Lecture 35: Spread of disease

Learning objectives

✓ Understand the mechanisms for modelling epidemics

Scientific examples

✓ Disease models

✓ Rubella

Maths skills

✓ Interpret the meaning of terms in a system of DE's

12.4 Epidemics and SIR models

- In this section, we will use systems of DEs to model the large-scale spread of communicable disease through a population over time.

Epidemic

Informal

A large-scale occurrence of disease in a human population is called an **epidemic** if new cases of the disease arise at a rate that "substantially exceeds what is expected" in a given time period. Localised occurrences are called **outbreaks**, and global occurrences are often called **pandemics**.

- Modelling diseases is important to understanding how they spread, and how their impact may be mitigated through approaches such as quarantine, vaccination and public health campaigns.

Photo 12.8: Images commemorating the bubonic plague in Eyam, the "Plague Village", UK. Left: stained glass 'Plague Window'. Centre: first page of the list of names of villagers who died from plague in 1665–6. Right: tombstone. (Source: PA.)

30%

- Modelling disease spread often begins with estimates of the number of secondary infections that typically arise from an individual with the disease, and the rate at which individuals recover from the disease.

Basic reproduction number and infectious period

The **infectious period** of a disease is the average length of time during which an infective individual can infect a susceptible individual. Many diseases are infectious before symptoms become apparent.

The **basic reproduction number** R_0 of a disease is the average number of secondary infections caused by a single infective individual in a completely susceptible population, in the absence of any preventive interventions.

constant for a given disease

The value of R_0 is determined by factors including how infectious the disease is, how it is spread and the duration of the infectious period.

- Table 12.3 gives information for some well-known communicable diseases.

> 1

Disease	Transmission method	R_0	Infectious period
Rubella	Airborne droplet	≈ 5	2 weeks
Measles	Airborne droplet	$12 - 18$	10 days
Whooping cough	Airborne droplet	$12 - 17$	3 weeks
Mumps	Airborne droplet	$4 - 7$	14 days
Swine flu	Airborne droplet	$1.4 - 1.6$	6 days
Seasonal influenza	Airborne droplet	$2 - 3$	6 days
COVID-19	Airborne droplet	$1 - 3$	7 – 14 days
Polio	Faecal/oral	$5 - 7$	6 – 20 days
HIV/AIDS	Sexual contact	$2 - 5$	unlimited
Syphilis	Sexual contact	≈ 1.5	up to 2 years
Human papillomavirus	Sexual contact	$1 - 3$	very variable (up to decades)
Pneumonic plague	Airborne droplet	≈ 1.3	2 days (100% death rate)

alpha ?

Table 12.3: Transmission methods, infectious periods and values of R_0 for some communicable diseases.

UQ vaccine

Infection rate and recovery rate

The **infection rate** a is the rate at which secondary infections arise from a single infective individual, and is defined to equal the basic reproduction number divided by the infectious period (IP). Thus, $\boxed{a = \frac{R_0}{IP}}$. infected over this time period

The **recovery rate** b is the rate at which an infective individual recovers, and is defined to equal 1 divided by the infectious period. Thus, $\boxed{b = \frac{1}{IP}}$.

Question 12.4.1

Calculate the infection rate and the recovery rate for rubella.

From the table $R_0 = 5$, $IP = 2$ weeks

Infection rate $a = \frac{5}{2} = 2.5$ week^{-1}

Recovery rate $b = \frac{1}{2} = 0.5$ week^{-1}

- The $\boxed{\text{SIR}}$ (**S**usceptible, **I**nfective, **R**ecovered) epidemic model is used to model many infectious diseases including rubella, measles, and Covid-19.

SIR model of epidemics

The *SIR* epidemic model classifies a population into three distinct **compartments** or groups, and uses a system of DEs to predict the changes in the number of people in each group. At any time t:

(1) The *susceptible* compartment $S(t)$ is the group of people who are susceptible to the disease.

(2) The *infective* compartment $I(t)$ is the group of people who have the disease and can infect susceptible people.

(3) The *recovered* compartment $R(t)$ is the group of people who had the disease but are no longer infectious. We assume that recovered individuals are not able to be re-infected.

SIR model of epidemics *(continued)*

The only possible transitions in the simple SIR model are that: a susceptible person can become infective; and an infective person can become recovered.

$S \to I \to R$

The model assumes that there are no births or deaths from any causes, and that the disease is spread by contact between susceptible and infective individuals in a sufficiently large population that mixes homogeneously.

R_0 – basic reproduction number (constant)

Effective reproduction number and infection rate

Often, not everyone in a population is susceptible to a disease. The **effective reproduction number**, R_e, is an estimate of the *average* number of secondary infections arising from an infective individual. If a population of size N contains S susceptible individuals then

constant

$$R_e(t) = R_0 \times \frac{S(t)}{N}.$$

← total number

Similarly, if a is the infection rate in a fully susceptible population, then the **effective infection rate** in a population that is not fully susceptible is

$$\boxed{a_e(t) = a \times \frac{S(t)}{N}.}$$

The equations for the SIR model

If a population of N people at time t is divided into three compartments, susceptible $S(t)$, infective $I(t)$ and recovered $R(t)$, then the SIR model is:

$$\begin{aligned} S' &= -a \times \frac{S}{N} \times I \\ I' &= a \times \frac{S}{N} \times I - bI \\ R' &= b\, I \end{aligned}$$

where a is the infection rate and b is the recovery rate.

constant constant

Question 12.4.2

Explain the meaning of each of the terms in the SIR model.

$a\frac{S}{N}I$ — interaction S meets I leading to S become infected per unit time

bI = number that recover per unit time

Question 12.4.3

Write equations for the total population size, $N(t)$, and the rate of change in population, $N'(t)$. Comment on whether these population dynamics are appropriate for a model of disease spread.

Total population $N = S + I + R$

Rate of change

$$N' = S' + I' + R'$$

$$= \underbrace{-a\frac{S}{\cancel{N}}\cancel{I}}_{S'} + \underbrace{a\frac{S}{\cancel{N}}\cancel{I} - b\cancel{I}}_{I'} + \underbrace{b\cancel{I}}_{R'}$$

$$= 0$$

Total population is constant throughout

Case Study 31: **Rubella**

- **Rubella** (or **German measles**) was (and in some countries, still is) a common disease, particularly in childhood.
- In most cases, symptoms are very mild, and may even pass unnoticed. However, if a woman is infected during the first 20 weeks of pregnancy then spontaneous abortion can occur (in about 20% of cases), or the child may be born with congenital rubella syndrome (CRS), which is a range of incurable conditions including deafness, blindness and intellectual impairment.
- The risk of developing CRS in an unborn child is as high as 90% if the mother is infected during the first 10 weeks of pregnancy.

- There was a rubella epidemic in the USA between 1962 and 1965. Data from [40] show that during 1964–65 there were:

  - 12.5 million rubella cases
  - 11000 abortions (spontaneous and surgical)

  - 20000 infants born with CRS (12000 deaf, 3580 blind, 1800 with intellectual impairment)
- During that epidemic, 1% of all children born in New York were affected.
- A vaccine was introduced in 1969 and is routinely administered in many countries. In Queensland, the Department of Health recommends all children have combined MMR (measles, mumps and rubella) vaccines at the ages of 12 months and 4 years.
- Vaccination campaigns have greatly reduced the incidence of rubella and the frequency of outbreaks. The Centers for Disease Control and Prevention announced that rubella was eliminated from the USA in 2004.
- In January 2008, at least four babies in Sydney became infected with rubella. All were less than 12 months old, so were under the age for vaccination with the MMR vaccine.

Example 12.4.4

Assume that a population of 10000 people contains $\boxed{10}$ people infective with rubella, and that everyone else is susceptible. Using the values of a and b from above, the SIR equations for rubella are:

$a = 2.5$ /week

$b = 0.5$ /week

$$
\begin{aligned}
S' &= -2.5 \times \frac{S}{10000} \times I \\
I' &= 2.5 \times \frac{S}{10000} \times I - 0.5I \\
R' &= 0.5I
\end{aligned}
$$

where $I(0) = 10$, $S(0) = 9990$ and $R(0) = 0$. ← Initial conditions

Question 12.4.5

Use Euler's method and a step size of one week to estimate the number of people in each category after one week.

$S(0) = 9{,}990$ $I(0) = 10$, $R(0) = 0$

we have $h = 1$ week

$$
\begin{aligned}
S(1) &= S(0) + h\,S'(0) \\
&= S(0) + h\left[-2.5 \times \frac{S(0)}{10{,}000} \times I(0)\right] \\
&= 9990 + h\left[-2.5 \times \frac{9990}{10000} \times 10\right] \\
&= 9965
\end{aligned}
$$

$$
\begin{aligned}
I(1) &= I(0) + h\,I'(0) \\
&= 10 + 1\left[2.5 \times \frac{9990}{10000} \times 10 - 0.5(10)\right] \\
&= 30
\end{aligned}
$$

$$
\begin{aligned}
R(1) &= R(0) + h\,R'(0) \\
&= 0 + 1\left[0.5 \times 10\right] \\
&= 5
\end{aligned}
$$

Now we can develop a computer program to model a rubella epidemic.

Program 12.3: SIR model of rubella.

```
# This program uses Euler's method and the SIR equations to
# model the spread of rubella in a succeptible population
from pylab import *

# Initialise variables for rubella; values of a and b are per week.
N = 10000
a = 2.5
b = 0.5
```

total no. of people

rates (per week)

```
# Initialise variables for Euler's method. The stepsize is 0.1 week.
weeks = 12
stepsize = 0.1
time = arange(0, weeks + stepsize, stepsize)
steps = size(time)
SA = zeros(int(steps))
IA = zeros(int(steps))
RA = zeros(int(steps))
```

maximum time

step size

arrays


```
# Set the initial number of people in each category.
IA[0] = 10
SA[0] = N - IA[0]
RA[0] = 0
```

initial


```
# Step through Euler's method.
i = 0
while i < steps-1:
    dS = -a * SA[i] * IA[i]/N
    dI = a * SA[i] * IA[i]/N - b * IA[i]
    dR = b * IA[i]
    SA[i+1] = SA[i] + stepsize * dS
    IA[i+1] = IA[i] + stepsize * dI
    RA[i+1] = RA[i] + stepsize * dR
    i = i+1
```

DEs

Euler

```
# Output
plot(time, SA, "b--", linewidth=3,label="Susceptible")
plot(time, IA, "r-", linewidth=3,label="Infective",)
plot(time, RA, "k-.", linewidth=3,label="Recovered")
xlabel("Time (weeks)")
ylabel("Number of people")
xlim(0,12)
ylim(0,10000)
legend(loc="center right")
grid(True)
show()
```

plot

Example 12.4.6

Figure 12.7 shows the program output for a period of 12 weeks.

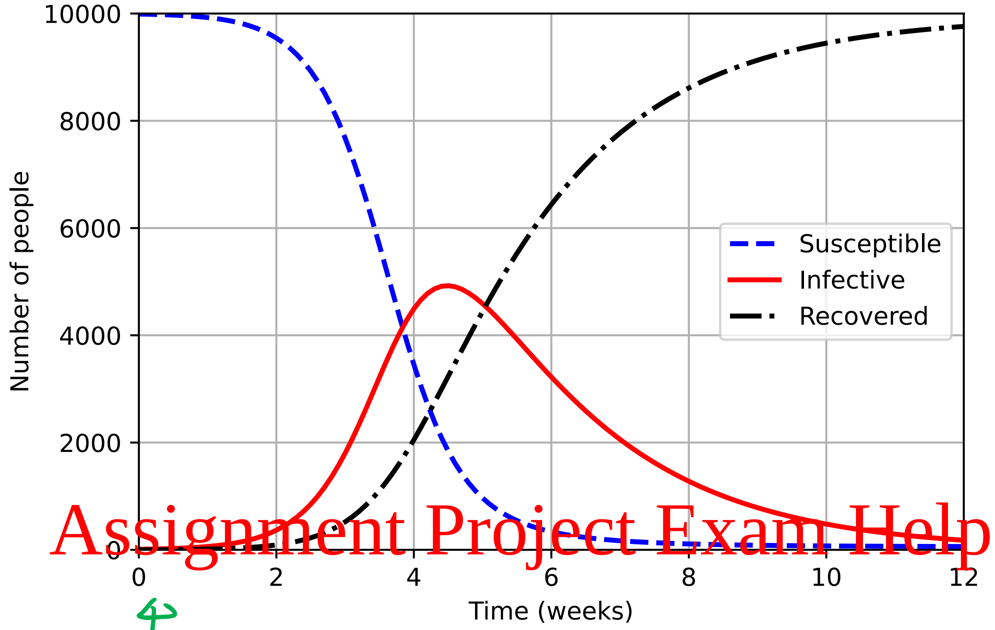

Figure 12.7: A rubella epidemic modelled using Euler's method, showing the numbers of people who are susceptible, infective and recovered.

Figure 12.7 shows that:

- An epidemic occurs. This is expected, because the population mostly comprises susceptible individuals.
- The epidemic lasts for approximately 12 weeks.
- The peak number of infectives is 4925, which occurs approximately 4.4 weeks after infectives first entered the population.

End of Case Study 31: Rubella.