Lecture 32: Approximating solutions to DEs

Learning objectives

✓ Use Euler's method to find approximate solutions to differential equations

"oiler"

Scientific examples

✓ Population growth

Maths skills

✓ Understand how and why Euler's method works

11.4 Euler's method

- We have seen some examples of DEs and their solutions, but for more complex DEs finding an exact solution is often not possible analytically.
- Instead, we can often find approximate solutions using numerical algorithms.
- One numerical algorithm for finding an approximate solution to a DE is called *Euler's method*.
- Euler's method is a useful approach, that you have possibly used before without knowing, based on a simple observation: if the value of some quantity is changing at a certain amount per time period, then it is possible to estimate the future value as follows:

P(0) know

P(1) (future value) = (current value) +
(estimated change per time period) × (number of time periods)

solve for

$\frac{\Delta P(0)}{\Delta t}$ know

t set

Question 11.4.1

(See Question 11.1.1 and Example 11.1.2.) The human population of Earth in July 2014 was 7.295 billion, and was expected to grow by about 80.245 million over the next year (note that this is a growth of 1.1%).

(a) Assuming the population increases by the same number each year, predict the population in July 2017.

$$P(2017) = P(2014) + \Delta P/\text{year} \times 3\,\text{years}$$
$$= 7.295\times 10^9 + 80.245\times 10^6 \times 3$$
$$= 7.536\times 10^9$$

(b) Instead of growing by a fixed number each year, the global population has a growth rate of about 1.1% per annum. Estimate the population in July 2015, July 2016 and July 2017.

$$P(2015) = P(2014) + \Delta P/\text{year} \times 1\,\text{year}$$

$$= 7.295\times 10^9 + \frac{1.1}{100} \times 7.295\times 10^9 \times 1$$
$$= 7.375\times 10^9$$

$$P(2016) = P(2015) + \Delta P/\text{year} \times 1\,\text{year}$$
$$= 7.375\times 10^9 + 0.011\times 7.375\times 10^9 \times 1$$
$$= 7.456\times 10^9$$

$$P(2017) = P(2016) + \Delta P/\text{year} \times 1\,\text{year}$$
$$= 7.456\times 10^9 + 0.011\times 7.456\times 10^9 \times 1$$
$$= 7.538\times 10^9$$

***Question* 11.4.1** *(continued)*

(c) Compare your answers to Parts (a) and (b), and explain the difference.

Part (b) is slightly larger
as account for growth each year.
Growth number is bigger each year
in (b) but constant in (a)

- Euler's method proceeds by approximating the unknown function as a series of short **straight lines**, starting from the initial point, each with:

  time step

  - **width** equal to a chosen step size h;

    known

  - **slope** equal to the slope calculated using the DE; and
  - **height** equal to the width multiplied by the slope. (calculate)

- The following is a more formal description of Euler's method.

Euler's method

Given an unknown quantity y, a DE of the form $y' = \ldots$, and a value of y (say y_0) at a given time (say t_0):

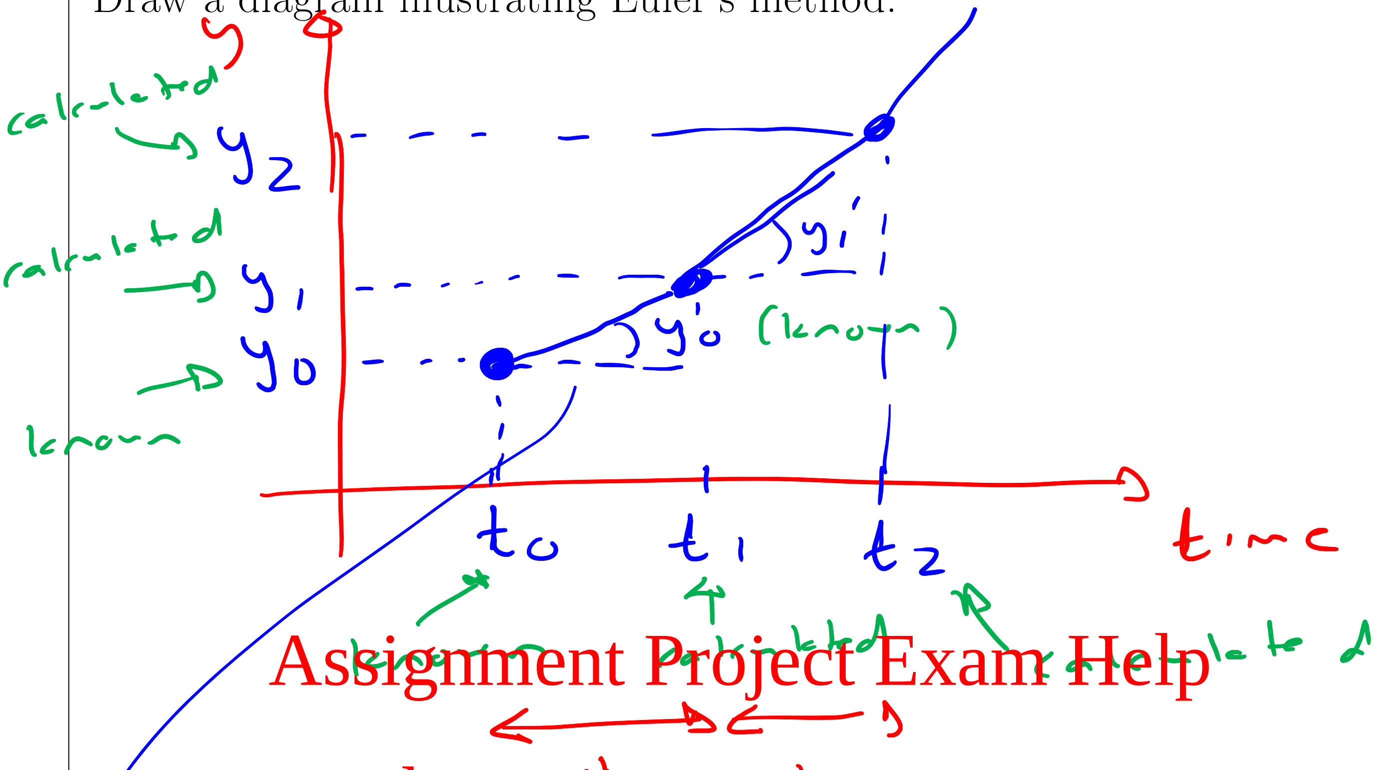
1. Choose a small *step size* h, and start at the initial point $\boxed{(t_0, y_0)}$.
2. Use the DE to find the slope y_i' at the current time, by substituting the current values of t_i and y_i into the DE.
3. Advance the current point to the end point of a short straight line, by setting $t_{i+1} = t_i + h$ and $y_{i+1} = y_i + h \times y_i'$.

   The new point $\boxed{(t_{i+1}, y_{i+1})}$ is the next approximate function value.
4. If t has reached the desired end-point then stop, else return to Step 2.

either done or repeat!

***Question** 11.4.2*

Draw a diagram illustrating Euler's method.



$$y_0' = \frac{\text{rise}}{\text{run}} = \frac{y_1 - y_0}{t_1 - t_0} = \frac{y_1 - y_0}{h}$$

$$\Rightarrow \quad y_1 - y_0 = h y_0'$$

Euler $\boxed{y_1 = y_0 + h y_0'}$

knowns

(repeat)

exponential

r - growth rate

r = 0.1

Example 11.4.3

Consider a population of algae growing at a rate of 10% per day, with an initial population of algal cells of 100 per mL. Thus, this population can be modelled by the DE

At t_0 assume $t_0 = 0$

DE
$$\boxed{y' = 0.1y} \quad \text{where} \quad \boxed{y_0 = 100.}$$

Use Euler's method to estimate the population after 5 days, using a step size of $h = 1$ day.

Answer: (0, 100) $y' = 0.1y$

We use (t_0, y_0) and the DE to approximate the next point (t_1, y_1). We compute:
$t_1 = t_0 + h = 0 + 1 = 1$ days
$y_1 = y_0 + hy_0' = y_0 + h(0.1y_0) = 100 + \overset{h}{1} \times \overset{DE}{\boxed{0.1 \times 100}} = 110 \text{ mL}^{-1}$ y_1
Hence the new point is $(t_1, y_1) = (1, 110)$.
Our approximation for the concentration of algal cells after 1 day is $y(1) \approx 110 \text{ mL}^{-1}$.

repeat

We use (t_1, y_1) and the DE to approximate the next point (t_2, y_2). We compute:
$t_2 = t_1 + h = 1 + 1 = 2$ days
$y_2 = y_1 + hy_1' = y_1 + h(0.1y_1) = 110 + \overset{h}{1} \times \overset{DE}{\boxed{0.1 \times 110}} = 121 \text{ mL}^{-1}$.
Hence the new point is $(t_2, y_2) = (2, 121)$.
Our approximation for the concentration of algal cells after 2 days is $y(2) \approx 121 \text{ mL}^{-1}$.

We use (t_2, y_2) and the DE to approximate the next point (t_3, y_3). We compute:
$t_3 = t_2 + h = 2 + 1 = 3$ days
$y_3 = y_2 + hy_2' = y_2 + h(0.1y_2) = 121 + 1 \times 0.1 \times 121 = 133.1 \text{ mL}^{-1}$.
Hence the new point is $(t_3, y_3) = (3, 133.1)$.
Our approximation for the concentration of algal cells after 3 days is $y(3) \approx 133.1 \text{ mL}^{-1}$.

We use (t_3, y_3) and the DE to approximate the next point (t_4, y_4). We compute:
$t_4 = t_3 + h = 3 + 1 = 4$ days
$y_4 = y_3 + hy_3' = y_3 + h(0.1y_3) = 133.1 + 1 \times 0.1 \times 133.1 = 146.41 \approx 146.4 \text{ mL}^{-1}$.
Hence the new point is $(t_4, y_4) = (4, 146.4)$.
Our approximation for the concentration of algal cells after 4 days is $y(4) \approx 146.4 \text{ mL}^{-1}$.

We use (t_4, y_4) and the DE to approximate the next point (t_5, y_5). We compute:
$t_5 = t_4 + h = 4 + 1 = 5$ days
$y_5 = y_4 + hy_4' = y_4 + h(0.1y_4) = 146.4 + 1 \times 0.1 \times 146.4 = 161.04 \approx 161.0 \text{ mL}^{-1}$.
Hence the new point is $(t_5, y_5) = (5, 161.0)$. ← after 5 days
Our approximation for the concentration of algal cells after 5 days is $\boxed{y(5) \approx 161.0 \text{ mL}^{-1}}$.

After 5 days, the concentration of algal cells is approximately 161 mL^{-1}.

Example 11.4.3 *(continued)*

The following table summarises the above calculations:

i	t_i (days)	y_i (mL^{-1})	$t_{i+1} = t_i + h$ (days)	$y_{i+1} = y_i + h \times y_i' = y_i + h \times 0.1 \times y_i$ (mL^{-1})
0	0	100.0	1	110.0
1	1	110.0	2	121.0
2	2	121.0	3	133.1
3	3	133.1	4	146.4
4	4	146.4	5	161.0

approximate the function

Table 11.1: Five steps of Euler's method.

Figure 11.8 plots the five points calculated above, marked as circles ~~asterisks~~, with straight lines approximating the function between these points. This graph shows an approximate solution to the differential equation over these five days.

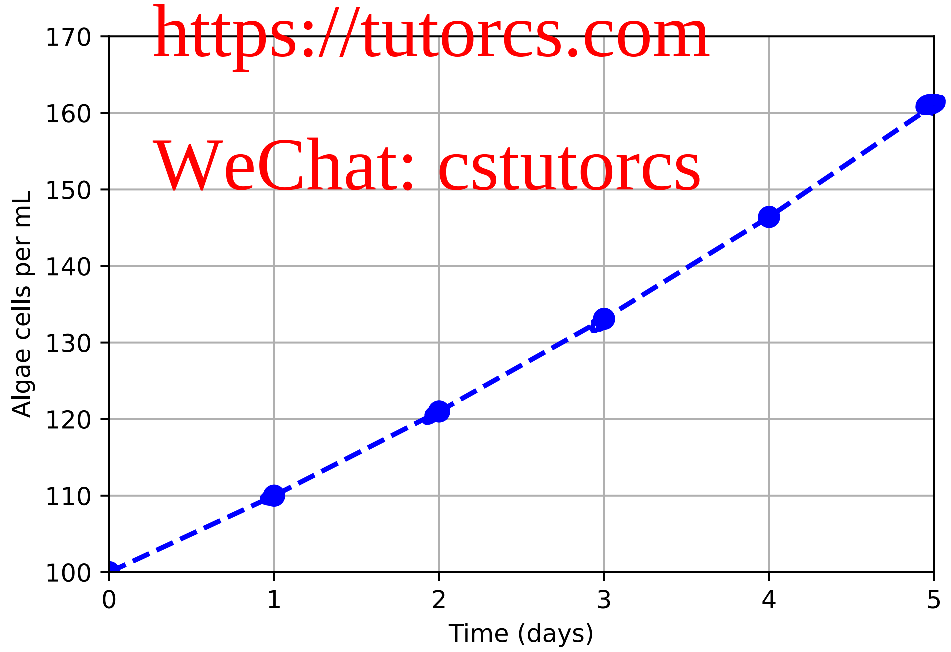


Figure 11.8: Approximate solution to the DE $y' = 0.1y$ with $y(0) = 100$ per mL.

If we used $h = 5$ days

$$
\begin{aligned}
P(5) &= P(0) + h\Delta P \\
&= 100 + 5\,(0.1 \times 100) \\
&= 150 \ \text{mL}^{-1}
\end{aligned}
$$

Question 11.4.4

How will our answer differ if we repeat the problem in the previous example, but using $h = 2.5$ days instead of $h = 1$ day?

if we used $h = 2.5$ days :
- answer likely to be less accurate

Repeat the problem in the previous example, but using $h = 2.5$ days.

we had $y_0 = 100 \text{ mL}^{-1}$ $h = 2.5$ days

DE $y' = 0.1y$

step 1

$$y_1(2.5) = y_0(0) + h y_0'(0)$$

$$= 100 + 2.5(0.1 \times 100)$$

$$= 125 \text{ mL}^{-1}$$

$$t_1 = t_0 + 2.5 = 0 + 2.5$$

$$= 2.5 \text{ days}$$

Step 2

$$t_2 = t_1 + 2.5 = 2.5 + 2.5 = 5 \text{ days}$$

$$y_2 = y_1 + h y_1'$$

$$= 125 + 2.5(0.1 \times 125)$$

$$\simeq 156 \text{ mL}^{-1}$$

After 5 days, the population is 156 mL^{-1}

There are some important things to know about Euler's method.

- It gives an **approximate** solution, not an exact one. There will be numerical inaccuracies in the answer, particularly over a large range of t values.
- The choice of step size is very important: smaller values will give a more accurate answer, but take longer to calculate.
- Despite these limitations, the method can give very good approximate solutions to quite difficult problems.

Example 11.4.5

In Example 11.4.3 we used a step size of $h = 1$ day to solve $y' = 0.1y$.

Figure 11.9 shows approximate solutions with a step size of $h = 2.5$ days (bottom curve), $h = 1$ day (middle curve) and the exact, true solution (top curve).

As h becomes smaller, the solution becomes more accurate (that is, it moves closer to the true solution).

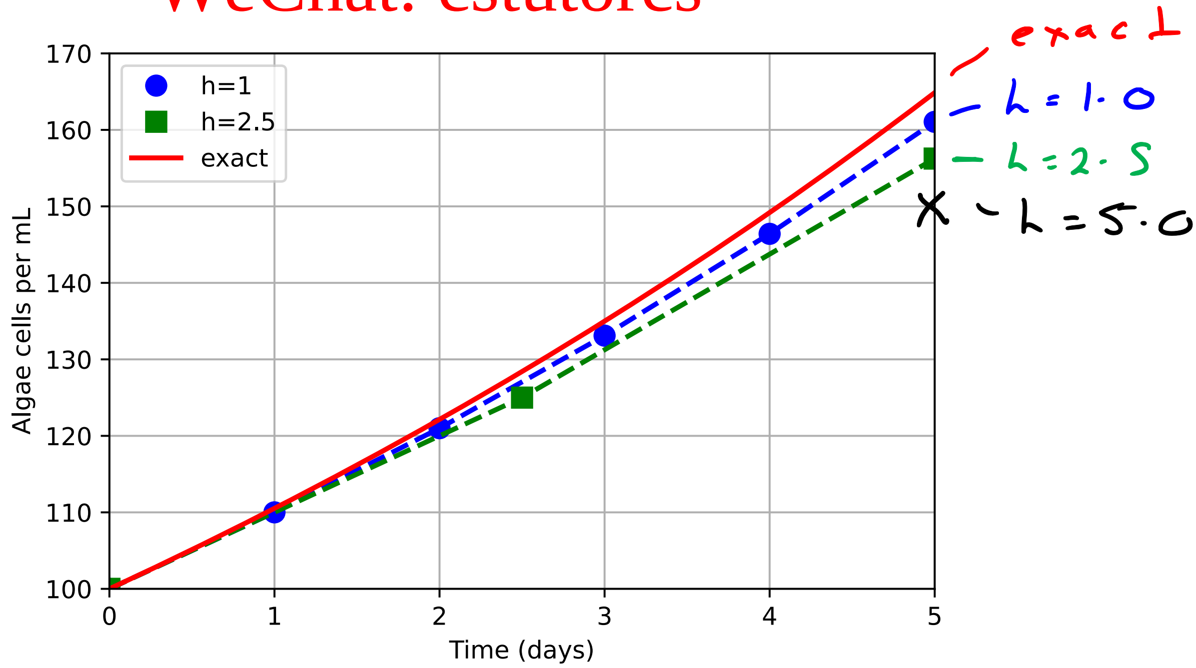


Figure 11.9: Approximate and exact solutions to the DE $y' = 0.1y$ with $y(0) = 100$ cells per mL.

Chapter 12: Systems of DEs

Image 12.1: *The wild hunt: Åsgårdsreien* (1872), Peter Nicolai Arbo (1831 – 1892), Nasjonalgalleriet, Oslo. (Source: en.wikipedia.org)

Here we introduce some simple systems of DEs that allow us to model organisms with multiple life stages, and situations in which multiple populations interact, such as predator/prey relationships.

We will use systems of DEs to model the spread of infectious diseases, and investigate the potential impact of pandemics. We approximate solutions to systems of DEs using Euler's method.

12.1 Introduction to systems of DEs

- The DE models we have studied so far have all modelled a single, distinct phenomenon.
- Often, multiple factors interact, requiring more sophisticated models.
- For example:
  - in predator-prey relationships, changes in population sizes of *two* species are interrelated;
  - in species with multiple distinct life stages, changes in the population sizes within each stage depend on the numbers in other stages; and

– the rates at which epidemics spread through populations are influenced by the number of infected individuals **and also** by the number of susceptible individuals.

- Typically, models for these more complex situations use a *system* of DEs (that is, more than one DE).
- Just as with single DEs, analytical solutions exist for some systems of DEs, but other systems require approximate solutions.
- Euler's method can be used to solve a system of DEs approximately, by applying a single iteration to each equation in turn, and then repeating.

Photo 12.1: Predators: Siberian tigers, *Panthera tigris altaica*. (Source: PA.)

***Question** 12.1.1*

If we were to study all humans on Earth, what would be some of the important life-stages to consider, and what would be some important transitions to keep track of?

$$\begin{tikzcd} \text{infant} \arrow[r] \arrow[d] & \text{youth} \arrow[r] \arrow[d] & \text{adult} \arrow[r] \arrow[d] & \text{elderly} \arrow[d] \\ \text{die} & \text{die} & \text{die} & \text{die} \end{tikzcd}$$

$I' =$ $Y' =$ $A' =$ $E' =$

$\underbrace{\qquad}_{4}$ function infant, youths, adults