Chapter 11: **Differential equations and populations**

Lecture 29: **What's a DE?**

Learning objectives

✓ Understand what a differential equation (DE) is and how they arise naturally in scientific contexts

Scientific examples

✓ Human population growth

✓ E. coli

Maths skills

✓ Understand what a DE is

✓ Know how to check if a function is a solution to a DE

✓ Know the exponential DE and its solution

Image 11.1: *Lascaux Cave*, Depiction of aurochs, horses and deer, UNESCO World Heritage Site, Part of prehistoric sites and decorated caves of the Vésère Valley (Source: en.wikimedia.org.)

Differential equations are very useful, "natural" tools for modelling a huge range of phenomena in science and other areas. For example, if you know that a population is changing in size at a rate proportional to its current value, then you can write and solve a simple DE to represent what is happening. Many natural phenomena display this kind of relationship, so knowing how to write and solve the DE allows the phenomena to be studied.

Here we investigate changing phenomena in the context of life cycles and populations, and explore how to model these phenomena. The mathematics that underpins this section is the *differential equation* or DE. Differential equations describe how the value(s) of one or more quantities are changing, and typically involve a function and its derivative(s). We will begin by introducing differential equations (DEs), and then study the exponential and logistic DEs, and use them to model changes in the sizes of simple populations. We will also learn how to use *Euler's method* to solve DEs numerically.

11.1 Introduction to differential equations

- Typically, developing a mathematical model of a phenomenon involves deriving one or more equations that predict the *value* of the phenomenon.
- Sometimes, this is difficult or impossible. Instead, it may be possible to write an equation for how the value is *changing*, then use mathematical techniques to deduce information about the *value*.
- An equation for the rate at which the value of a phenomenon is changing is called a *differential equation*.

  population $N \Rightarrow$ change N'

- To understand differential equations, it is essential to be clear about the distinction between the *value* of a phenomenon, and the *rate at which that value is changing*.

$y = y(t)$ $\quad y' \quad N' \quad N$ $\qquad y' = f(y,t)$

Differential equations

If y is an unknown function of t, then a **differential equation** is an equation that involves a combination of t, y and/or the derivatives of y.

In all of the examples we will study, the DE will be of the form $y' = \ldots$. (That is, the DE will be an equation for the rate of change of y.)

A particular function y is called a **solution** to a DE if the DE is true when y and its derivative(s) are substituted into the DE.

Thus, a solution to a DE is **another function** which, when substituted into the DE, makes the DE **true**.

maths

$y = y(t) \Rightarrow y = Ce^{kt}$

$y' = f(y,t)$

- Just as with any mathematical model, there are two steps to modelling with DEs: *writing* the equations, and then *solving* them.

  $\hookrightarrow$ physical situation (science) $\quad y$

- DEs are often very "natural" ways of representing phenomena. That is, it often makes "more sense" to write an equation for a rate of change of some value than to write an equation for the value.

***Question** 11.1.1*

Write a DE to model each of the following.

function

(a) The ~~straight line~~ distance $D(t)$ travelled by a car increases by 10 m/s.

DE $\boxed{D' = 10}$

Units – RHS: m/s
– LHS: $D' \to \frac{\Delta D}{\Delta t}$ $\frac{m}{s}$

(b) The human population P of Earth is increasing at about 1.1% per annum. Assume this growth rate continues.

DE $\boxed{P' = \frac{1.1}{100} P = 0.011 P}$

Units – RHS year^{-1}
– LHS $\frac{\Delta P}{\Delta t} \cdot \frac{1}{\text{year}} = \text{year}^{-1}$

(c) According to the Australian Bureau of Statistics, during 2013 Australia had: birth rate 1.34%; death rate 0.64%; 504000 people inward migration; 268000 people outward emigration. If these changes continue indefinitely, write a DE for the Australian population $P(t)$ in year t.

(inward) (outward)

$$P' = 0.0134\,P - 0.0064\,P + 504000 - 268000$$

DE $\boxed{P' = 0.007P + 236000}$

Units: per year

(d) The *von Bertalanffy growth model* states that the rate of increase of the length $L(t)$ of a shark of age t years is proportional to the difference between a fixed maximum length M and its current length $L(t)$. The constant of proportionality is an intrinsic positive growth rate r.

$L' \propto (M - L)$ or ~~$(L - M)$~~ $(M-L)$

Normally $L < M$, $L' > 0$

"$\propto$" proportional to

DE : $\boxed{L' = r(M - L)}$

Units : $\frac{\text{length}}{\text{time}}$

(e) *Newton's Law of Cooling* states that the rate of change of the temperature $T(t)$ of an object placed in an environment with fixed temperature F is proportional to the relative difference in the temperatures.

$T' \propto (F - T)$

coffee in a cold room
$T(\text{coffee}) > F(\text{room})$
$(F - T) < 0$, $T' < 0$

DE : $\boxed{T' = k(F - T)}$

$L'(L, t) = r(M - L(t))$

- Once a DE has been written for the *rate of change* of a phenomenon, that DE can (sometimes) be *solved* to give the *value* of the phenomenon.
- This (usually) requires an additional piece of measured information, such as the value of the phenomenon at some time, often $t = 0$.

Example *11.1.2*

Refer to Question 11.1.1. The following additional information applies in each case.

(a) Initially the car has travelled 0 metres. $D(0) = 0$

(b) The human population of Earth in July 2014 was about 7.24 billion.
$P(2014) = 7.24 \times 10^9$

(c) The human population of Australia at the start of 2013 was 22.9 million.

(d) In [22], it is shown that:

- the maximum length of a female *sand tiger* or *grey nurse* shark is $\boxed{M = 295.8 \text{ cm}}$;
- the intrinsic growth rate is $\boxed{r = 0.11 \text{ yr}^{-1}}$; and
- the length of a typical shark at birth is $\boxed{110 \text{ cm}}$.

$L(0) = 110\text{cm}$

(e) An item with temperature $\boxed{85\ °\text{C}}$ is placed in a room with constant temperature $\boxed{25\ °\text{C}}$.

$F = 25°C$ $T(0) = 85°C$

- We can solve some DEs *analytically*, using integration and algebra.
- We will next look at two very well-studied types of $\boxed{\text{DEs}}$ and their solutions.

11.2 The exponential DE

- Earlier we studied exponential growth and decay. On Page ~~113~~ 114 we state "Any phenomenon that has a rate of change proportional to the current amount follows an exponential function". $N' \propto N$
- This occurs precisely because such phenomena satisfy simple DEs whose solutions are exponential functions.

DE for exponential growth and decay

Any quantity $N(t)$ whose rate of change at any time is proportional to the value of N, with rate of change equal to a constant r per time period, follows the DE

DE :
$$\boxed{N' = rN.}$$

The solution to this DE is the exponential function

Solution :
$$\boxed{N(t) = N_0e^{rt},}$$

where N_0 is the value of N at $t = 0$.

LHS = ...
RHS = ...
LHS = RHS

***Example* 11.2.1**

It can be shown that $N(t) = N_0e^{rt}$ is the solution to $N' = rN$.

The derivative of $N(t)$ is: $N'(t) = \boxed{r \times N_0e^{rt}.}$

We can see that this is equivalent to $N'(t) = \boxed{rN(t).}$

- Many populations can be modelled effectively using exponential functions, for some periods of time.

Case Study 25: **Poo**

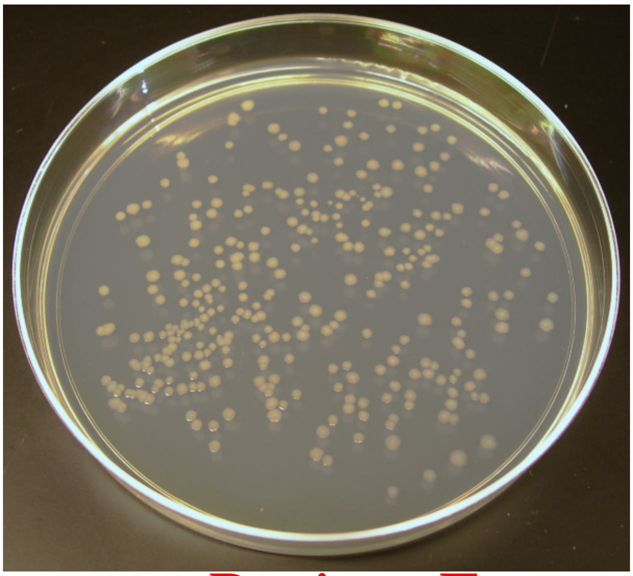


Photo 11.2: E. coli colonies on a plate (Source: commons.wikimedia.org.)

- *Escherichia coli* (usually shortened to *E. coli*) are bacteria commonly found in the lower intestine of warm-blooded animals, including humans.
- Most strains of *E. coli* are harmless in the digestive system, or even beneficial to the host. However, some strains produce toxins, and can cause food poisoning, gastrointestinal infections and urinary tract infections.
- One such strain is O104:H4, which caused outbreaks of illness in Europe in 2011. Around 50 people died and more than 4000 became ill. Contamination was traced to a farm that grew organic sprouted vegetables.
- Because *E. coli* can survive outside the body for some time, tests for *E. coli* are often used to identify faecal contamination in environmental samples or foods during hygiene checks.
- Under simplifying assumptions (such as relatively unlimited resources), the rate of increase of a population of *E. coli* at any time is proportional to the population size at that time.
- Hence the population follows an exponential function, and it makes sense to discuss the *doubling time* of the population.

- Under favourable conditions, the doubling time for a population of *E. coli* may be an hour, or even shorter. The rapid growth rate is one reason why good hygiene standards are important in food preparation.

- When studying populations of bacteria, microbiologists commonly count *colony-forming units* (CFU), which is the number of *live* bacterial cells. (Direct counts of individuals include both dead and living cells.)

***Question** 11.2.2*

A population of *E. coli* has a growth constant of 1% per minute. Let $N(t)$ be number of CFU's of E. coli at time t minutes, and assume at time $t = 0$ there are 1000 CFU's. Write and solve a DE to model this population.

Given : r = 1% per minute = $0.01\ \text{min}^{-1}$

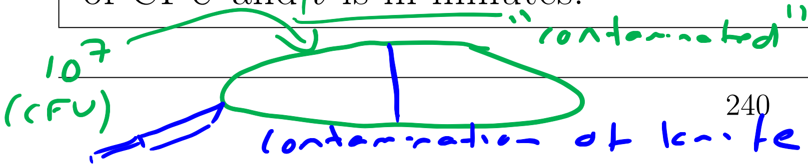
$N(0) = 1000$ CFUs $N_0 = 1000$

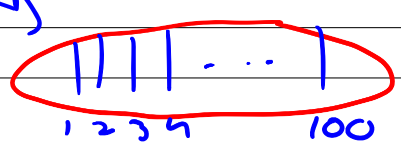
DE : $N' = rN \Rightarrow \boxed{N' = 0.01N}$

Solution : $N = N_0 e^{rt} \Rightarrow \boxed{N = 1000e^{0.01t}}$

***Question** 11.2.3*

A study [51] investigates *E. coli* contamination of pre-cooked meat. Researchers contaminated some ham with 10^7 CFU of *E. coli* and then sliced the ham. The same blade was then used to slice clean ham. The number of CFU were counted on each of 100 slices of the second ham, showing that Slice 1 contained around 580 CFU and Slice 100 contained 9 CFU. To answer the following questions, assume the number of *E. coli* on the ham can be modelled by $\boxed{N(t) = N_0e^{0.01t}}$ where N_0 is the initial number of CFU and t is in minutes.





Question 11.2.3 *(continued)*

(a) How many CFU will be on Slice 1 after 24 hours, assuming it is stored under ideal growing conditions for *E. coli*?

For slice: $N_0 = 580$ (initial contamination)

Model (soln DE) $N = N_0 e^{rt}$

$N = 580e^{0.01t}$

Solve for N when $t = 24$ hours

$t = 24 \text{ hours} \times \frac{60 \text{ min}}{1 \text{ hr}} = 1440 \text{ min}$

$N(t = 24 \text{ hours}) = 580e^{0.01(1440)}$

$\simeq 1.0 \times 10^9$

CFUs: 580 → 1 billion in 24 hours

(b) At what time does Slice 100 contain 580 CFU?

For slice 100, $N_0 = 9$

Solution: $N = 9e^{0.01t}$

Find t when $N = 580$

$580 = 9e^{0.01t}$

$\ln\left(\frac{580}{9}\right) = 0.01t$

$t = \frac{1}{0.01}\ln\left(\frac{580}{9}\right) \simeq 417 \text{ min}$ (~7 hours)

(c) Human faeces can contain 10^9 CFU per gram. Given the results of this study, what does this mean for hygiene practices in food preparation?

- wash your hands
- clean preparation area
- Don't cross contaminate
- wash fresh food

Question *11.2.3 (continued)*

(d) Consider the E. coli on Slice 1. The mass of 1 CFU is approximately 1×10^{-12} g and the mass of the Earth is approximately 6×10^{24} kg. According to the exponential model, how long it would take for the mass of the E.coli on Slice 1 to reach the mass of the Earth?

For slice 1, $N = 580 e^{0.01t}$

Find t when N is the number of CFUs that have the same mass of the Earth

1 CFU — mass 1×10^{-12} g

N CFU — mass 6×10^{24} kg $= 6 \times 10^{27}$ g

$$N = \frac{6 \times 10^{27}}{1 \times 10^{-12}} \Rightarrow N = 6 \times 10^{39}$$

Use exp. equation to find t

$$6 \times 10^{39} = 580 e^{0.01t}$$

$$t = \frac{1}{0.01} \ln\left(\frac{6 \times 10^{39}}{580}\right)$$

$$\simeq 8500 \text{ min}$$

↳ about 6 days.

End of Case Study 25: Poo.

Problem with model! unconstrained growth