Lecture 16: **Chill out with logs**

Learning objectives

✓ Interpret exponential function models of real-world phenomena

✓ Understand the form of log-plots

Scientific examples

✓ Newton's Law of Heating and Cooling

✓ Atmospheric pressure

Maths skills

✓ Understand and interpret exponential functions and their graphs

✓ Interpret log-lin and log-log plots

Case Study 13: **Hot stuff, cold stuff**

- Moving an object with one temperature to a location with a different (but constant) temperature leads to a gradual change in the temperature of the object to match that of the new location.

- Energy (also called heat) is transferred to/from the object from/to the surrounds through processes such as conduction, convection and radiation.

- amount per unit time

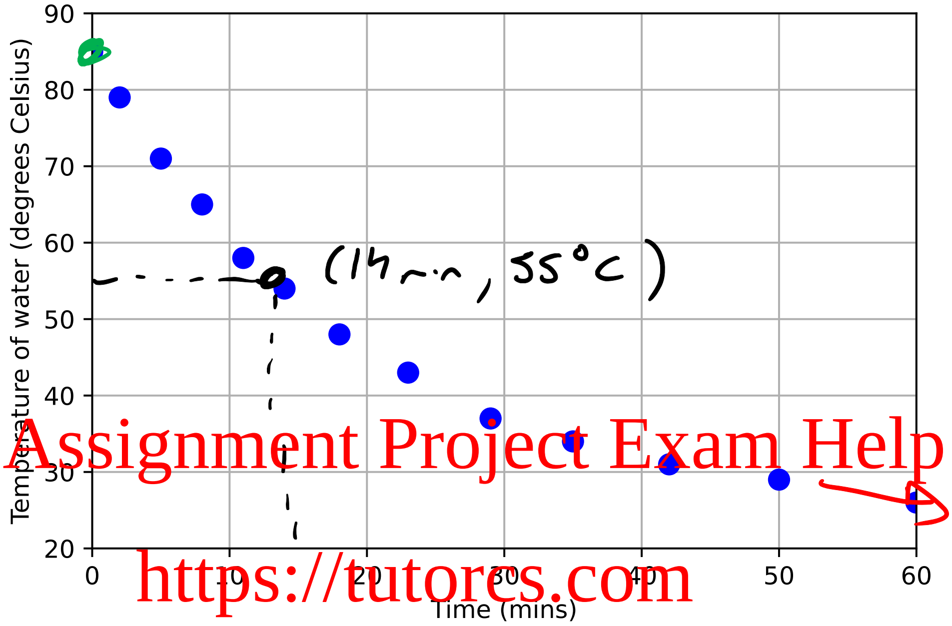
- The rate at which this energy is transferred depends on the *temperature difference* between the object and its surrounds.

- Hence the temperature of the object as a function of time can be described by an *exponential function*.

$$\frac{\Delta T}{\Delta t} \propto (T_s - T_{object})$$

Question 6.2.4

In an experiment, the temperature of hot water in a cooler, constant temperature container was recorded at various times over one hour; see Figure 6.2. The room (and container) temperature was measured to be 25 °C.



Ae^{kt}

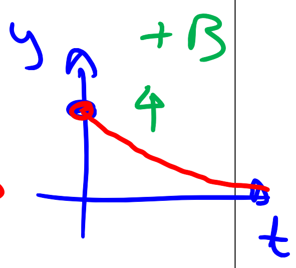


Figure 6.2: A graph of the measured temperatures.

Determine an equation for the water temperature at any time in minutes. Note that the water approaches room temperature over a long time.

Assume exponential $\quad T = Ae^{kt} + B, \; k<0$

Find A, k, B

At large times, $T \to$ surrounding temperature $\quad B = 25°C \quad$ ($e^{kt} \to 0$ for $k<0$ and t large)

At $t=0$, $T = 85°C$,

$e^0 = 1 \qquad 85 = Ae^{k(0)} + 25 \Rightarrow A = 85 - 25 = 60°C$

Use point (14 min, 55°C)

$$T = 60e^{kt} + 25$$
$$55 = 60e^{k(14)} + 25$$
$$30 = 60e^{14k} \Rightarrow \ln\left(\frac{30}{60}\right) = \ln e^{(14k)}$$
$$\ln\tfrac{1}{2} = 14k \ln e$$
$$k = \frac{\ln\frac{1}{2}}{14} \simeq -0.05\ \text{min}^{-1}$$

Summary

$$T = 60e^{-0.05t} + 25$$

where t is in minutes and T is in °C

We can develop a computer program to model the temperature.

Program specifications: Write a program that plots the measured water temperatures and the function that models these temperatures.

Program 6.1: Temperatures

```
# Program to plot measured and modelled water temperatures.
from pylab import *

# Measured temperatures (minutes, degrees C)
times = array([0,2,5,8,11,14,18,23,29,35,42,50,60])
temperature_data = array([85,79,71,65,58,54,48,43,37,34,31,29,26])

# Model
temperature_model = 60 * exp(-0.05 * times) + 25

# Draw graph
plot(times, temperature_data, 'bo', markersize=8, label="Data")
plot(times, temperature_model, 'r-', linewidth=3, label="Model")
xlabel("Time (mins)")
ylabel("Temperature of water (degrees Celsius)")
xlim(0,60)
ylim(20,90)
grid(True)
legend()
show()
```

data input

model

output graph

Output from the program is shown in Figure 6.3.

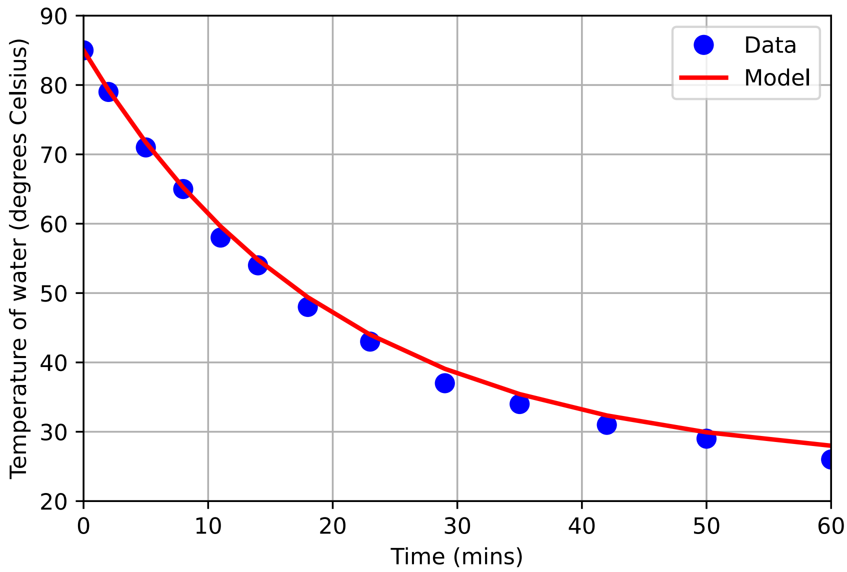


Figure 6.3: Modelled and measured water temperatures.

Question 6.2.5

Do you think the model shown in Figure 6.3 is a good fit to the given data? If you were to use this model, justify your choice. If you were to modify the model, what change or changes would you suggest and why?

Generally a good fit but perhaps over-predicts for larger times.

Could correct by assuming that the room temperature was 24°C ← re-measure

End of Case Study 13: Hot stuff, cold stuff.

6.3 Logarithms in action

- Logarithms provide a convenient mechanism for converting exponential data into a form that can make data analysis easier.

Question 6.3.1

Assume some data are modelled by the exponential function $y(x) = Ce^{kx}$. Demonstrate how a logarithmic transformation of the data values results in a linear model. Interpret the y-intercept and gradient of the linear model. (Hint: if x and y are positive then $\ln(xy) = \ln x + \ln y$.)

Our model follows $y = Ce^{kx}$

Transform - take ln of both sides

$$\ln y = \ln(Ce^{kx})$$

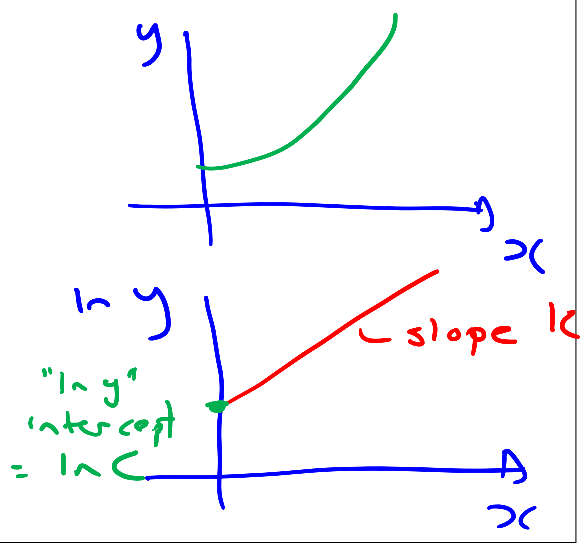
$$= \ln C + \ln e^{kx}$$

$$= \ln C + kx \underbrace{\ln(e)}_{=1}$$

$$\underbrace{\ln y}_{\text{dep. variable}} = \underbrace{\ln C}_{\text{"y"-intercept}} + \underbrace{k}_{\text{slope}} \underbrace{x}_{\text{indep. variable}}$$

This a linear function

"linearising the equation"



Question 6.3.2

Earlier we saw that the *International Standard Atmosphere* (ISA) [27] models various atmospheric properties, including temperature, pressure and density. Figure 6.4 shows atmospheric pressures in kilopascals (kPa) at various altitudes in the ISA in a linear-linear and log-linear form.

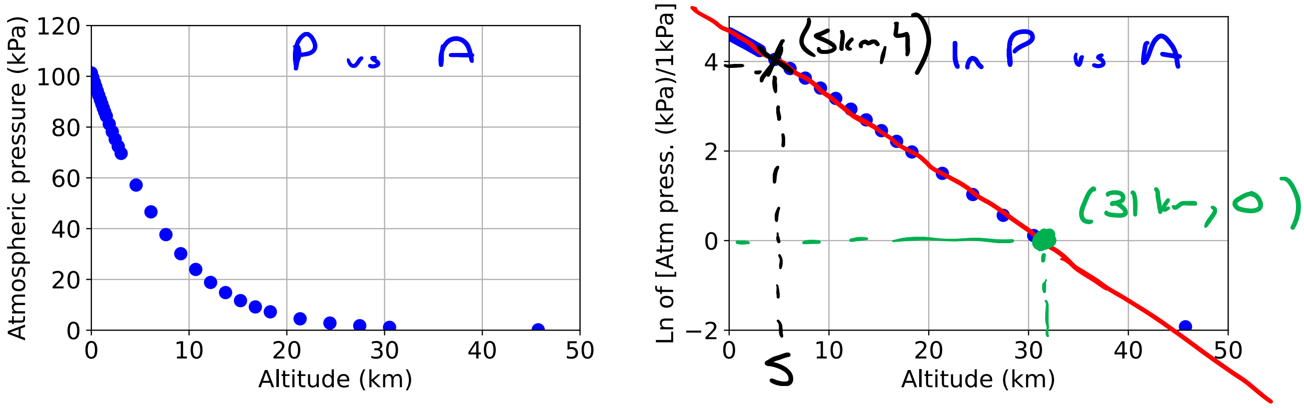


Figure 6.4: ISA pressure (linear and transformed data).

(a) Use Figure 6.4 and Question 6.3.1 to find an exponential model of pressure in the ISA.

Two points $(31, 0)$ & $(5, 4)$

Slope $= \frac{\text{rise}}{\text{run}} = \frac{4-0}{5-31} \simeq -0.15 \text{ km}^{-1}$

Equation $(\ln P) = (\ln P_0) - 0.15 A$

Substitute $(31, 0)$ A & ln P

$$0 = \ln P_0 - 0.15(31)$$

$$\ln P_0 = 0.15 \times 31 \sim 4.65$$

$$P_0 = e^{4.65} \simeq 105 \text{ kPa}$$

Our equation is

$$P = P_0 e^{kA}$$

$$\underline{P = 105\, e^{-0.15A}}$$

— A is in km, P is in kPa

Question *6.3.2 (continued)*

b) When a jetliner is in flight, the pressure in the cabin is artificially raised to a higher level than the pressure outside. The *cabin altitude* is the altitude at which atmospheric pressure matches the pressure inside the cabin.

pressure in cabin = air pressure at a height of 2,000m

Modern planes typically cruise at an altitude of 12,000 m, but maintain a cabin altitude of about 2,000 m. Determine the pressure inside and outside the cabin when cruising. Note that, on the ground, atmospheric pressure is around 100 kPa.

$P = 105e^{-0.15A}$ ← ISA model

cabin pressure ⇒ P when $A = 2{,}000$m $= 2$ km

$P = 105e^{-0.15(2)} \simeq 78$ kPa

(c.f. ~101 kPa)

outside pressure ⇒ P when $A = 12{,}000$m $= 12$km

$P = 105e^{-0.15(12)} \simeq 17$ kPa

Photo 6.2: Bang? (Source: PA.)

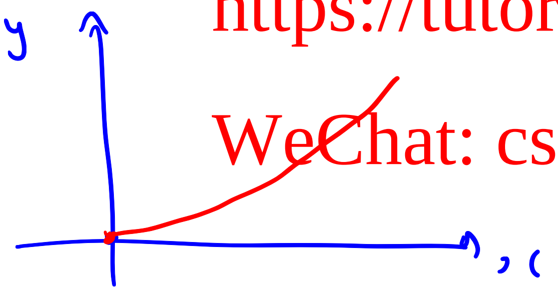
- Logarithms can also be used to "linearise" power functions.

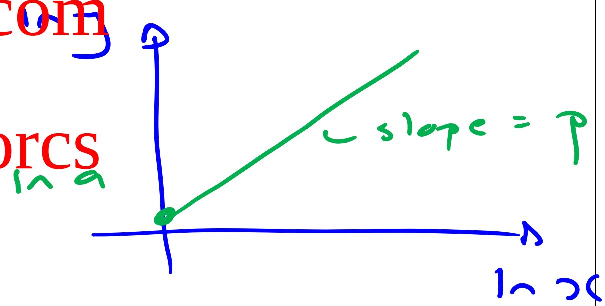
Question 6.3.3

Assume we now have some data modelled by a power function of the form $y(x) = ax^p$. Demonstrate how a logarithmic transformation of this data can also result in a linear model. Again interpret the y-intercept and gradient of the linear model. (Reminders: if x and y are positive then $\ln(xy) = \ln x + \ln y$ and $\ln x^p = p \ln x$)

Transform - take ln of both sides

$$\ln y = \ln(ax^p)$$
$$= \ln a + \ln(x^p)$$
$$\underbrace{\ln y}_{\text{dep. variable}} = \underbrace{\ln a}_{\text{intercept}} + \underbrace{p}_{\text{slope}}\ \underbrace{\ln x}_{\text{indep. variable}}$$





In summary

- A log-linear plot is useful in examining data that may be modelled by an *exponential function*.
- A log-log plot is useful in examining data that may be modelled with a *power function*.

Use the online modules on functions to help your understanding of this process of "linearising" data.

SOMSE

Question 6.3.4

Keeling Model 3: Figure 6.5 shows two plots: a graph of the function $y(t) = 280 + 35e^{0.022t}$, and the Keeling curve.

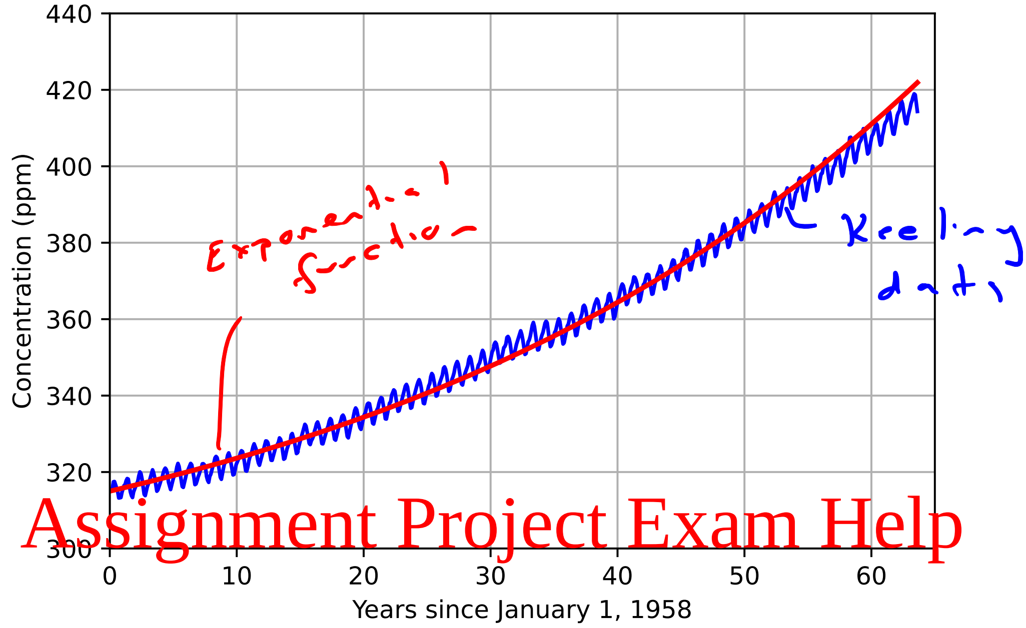


Figure 6.5: The Keeling curve and an exponential model.

(a) Explain mathematically how each term in $y(t)$ impacts on its graph.

$y = 280 + 35e^{0.022t}$

vertical shift

vertical scale

exponential growth

(b) Data from ice-core samples show that long-term atmospheric CO_2 levels remained relatively constant at 280 ppm. Explain the physical significance of the constants 280 ppm and 35 ppm.

As $t \to -\infty$ (large negative)

$e^{0.022t} \to 0$

280 ppm – historical levels

35 ppm – increase until $t = 0$ (Jan 1, 1958)

At $t = 0$

$280 + 35e^0$

$= 280 + 35$

$= 315$ ppm

(c) How effectively does $y(t)$ model the underlying Keeling curve trend?

Fits the growth very well

Need oscillations.