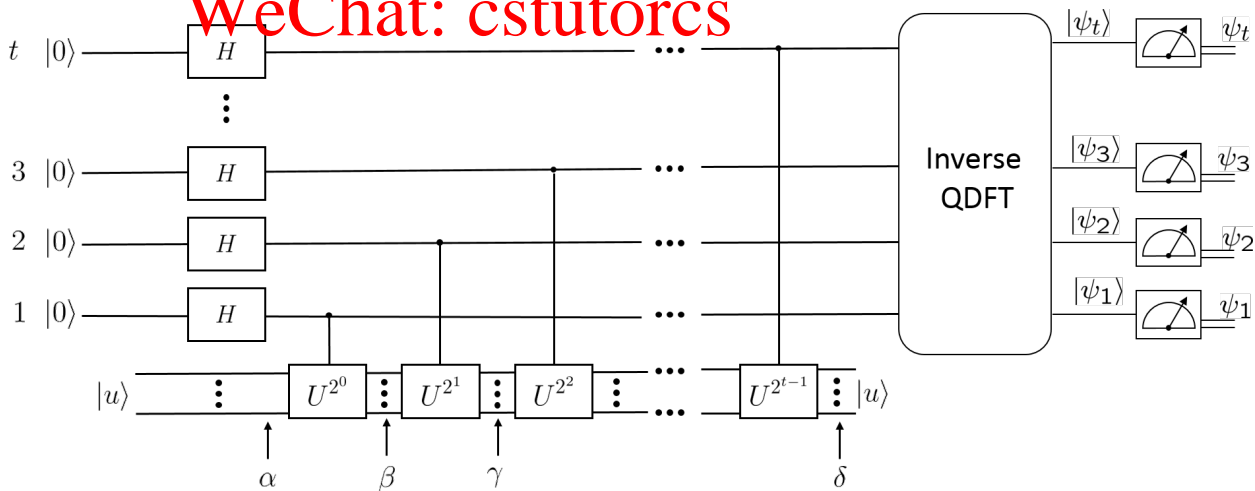


Figure 4: Quantum circuit for phase estimation