Stat 310, 2020, Homework 2. Due at the beginning of the class, Wednesday, January 22.

Problem 1: (40 points)

An essential part of optimization (as it will turn out later) is to efficiently determine the *inertia* of a symmetric matrix that is, the number of positive, negative, and zero eigenvalues. An *inertia matrix* is a matrix that has only 1,0 and -1 on the diagonal.

a) A matrix A is said to be *congruent* with a matrix B if there exists an invertible S such that A= SBS^T
b) Show that if A is congruent with B then B is congruent with A and if A is congruent with B and B is congruent with C then A is congruent with C.
c) Show that any symmetric matrix A is congruent with an inertia matrix N,
d) Let N_1 and N_2 be 2 inertia matrices that are congruent. Prove that their number of zeros are equal (hint: find a 1 to 1 mapping between their null spaces, based on their congruence relationship).
e) (the hard part). Show that if N_1 and N_2 are 2 congruent inertia matrices then their number of positive entries is also equal to each other. (Hint: Suppose $N_2 = S^T N_1 S$ is the congruence relationship. Without loss of generality we can assume that the matrix S is such that the first entries in N_1 and N_2 are positive, the following group are the negative ones, and the last group is made of the zero entries. Otherwise the matrix S can be multiplied by the proper permutations that ensures that to be the case. Suppose now that s is the number of positive entries in N_1 and that t is the number of positive entries in N_2 and that $s < t$. Show then that there exists a vector $0 \neq x \in \mathbb{R}^n$ such that $x_{t+1} = x_{t+2} = \ldots x_n = 0$ and that $(Sx)_1 = (Sx)_2 = \ldots = (Sx)_s = 0$. For this vector, show that $x^T N_2 x > 0, \quad (Sx)^T N_1 (Sx) \leq 0$ which is a contradiction). Then prove that $N_1 = N_2$
f) Propose a way that is guaranteed to terminate in a finite number of operations, and that computes the inertia (the number of zero eigenvalues, positive eigenvalues and negative eigenvalues) of a symmetric matrix A. What seems to be the asymptotic effort of doing so for the matrix W(n) in Problem 2, if we do sparse calculations?

Problem 2: (computation; Newton's method, 30 points)

I. Implement Newton's method ((3.38) in Nocedal and Wright, or from lecture notes). Allow the number of iterations to be a parameter.

II. Apply the method to the following function (Fenton's function);

$$f(x)=\left\{12+x_1^2+\frac{1+x_2^2}{x_1^2}+\frac{x_1^2x_2^2+100}{(x_1x_2)^4}\right\}\left(\frac{1}{10}\right)$$

If you code the derivatives by hand, then implement the function (in Matlab) [f,g,H]=fentonfgH(x), where x is the point of function, gradient, and Hessian evaluation.

III. Initialize the method at $x=[3,2]$. Describe what you observe.

IV. Initialize the method at x=[3,4]. Describe what you observe.

V. For the cases where the method converges, estimate the order of convergence numerically.

VI. Submit your code to the Matlab submission folder as follows. Implement a function whose name is hwk2p2.m in Matlab, that should take in the initial point and the number of iterations, and return the final point. In Matlab, the calling sequence should be: [x]=hwk2p2(x0,n).