Chapter 8: **Combining functions**

Lecture 21: **Merging models**

Learning objectives

✓ Analyse more complex models of real-world phenomenon

✓ Compare models and critically evaluate predictions made by extrapolating

✓ Understand interactions between different factors in models

Scientific examples

✓ Atmospheric carbon dioxide

✓ Wind chill and apparent temperature

Maths skills

✓ Understand when and how to combine functions

✓ Interpret functions with multiple variables

Image 8.1: Right panel of *The Garden of Earthly Delights* (1503 – 1504), Hieronymus Bosch (c. 1450 – 1516), Museo del Prado, Madrid. (Source: en.wikipedia.org)

In this chapter we will explore some more complex functions, such as those with several variables and those which are combinations of functions we have previously seen. We will model the Keeling curve again, using combinations of the functions we saw in previous sections and we will model apparent temperature a function of several variables.

In this chapter, we introduce a type of function (which is the product of two functions we have already studied) called a **surge function**. These functions are most often used in pharmacokinetics, the study of what happens to a drug inside the body.

8.1 Keeling revisited

- Now we develop a more accurate mathematical model of the Keeling curve. Recall that the Keeling curve graphs the concentration of atmospheric carbon dioxide (in parts per million by volume) over time, since 1958.
- When we discussed power functions, we saw that the following function modelled the general trend of the Keeling curve quite well (increasing and bending upwards, that is increasing at an increasing rate)

$$\boxed{y(t) = \frac{1}{3}t^{1.37} + 315.}$$

- What we were missing from our model was a reflection of the cyclic variation that occurs in the Keeling curve data each year. We can now combine our function with a sine function in order to model the cyclic variation.

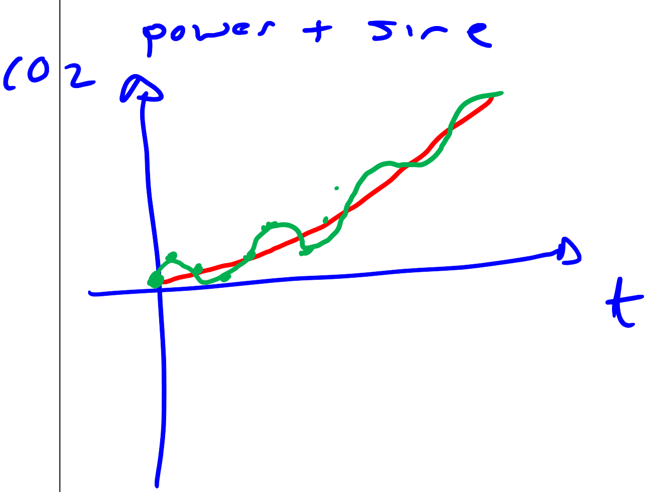
***Question** 8.1.1*

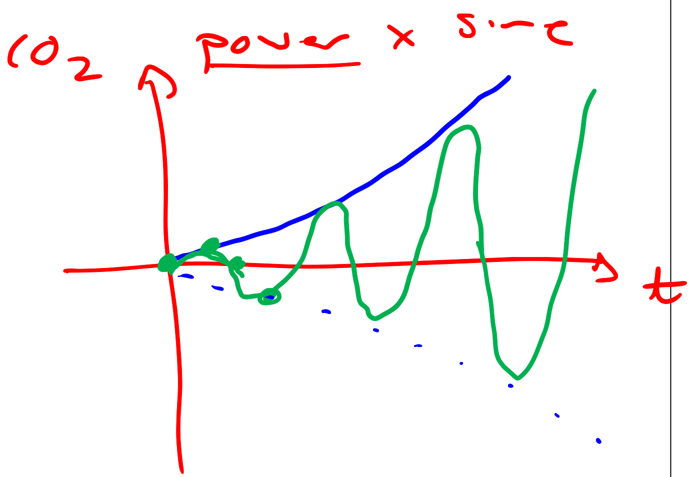
Give a rough sketch of the shape of the graph in each case:

(a) a power function added to a sine function;

(b) a power function multiplied by a sine function.

Which type of model is more appropriate to model the Keeling curve?





SomsE

Question 8.1.2

Keeling Model 4: Figures 8.1, 8.2 and 8.3 plot the Keeling curve and the following function $y(t)$ over three different time periods.

$$y(t) = \frac{1}{3}t^{1.37} + 315 + 3.5\sin\left(\frac{2\pi}{1}(t-0.15)\right).$$

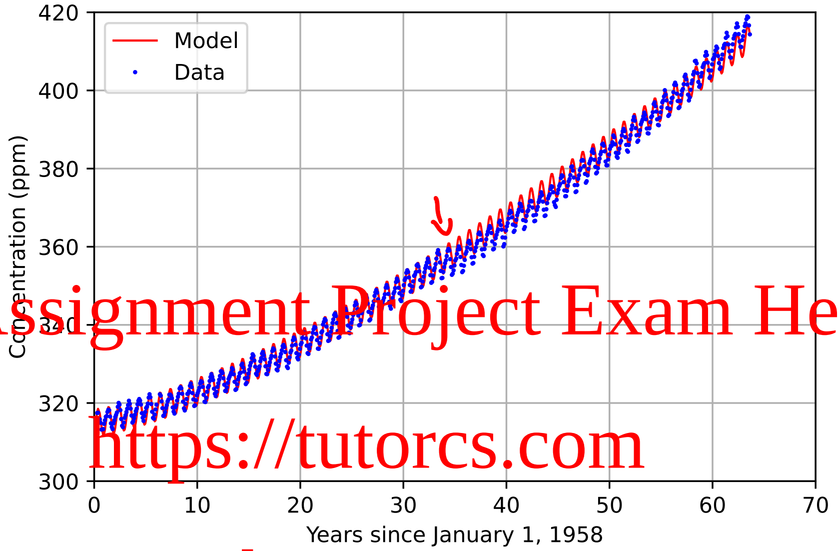


Figure 8.1: The Keeling curve and a model using sin and power functions.

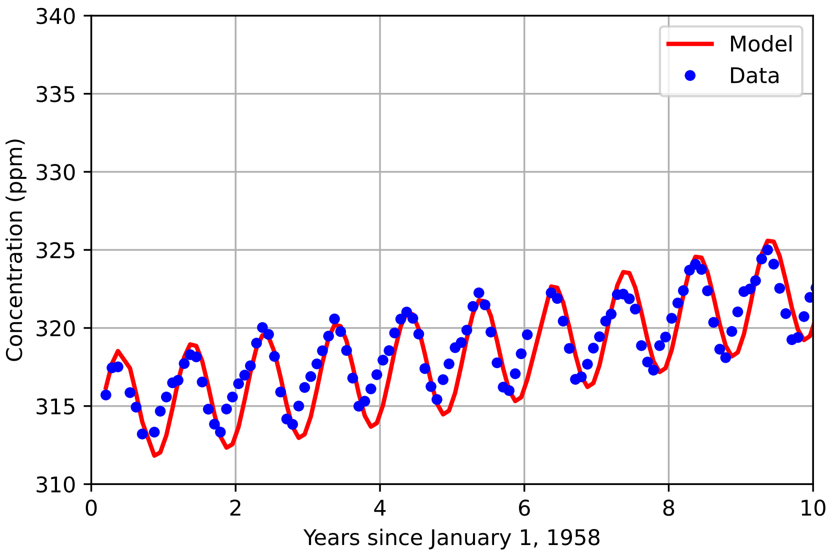


Figure 8.2: The Keeling curve and a model using sin and power functions (early years).

Question 8.1.2 (continued)

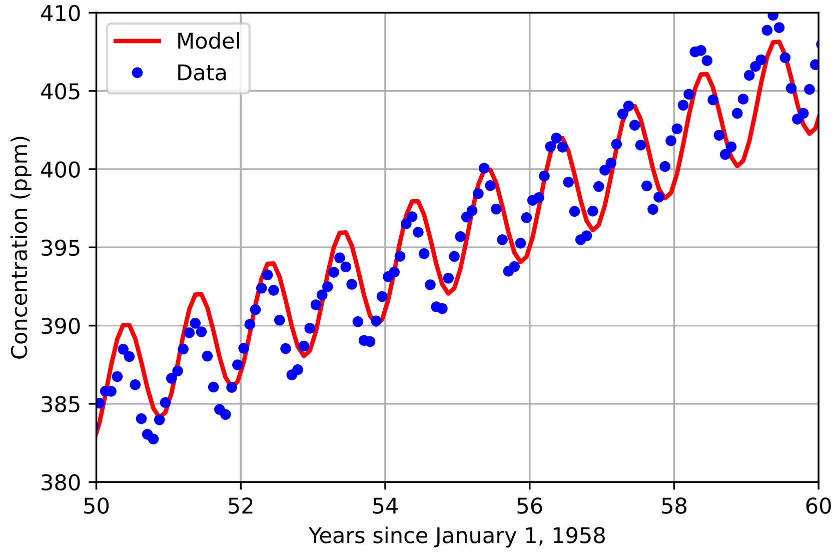


Figure 8.3: The Keeling curve and a model using sin and power functions (recent years).

(a) Explain how each term in $y(t)$ impacts on its graph. Recall that

p>1 - increasing

$$y(t) = \underbrace{\frac{1}{3}t^{1.37}}_{\text{vertical scale}} + \underbrace{315}_{\text{vertical shift}} + \underbrace{3.5\sin\left(\frac{2\pi}{1}(t-0.15)\right)}_{\text{oscillation}}.$$

(b) Given a physical justification for the term $(t - 0.15)$.

0.15 year horizontal shift
accounts for seasonal variation - maximum [CO_2] start of spring
equilibrium + increasing ~ Feb

(c) How effectively does $y(t)$ model the Keeling curve?

Very good!
(to available data)

Question 8.1.3

Consider the following three models of the Keeling curve.

- Model Q+S: $y(t) = 0.014t^2 + 0.7t + 315 + 3.5 \sin(2\pi(t - 0.15))$. quadratic
- Model P+S: $y(t) = 1/3t^{1.37} + 315 + 3.5 \sin(2\pi(t - 0.15))$. power
- Model E+S: $y(t) = 280 + 35e^{0.022t} + 3.5 \sin(2\pi(t - 0.15))$. exponential

Figure 8.4 plots graphs of the Keeling curve and all three models.

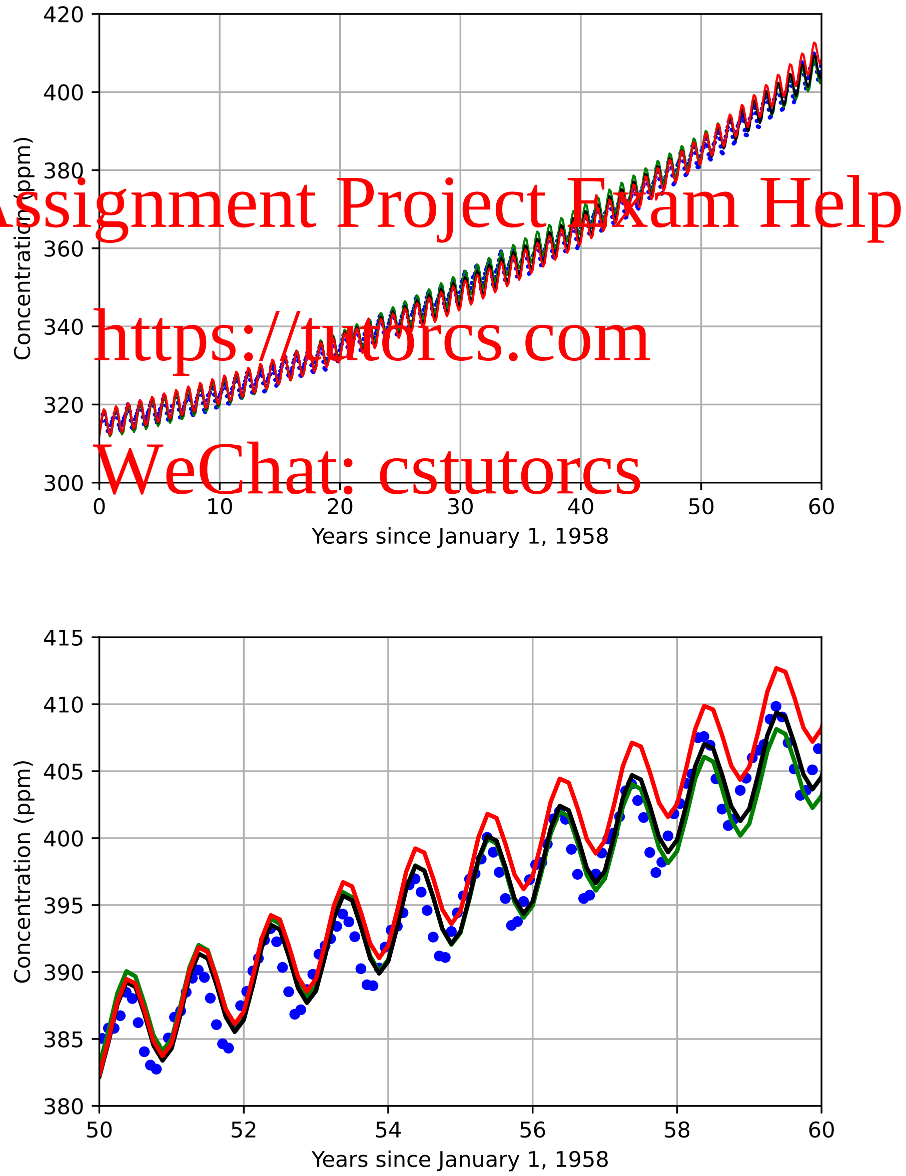


Figure 8.4: The Keeling curve and the three models for all years (top) and recent years (bottom).

***Question** 8.1.3 (continued)*

(a) Which of the three models of the Keeling curve is correct? Why?

"All models are false"
None are correct, but all potentially useful.

(b) Figure 8.5 extrapolates the models to the year 2058 (100 years after the Keeling study commenced). Which curve corresponds to each model?

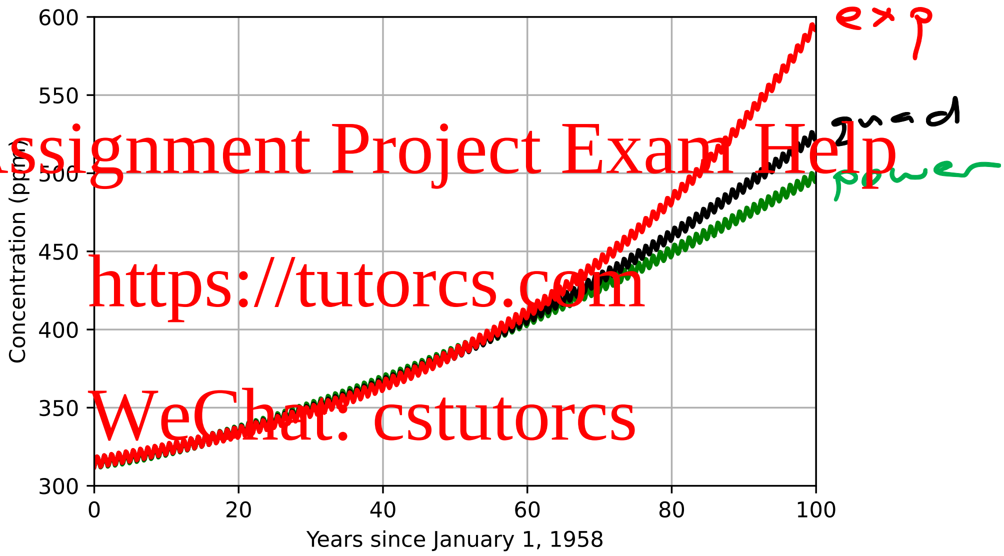


Figure 8.5: The three models of the Keeling curve, extrapolated to the year 2058.

(c) Briefly comment on the ramifications of the different predictions.

All serious
Exponential - severe

8.2 Apparent temperature

- Apparent temperature relates to how a person "perceives" the actual temperature.
- This feeling is impacted by the rate at which energy is lost from the body to the surrounds.

Case Study 14: **Apparent temperature for Aussies**

- Most weather apps now include a feature that explains what the "apparent temperature" is, or what the temperature "feels like".

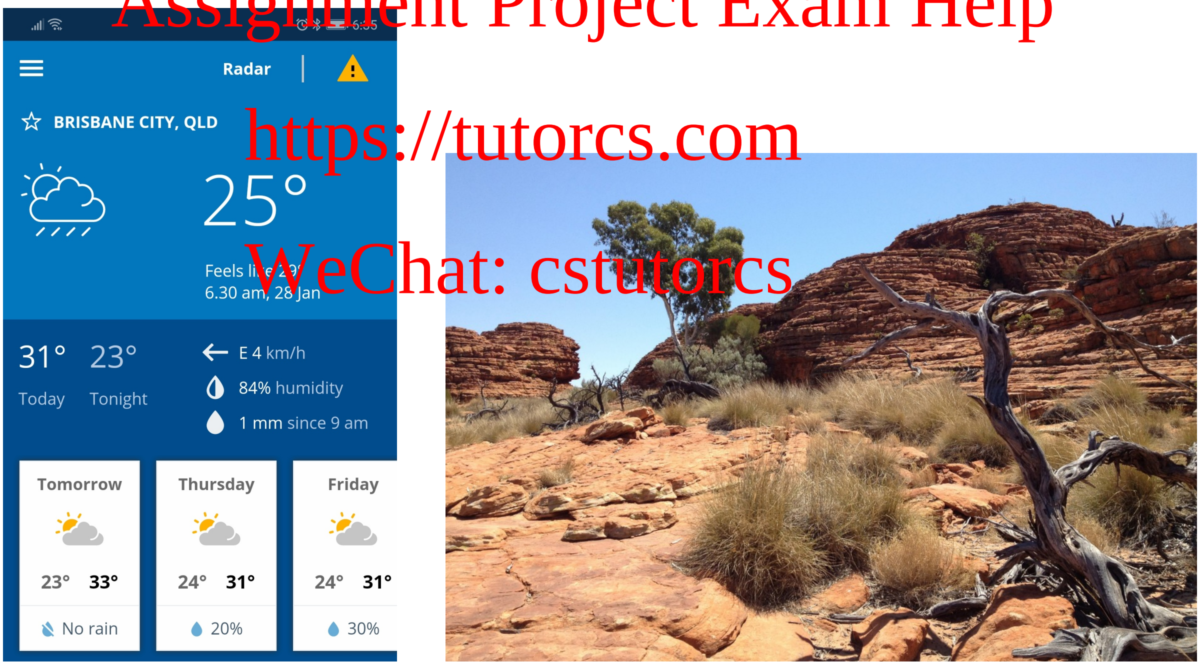


Photo 8.1: Left: Snapshot of BOM mobile weather app (source: SH.). Right: Outback landscape (source: https://pxhere.com/en/photo/949423, CCO)

***Question** 8.2.1*

Derive a plausible equation that models apparent temperature. (Hint: start by deciding which factors are important, whether they increase or decrease the apparent temperature.)

Apparent temperature AT

depends on:

Wind speed $W \uparrow$ $AT \downarrow$

Humidity $H \uparrow$ $AT \uparrow$

Temperature $T \uparrow$ $AT \uparrow$

[shade, clouds, ...]

Plausible model:

$$AT = T + H - W$$

or

$$AT = \frac{TH}{W}$$

$\vdots$

***Example** 8.2.2*

The model which is used by the Australian Bureau of Meteorology is based on the following function developed in [52]. Let T be the ambient air temperature in °C, H denote the relative humidity (%), and v be the wind speed in m/s. The perceived apparent temperature AT in °C can be modelled by:

interaction

$$\mathrm{AT} = T + 2.015\left(\frac{H}{100}\right)\exp\left(\frac{17.27T}{237.7+T}\right) - 0.7v - 4.00,$$

where we note that $\exp(x)$ is another way to write e^x.

***Question** 8.2.3*

A simplified version of the model for apparent temperature in Australia is given by

$$\mathrm{AT} = T + 2\left(\frac{\mathrm{H}}{100}\right)e^{0.06T} - 0.7v - 4.$$

(a) For each of the three factors affecting apparent temperature, describe how you would expect an increase in that factor to affect the apparent temperature, giving a physical reason. Is this consistent with the model?

$T \uparrow \quad AT \uparrow$ hotter → feels hotter

$H \uparrow \quad AT \uparrow$ higher humidity limits evaporative cooling

$v \uparrow \quad AT \downarrow$ increase evaporation increases convective cooling

Question *8.2.3 (continued)*

(b) Suppose that on one of the days during the 2009 heat wave in Mildura, at noon there was a relative humidity of 60%, wind speed of 3 m/s, and the apparent temperature AT was 4°C higher than the ambient air temperature T. What was the ambient air temperature T at that time?

$H = 60\%$, $v = 3\,m/s$

→ AT is $4°C$ higher than T ←

$T = ?$

$$AT = T + 2\left(\frac{H}{100}\right)e^{0.06T} - 0.7v - 4$$

we have $AT = T + 4$

$$\cancel{T} + 4 = \cancel{T} + 2\left(\frac{60}{100}\right)e^{0.06T} - 0.7(3) - 4$$

$$1.2e^{0.06T} = 10.1$$

$$0.06T = \ln\left(\frac{10.1}{1.2}\right)$$

$$T = \frac{1}{0.06}\ln\left(\frac{10.1}{1.2}\right)$$

$$\simeq 35.5^\circ C$$

End of Case Study 14: Apparent temperature for Aussies.

Case Study 15: **Wind chill**

- In cold climates, the *apparent* temperature to the human body is often called the *wind chill* temperature.
- Because wind chill can cause major discomfort, and in cold climates can lead to serious injuries such as frostbite or even death, it is important to measure, model and predict the severity of wind chill.

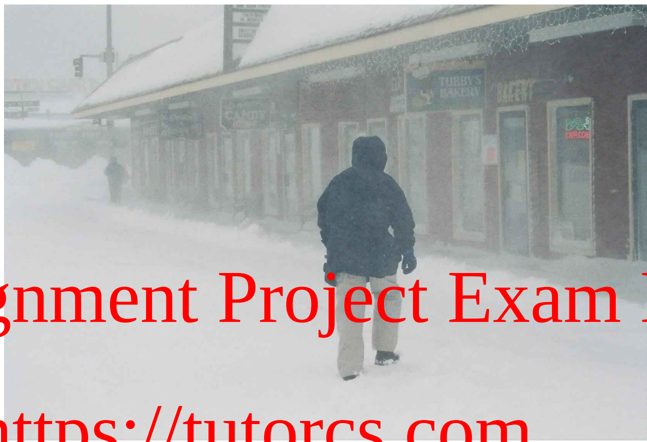


Photo 8.2: Blizzard, West Yellowstone, USA. (Source: PA.)

- It is possible to measure wind chill in a number of ways. In 2001, the US National Weather Service developed the most widely accepted model.
- Researchers exposed volunteers to various low temperatures and high wind speeds in a wind tunnel, recording their perceptions of temperatures, along with measurements of the physiological impact of wind chill on their faces.
- The researchers then formulated an equation that modelled the perceived wind chill temperature as a function of the ambient air temperature and the wind speed (for speeds of at least 5 km/h).
- Let T be the ambient air temperature in °C and v be the wind speed in km/h. The perceived wind chill temperature W in °C according to their model is:

$$\boxed{W = 13.112 + 0.6215T - 11.37v^{0.16} + \underbrace{0.3965Tv^{0.16}}_{\text{interaction}}}$$

Program specifications: Write a program that inputs wind speed in km/h and air temperature in °C, then calculates the apparent wind chill temperature.

Program 8.1: Wind chill

```
# A program to calculate apparent wind chill temperatures.
from pylab import *

air_temperature = float(input("Enter air temp. in degrees Celsius: "))
wind_speed = float(input("Enter wind speed in km/h: "))
x = pow(wind_speed,0.16)
wind_chill = 13.112 + 0.6215 * air_temperature - 11.37 * x + 0.3965 * air_temperature * x
wind_chill_1dec = round(wind_chill,1)

print("An air temp. of ",wind_speed," Celsius and wind speed of")
print( wind_speed,"km/h gives a wind chill of",wind_chill_1dec," Celsius.")
```

input

model

output

Here is the output from running the above program twice:


```
Enter air temp. in degrees Celsius: -19
Enter wind speed in km/h: 19
An air temp. of  -19  Celsius and wind speed of
19 km/h gives a wind chill of -29.0  Celsius.

Enter air temp. in degrees Celsius: -36
Enter wind speed in km/h: 135
An air temp. of  -36  Celsius and wind speed of
135 km/h gives a wind chill of -65.5  Celsius.
```


Photo 8.3: Mont Blanc. (Source: PA.)

- A common way to present information from the wind chill model is via a table of values, often with colour coding to show the risk of developing *frostbite*; see Figure 8.6.

		Air temperature (degrees Celsius)												
		10	5	0	-5	-10	-15	-20	-25	-30	-35	-40	-45	-50
Wind Speed (km/h)	5	10	4	-2	-7	-13	-19	-24	-30	-36	-41	-47	-53	-58
	10	9	3	-3	-9	-15	-21	-27	-33	-39	-45	-51	-57	-63
	15	8	2	-4	-11	-17	-23	-29	-35	-41	-48	-54	-60	-66
	20	7	1	-5	-12	-18	-24	-30	-37	-43	-49	-56	-62	-68
	25	7	1	-6	-12	-19	-25	-32	-38	-44	-51	-57	-64	-70
	30	7	0	-6	-13	-20	-26	-33	-39	-46	-52	-59	-65	-72
	35	6	0	-7	-14	-20	-27	-33	-40	-47	-53	-60	-66	-73
	40	6	-1	-7	-14	-21	-27	-34	-41	-48	-54	-61	-68	-74
	45	6	-1	-8	-15	-21	-28	-35	-42	-48	-55	-62	-69	-75
	50	5	-1	-8	-15	-22	-29	-35	-42	-49	-56	-63	-69	-76
	55	5	-2	-8	-15	-22	-29	-36	-43	-50	-57	-63	-70	-77
	60	5	-2	-9	-16	-23	-30	-36	-43	-50	-57	-64	-71	-78
	65	5	-2	-9	-16	-23	-30	-37	-44	-51	-58	-65	-72	-79
	70	5	-2	-9	-16	-23	-30	-37	-44	-51	-58	-65	-73	-80

Risk of developing frostbite:	
Low:	< 5% chance of developing frostbite
Increasing:	5% - 95% chance of developing frostbite in 10 to 30 mins.
High:	> 95% chance of developing frostbite in 5 to 10 mins.
Very high:	> 95% chance of developing frostbite in 2 to 5 mins.
Extreme:	> 95% chance of developing frostbite in 2 mins.

Figure 8.6: Wind chill temperatures at various ambient temperatures and wind speeds, colour-coded with frostbite risk factors.

- Frostbite is a medical condition in which intense cold causes tissues to freeze and die, most commonly in body extremities, particularly fingers and toes.
- Severe cases can lead to gangrene and the need for amputations.

End of Case Study 15: Wind chill.