Lecture 30: A limit to the madness

Learning objectives

✓ Analyse the form of population growth under limited resource constraints

✓ Understand the meaning and use of carrying capacity

Scientific examples

✓ Fish populations

✓ Human population

Maths skills

✓ Know the logistic DE and its solution

✓ Interpret graphs

11.3 Limited scope for growth

- Exponential growth models are *unconstrained*, so the growth continues indefinitely with a constant proportional rate of increase, say r.
- This can be accurate over limited time periods, but in reality populations cannot continue to show unconstrained growth forever. Many *constrained* growth models assume that there is a maximum population size.

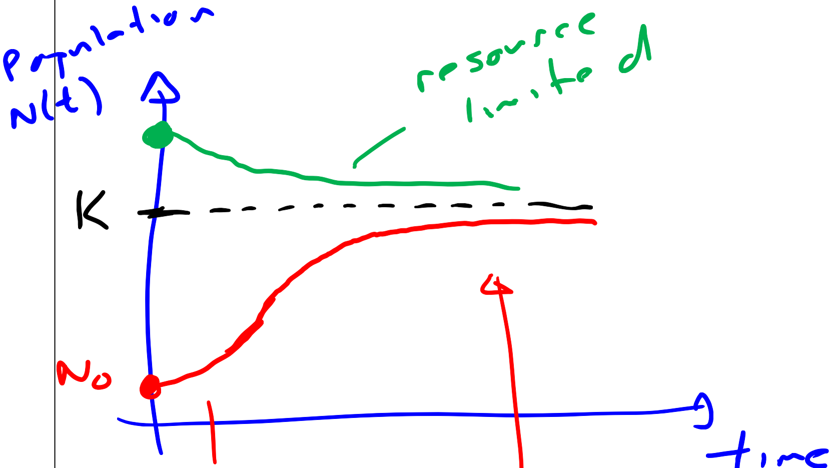
Carrying capacity of an ecosystem

The **carrying capacity** K of an ecosystem is the maximum population size of a particular organism that is supported by resources within the ecosystem. Resources may include food, water, shelter and sunlight. The carrying capacity for a particular organism often changes over time; for simplicity, we will assume it remains constant.

A population size below the carrying capacity will typically increase towards the carrying capacity, whereas a population size above the carrying capacity (which may occur when, for example, a lake is overstocked with fish) will typically decrease to the carrying capacity.

Question 11.3.1

Let $N(t)$ be the size of a fish population in a certain lake at any time t in months, and let the carrying capacity of the lake be K fish. Draw a rough sketch of $N(t)$ versus time.



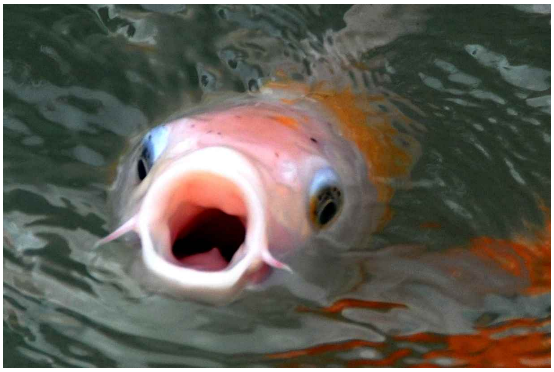


Photo 11.2: Kiss me, red. (Source: PA.)

exponential growth unconstrained

growth rate is significant

resource limited growth rate is small

growth rate depends on population

- We have modelled unconstrained growth with the exponential DE. This assumes a constant growth rate, r, giving an equation

$$\boxed{N' = rN}$$

r - growth rate "unconstrained"

- One way to model *constrained growth* is to assume that the growth rate varies with the population size N, rather than remaining constant. That is,

$$N' = g(N)N$$

varying growth rate

where $g(N)$ is an *effective growth rate* that changes as N changes.

Develop a model for g(N)

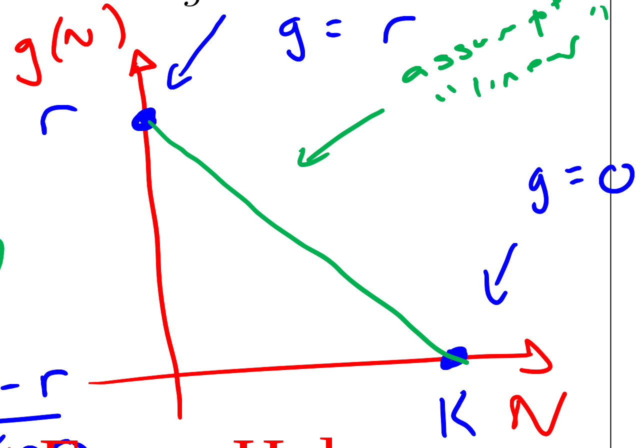
assumption

Question 11.3.2

Assume that a population has an unconstrained growth rate of r. As the population N increases, the effective growth rate g reduces linearly from r, until the population reaches the carrying capacity K at which point the effective growth rate is 0. Derive an expression for g as a function of N.

Given two points (N, g)

$(0, r)$ (y-int) $(K, 0)$ (x-int)



$g = mN + c$

m = slope = $\frac{\text{rise}}{\text{run}} = \frac{0-r}{K-0} = -r/K$

c = y-intercept = r

$g = \left(\frac{-r}{K}\right)N + r \Rightarrow g = r\left(1 - \frac{N}{K}\right)$

$N = K \Rightarrow g = 0$ ✓

$N = 0 \Rightarrow g = r$ ✓

Logistic DE

Any function $N(t)$ that changes at a rate proportional to its value (with unconstrained growth rate r), **and also** in reverse proportion to how close the value is to a carrying capacity K, is modelled by the logistic DE

$$\boxed{N' = r\left(1 - \frac{N}{K}\right)N.}$$

- In the logistic model, for $N < K$, the change in the population N' will:
  - **increase** as the population size gets larger and there are more individuals who can reproduce; and

    significant

    $N' > 0$, growing rapidly
  - **decrease** as the population size gets larger as individuals compete for scarce resources.

    $N' > 0$ but small, growing slowly

    extreme $N = K$, $N' = 0$

- The power of the logistic model is the interaction between two opposing factors, growth and competition.

Question 11.3.3

What does the logistic DE predict for N' in the following extreme cases:

$N' = r\left(1-\frac{N}{K}\right)N$

(a) N is approximately equal to K?

$N \simeq K \Rightarrow \frac{N}{K} \simeq 1 \Rightarrow \left(1-\frac{N}{K}\right) \simeq 0$

Thus $N' \simeq 0$

no growth (constrained)

(b) N is much less than K?

$N << K \Rightarrow \frac{N}{K} \simeq 0 \Rightarrow \left(1-\frac{N}{K}\right) \simeq 1$

"<<" => much less than

DE : $N' = rN$

unconstrained growth

Solution to the logistic DE

The logistic DE can be solved exactly. If N_0 is the value of N at time 0, the exact solution is

Soln:

$$\boxed{N(t) = \frac{K\,N_0}{N_0 + (K - N_0)e^{-rt}}}$$

satisfies the logistic DE

$r = 0.1 \text{ month}^{-1}$

Example 11.3.4

A fish species with initial population $N_0 = 30$ and unconstrained growth rate of 10% per month lives in a reef with a carrying capacity of K=1000 fish. The function $N(t)$ gives the number of fish at time t months, and Figure 11.1 graphs $N(t)$ for 80 months, showing the typical "S"-shaped logistic curve.

$$\boxed{N(t) = \frac{30000}{30 + 970e^{-0.1t}}}$$

t - units months

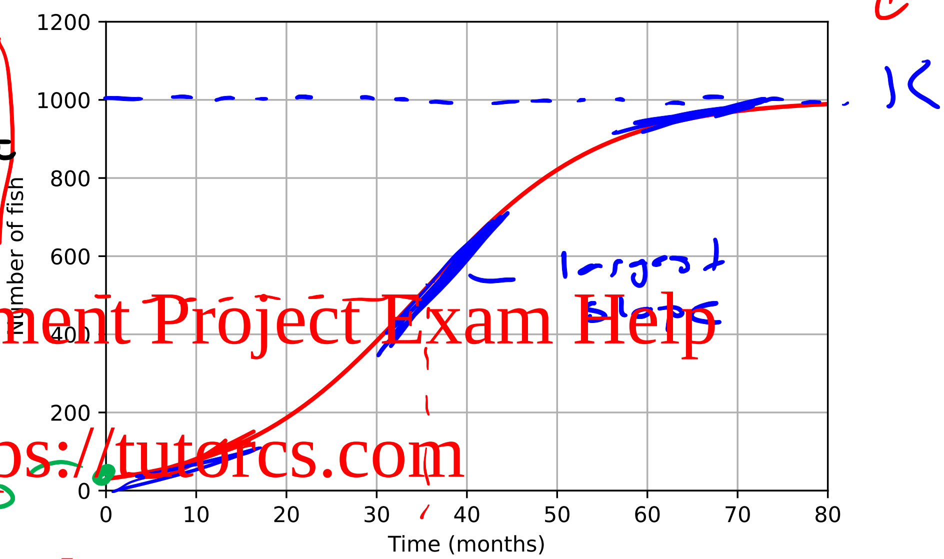


Figure 11.1: Logistic model with an initial population of 30 fish.

Question 11.3.5

With reference to Example 11.3.4, at what population size is the value of N' largest? Explain your answer briefly.

we note N' is the slope at time t

$N' > 0$

N' greatest (biggest slope)
at 30-50 months

Can show N' is greatest
at $K/2$.

here when $N = \frac{1000}{2} = 500$

Case Study 26: **Overpopulation annoys us all**

"worldometer.com" $\simeq$ 7.9 billion

- The human population of Earth is rising very rapidly. In 1950 the global population was about 2.5 billion, and in 2012 it first exceeded 7 billion.
- The doubling time has reduced in recent centuries: it took about 300 years for the population to double to one billion, then 120 years to double again, then 47 years to double again.

Question 11.3.6

(a) The following graphs show the human population of Earth over 1000 years (left) and transformed using natural logarithms (right). Is the population shown in Figure 11.2 growing exponentially?

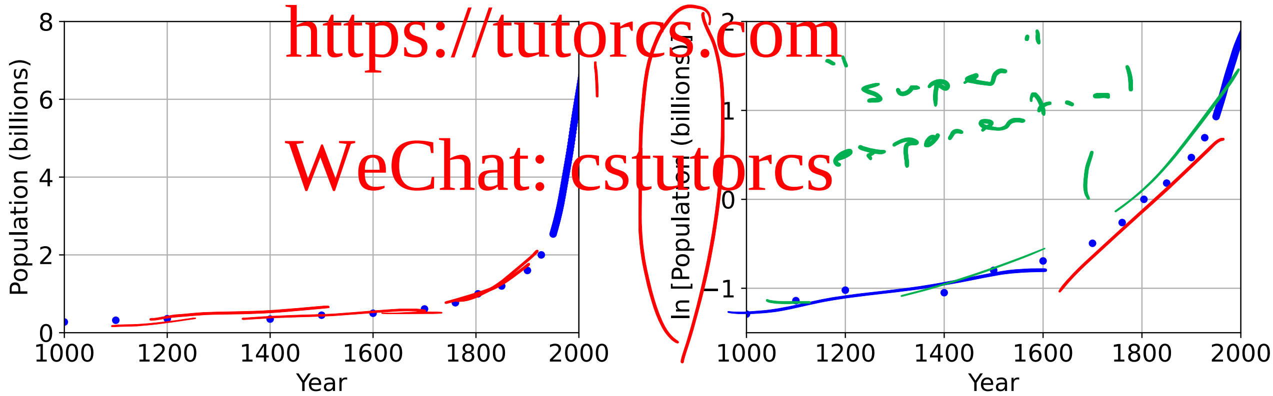


Figure 11.2: Human population of Earth over 1000 years (left), and transformed using ln (right).

log-linear plot => variables that is exponential will appear linear

If we had exponential growth then a log-linear plot would be a straight line

Also doubling time is not constant => not exponential growth.

change in population per unit time

Question *11.3.6 (continued)*

(b) If the population were growing exponentially, then the shape of the graph showing the annual change in global population would also be exponential. This is because $P(t) = P_0e^{rt}$ has $P'(t) = rP_0e^{rt}$.

Figure 11.3 shows the annual change in global population since 1951. Interpret this graph. What does it mean for global population since 1951, and what might it mean into the future?

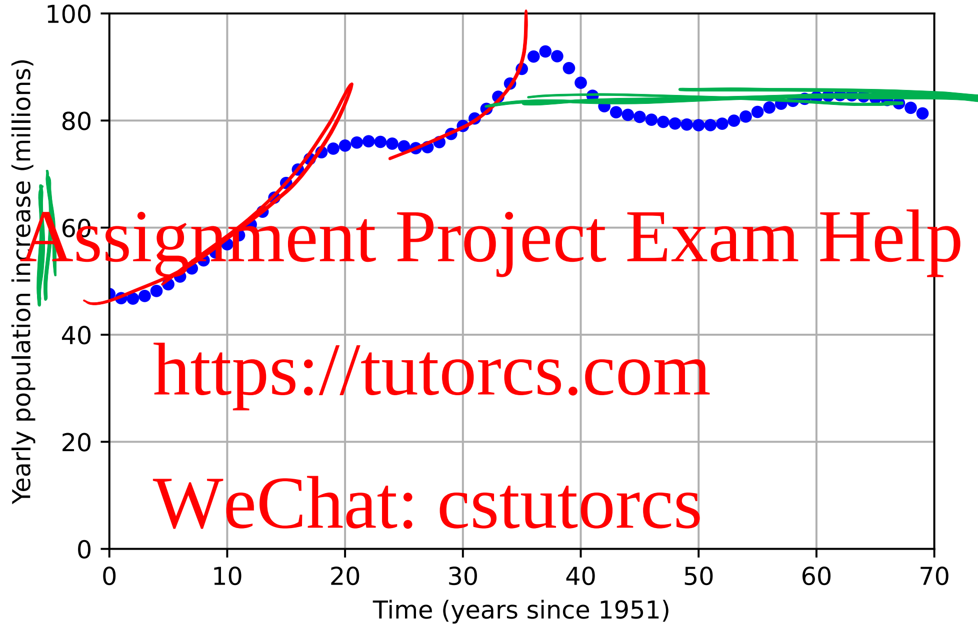


Figure 11.3: Annual change in global population, 1951 onwards.

Since about early 1980s, almost constant.

why?

- can't afford to have as many children
- contraception
- education, planning, social factors

- The United Nations has estimated the populations of all countries, and also globally, for each year until 2100; see [55]. Their models take into account predicted shifts in demographic patterns in each country.

- Figure 11.4 shows the projected global population based on different levels of fertility.

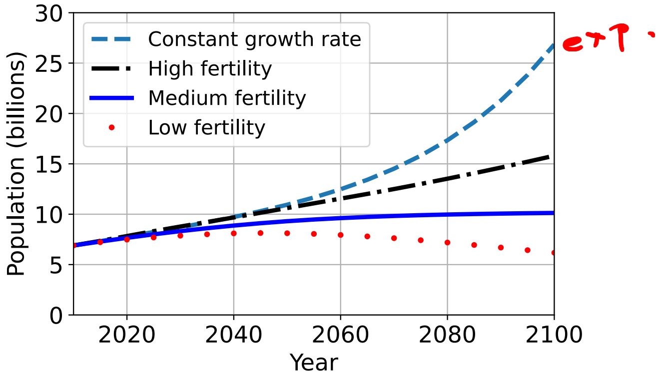


Figure 11.4: UN predictions of global population, 2010–2100.

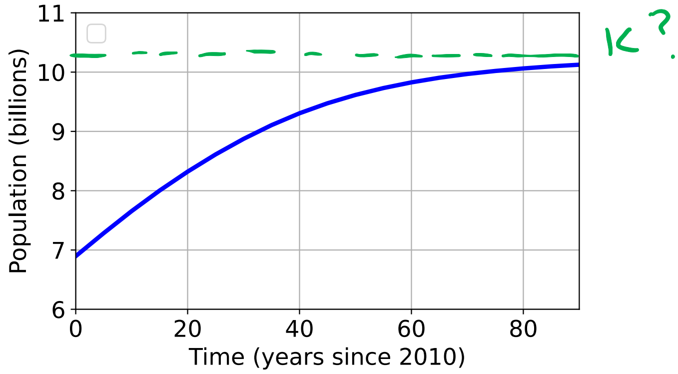


Figure 11.5: Predicted global population with medium fertility.

Question 11.3.7

Figure 11.5 shows the UN projected global population $P(t)$ from 2010 ($t = 0$) until 2100 ($t = 90$), assuming 'medium fertility'.

(a) Is it reasonable to model $P(t)$ using a logistic DE, $P' = rP\left(1 - \frac{P}{K}\right)$?

Mathematically - curve appears to have a logistic shape with a carrying capacity

Physically ?

(b) Estimate the value of K in the model. Discuss the statement: 'K is the carrying capacity of Earth for humans'.

From the graph $K \simeq 10 \cdot 3$ billion

Hopefully not resource limitation leading to competition

more a social limitation

End of Case Study 26: Overpopulation annoys us all.