Chapter 6: **All about that base**

Lecture 15: **Isotopes and exponents**

Learning objectives

✓ Interpret exponential function models of real-world phenomena

Scientific examples

✓ Radioactive decay

✓ Carbon dating

Maths skills

✓ Understand exponential functions and logarithms

✓ Doubling time and half-life

Functions : linear
quadratic
power
sin
exponentials + log

Image 6.1: *The Three Sphinxes of Bikini* (1947), Salvador Dalí (1904 – 1989), Morohashi Museum of Modern Art. (Source: Museum publication.)

Exponential functions are useful for modelling many natural phenomena such as growing populations or radioactive decay of isotopes, as well as many "un-natural" phenomena such as compound interest.

graphs - log-lin
log-log

Logarithms are closely related to exponential functions and you will have used them in previous mathematical study to solve exponential equations.

In this chapter we will review some of the properties of these important functions and discuss some of the scientific contexts in which they naturally arise. You should have encountered exponential and logarithmic functions in previous study of mathematics. See Section C.3 in Appendix C for the pre-requisite mathematical tools we will use in this chapter. Use the online modules, available through the course website, for further support. Sons E

We will also see how useful logarithms are for displaying and understanding data. Log plots and even log-log plots are extremely useful for communication and understanding data in many scientific contexts.

6.1 Growth and decay

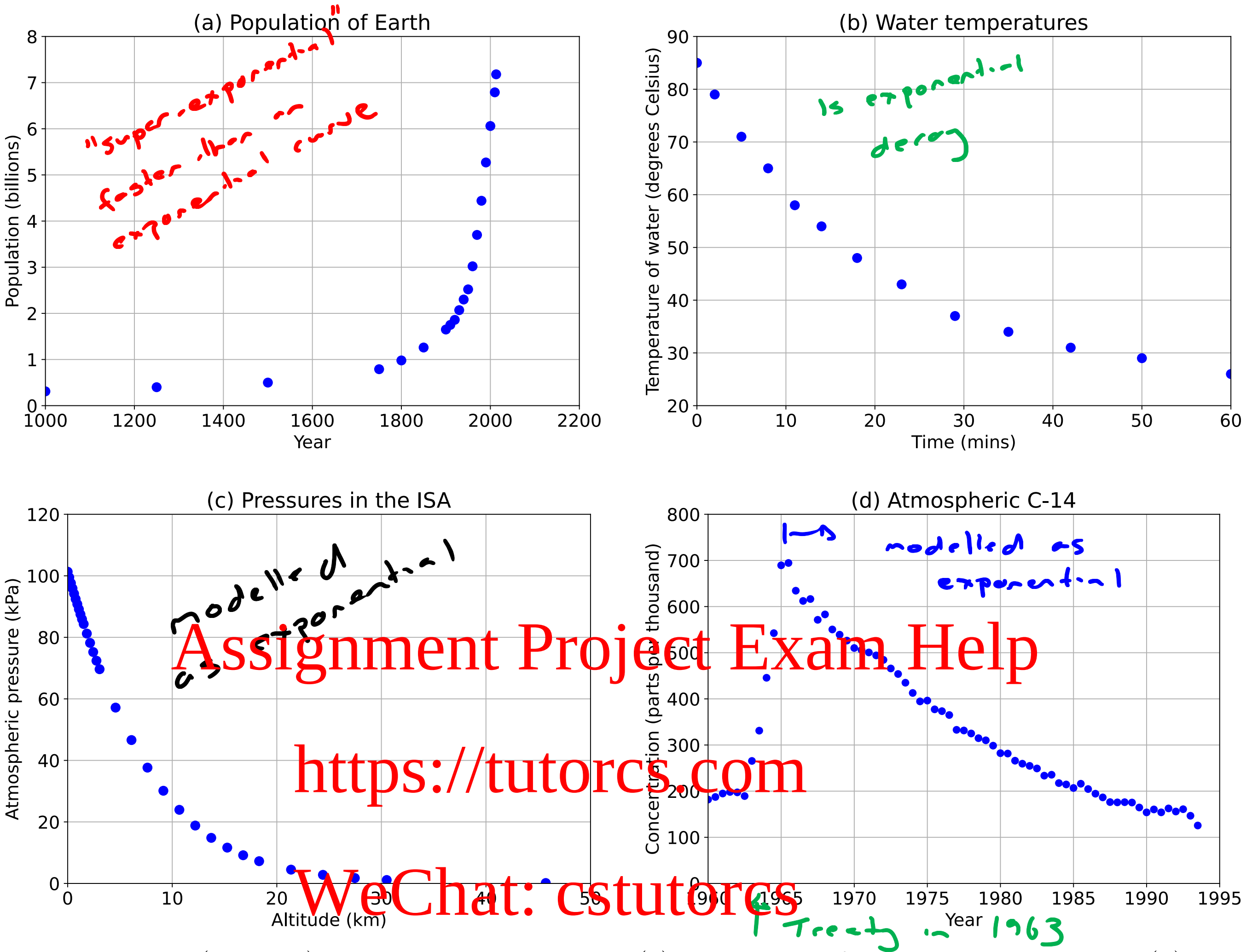


Figure 6.1: Four (possibly) exponential phenomena. (a) Population of Earth over 1000 years. (b) Measured water temperature in a simple experiment. (c) Atmospheric pressures in the international standard atmosphere. (d) Concentration of atmospheric radioactive Carbon-14.

***Question** 6.1.1*

What do you think could have caused the observed changes of the atmospheric radioactive Carbon-14?

In the early 1960s there were atmospheric tests of atomic bombs

Plausible - radioactive decay
but half-life ~ 5000-6000 years

Dispersion of C-14 throughout the atmosphere
taken up by plants & animals
(~ exponential)

- Science primarily studies phenomena that change. Often, the rate of change at any time is proportional to the amount that is currently there.
- This is typical of many populations. For example, each year the size of the global human population is increasing by around 1.5% of its current size.
- Any phenomenon that has a rate of change proportional to the current amount follows an *exponential* function. (We will see why later.)
- An exponential function is of the form $\boxed{y(t) = Ca^{kt}}$, where a is the *base* of the exponent. In many scientific contexts, Euler's number ($e \approx 2.718\ldots$) is used as the base, giving $y(t) = Ce^{kt}$.

a - base - 10 → 10^{kt}
- e → e^{kt}

k - is a constant
C - is a scaling constant

Doubling time/Half-life

The **doubling time** for an exponentially growing quantity is the time it takes to increase to twice its current size.

The **halving time** or **half-life** for an exponentially decreasing quantity is the time it takes to decrease to half its current size.

Many exponential phenomena in science have relatively constant doubling times or half-lives over extended periods; knowing these values provides useful information about the phenomena.

***Example** 6.1.2*

Exponential functions occur frequently in models of nature and the social sciences. Some examples include unconstrained and constrained population growth, radioactive decay and carbon dating, modelling drug concentrations in blood, and modelling *habituation* to a stimulus.

- *Logarithms* (or *logs*) are very closely related to exponential functions.
- Logarithms are the *inverse* of exponentiation (in much the same way that division is the inverse of multiplication).

$y = x^2$

$\sqrt{y} = x$

"natural log"

Logarithms and exponentials

The relationship between exponentials and logarithms is:

- If $\boxed{y = 10^x}$ then $\boxed{x = \log_{10} y}$ (and vice-versa).
- If $\boxed{y = e^x}$ then $\boxed{x = \ln y}$ (and vice-versa).

$\ln \Rightarrow \log_e$

$\log \Rightarrow \log_e$, $\log_{10}$

$\log(x^n) = n \log x$

Question 6.1.3

(a) Find $\log_{10} 1000$ and $\log_{10} 0.01$.

$\log_{10} 1000 = \log_{10} 10^3 = 3 \underbrace{\log_{10} 10}_{=1} = 3$

$\log_{10} 0.01 = \log_{10} 10^{-2} = -2 \log_{10} 10 = -2$

(b) If $y = e^{0.02t}$, find $\ln y$.

$\ln y = \ln\left(e^{0.02t}\right) = 0.02t \underbrace{\ln(e)}_{=1}$

$= 0.02t$

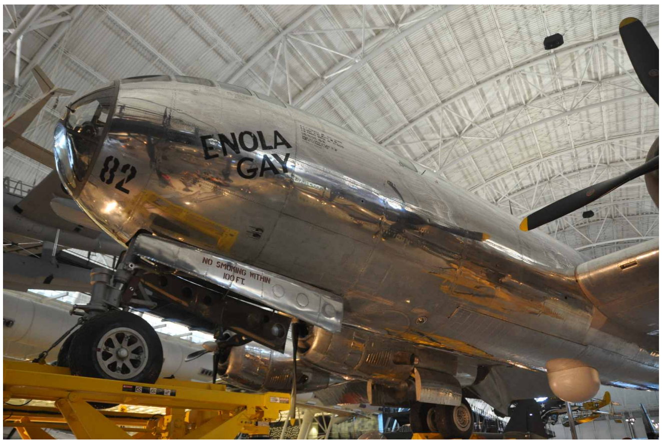
6.2 Exponentials in action

Case Study 12: Radioactive decay

Hydrogen H-1
− +
(stable)

Hydrogen H-2
− + n
(stable)

Hydrogen H-3
n − + n
(unstable)



Types of decay
α (particle) + + n n
β − (particle)
γ (wave)

Photo 6.1: The B-29 Superfortress bomber "Enola Gay", National Air and Space Museum, Virginia, USA. (Source: PA.)

- Isotopes of an element behave the same way chemically but have different numbers of neutrons in the nucleus of the atom.

- One standard way of denoting isotopes is to write the name or chemical symbol of the element, hyphenated with its atomic mass. For example, deuterium (an isotope of hydrogen and the main ingredient in "Heavy water") is written as Hydrogen-2 or H-2.

"Deuterium" - D
D_2O

- Not all atoms remain the same over time; some undergo *radioactive decay*, which involves rearrangement of the nucleus of the atom, sometimes changing it into a different element.
- Radioactive isotopes have useful applications in a range of sciences and industries, including chemistry, biology, medicine, physics and engineering. Therefore, it is important to understand how to model their decay.
- Radioactive decay is spontaneous, so there is no way of knowing *when* a *specific* individual atom is going to undergo decay.
- However, it *is* known that in any given time period a certain *proportion* of the total quantity in a sample will have decayed.

- Thus, radioactive material undergoes continuous decay at a rate **proportional** to the **quantity** of material, so the decay is an exponential process.

$\frac{\text{change}}{\text{time}} \propto \text{amount}$ ↑ "proportional to" $\frac{\Delta N}{\Delta t} \propto N$

Decay constant

For a radioactive element, the *decay constant* k reflects the rate of decay of the element, and is a property of the chemical element. The half-life can be calculated from the value of k, and vice-versa.

Example *6.2.1*

Decay constants and half-lives vary greatly between radioactive elements. For example:

- Polonium-212 has a half-life of about 3×10^{-7} s.
- Uranium-238 has a half-life of about 4.5×10^{9} years.
- Carbon-14 has a half-life of about 5730 years.

Example *6.2.2*

Carbon-14 (C-14, also known as *radiocarbon*) is used to determine the age of organic-based artefacts (up to around 60,000 years).

Cosmic rays striking nitrogen in the upper atmosphere produce C-14. It then reacts chemically with oxygen to form radioactive carbon dioxide which permeates living creatures in a fixed proportion, either directly (by absorption from the atmosphere), or indirectly (via food chains).

When an organism dies, it ceases to accumulate C-14, and the remaining amount undergoes net decay over time. *Carbon dating* is the process of measuring the residual level of C-14 in organic artefacts, and thus deducing their age.

Alive : C-14 is continually replenished
proportion of C-14 out of all carbon in the body is the same as the atmospheric proportion

Dead : C-14 no longer replenished - decreases due to radioactive decay

Question 6.2.3

The half-life of C-14 is 5730 years.

(a) Find the decay constant of C-14. $\Rightarrow$ Find k

Let A = number of C-14 atoms in a sample at a time t

A_0 = initial number of C-14 atoms in the sample

Thus $A = A_0 e^{kt}$ (expect $k < 0$)

Find k given the half-life = 5730 years

When $t = 5730$ years, $A = \frac{A_0}{2}$

Hence

$$\frac{\cancel{A_0}}{2} = \cancel{A_0} e^{k(5730)}$$

$$\ln\left(\frac{1}{2}\right) = \ln\left(e^{5730k}\right)$$

$= 1$

$$= 5730k \ln e$$

$$k = \frac{\ln(1/2)}{5730} = -1.2 \times 10^{-4} \text{ year}^{-1}$$

unitless

$e^{\boxed{x}}$

$\Rightarrow$ kt must be unitless

Hence

$$A = A_0 e^{(-1.2\times10^{-4})t}$$

where t is in years

A is unitless (number of C-14 atoms)

Question *6.2.3 (continued)*

(b) Consider the following extract from the paper [8]. 1989

> *"The Shroud of Turin, which many people believe was used to wrap Christ's body, bears detailed front and back images of a man who appears to have suffered whipping and crucifixion. It was first displayed at Lirey in France in the 1350s . . . Very small samples from the Shroud of Turin have been dated by accelerator mass spectrometry in laboratories at Arizona, Oxford and Zurich. As Controls, three samples whose ages had been determined independently were also dated."*

carbon dating

Researchers discovered that 91.9% of the expected original amount of C-14 was present (compared to that in new organic garments). Deduce the (approximate) age of the Shroud based on the dating process, and comment on your answer.

Use C-14 decay where

$$A(t) = A_0 e^{-1.2\times 10^{-4} t}$$

Find t when $A = 91.9\% A_0$
$= 0.919 A_0$

$$0.919 \cancel{A_0} = \cancel{A_0} e^{-1.2\times10^{-4} t}$$

$$\ln(0.919) = \ln\left(e^{-1.2\times10^{-4} t}\right)$$

$$= -1.2\times10^{-4} t (\ln e)$$

$$t = \ln(0.919) / -1.2\times10^{-4}$$

$$\approx 698 \text{ years}$$

End of Case Study 12: Radioactive decay.

Indicates dates to 1200-1300 (year)