cs2001

Introduction Theory of Computation

ecture 2

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Basic mathematical notation and terminology

Reading: ITC Section 0.2

Set-theory is the foundation of mathematics.

We will start with the set of natural numbers as a basic given set to start with:

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

Further, we will assume the existence of the empty set: $\emptyset$

Sets

Set-theory is [illegible] of mathematics.

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Further, we will assume the existence of the empty set: $\emptyset$

$\emptyset$ does not contain any elements

$\emptyset$ is a subset of [illegible] other set

Elements of sets

A set consists of elements, denoted by the $\in$-relation:

$$a \in A$$

means a is an element of set A.

To state that a is not an element of set A, we use the notation

$$a \notin A.$$

Important properties:

- Sets are not ordered and contain each element only once!
- The empty set has no elements.

Elements of sets

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Important properties:

- Sets are not ordered and contain each element only once!

$\{2,4,6,4\} = \{2,4,6\} = \{4,6,2\} = \{6,4,2,2\}$

Subsets

Using the element re can define what subsets are:

- $A \subseteq B$
- $A \subsetneq B$
- $A = B$

Subsets

Using the ele… we can define what subsets are:

- $A \subseteq B$ "… set of B"

  if $c \in A$ then $c \in B$.

  $\{1, 2, 4\} \subseteq \mathbb{N}$ $\{1,2,4\} \subseteq \{1,2,4\}$

- $A \subsetneq B$ "A is a proper subset of B"

  $A \subseteq B$ and … $b \notin A$.

  $\{1,2,4\} \subsetneq \mathbb{N}$, $\{n \in \mathbb{N} \mid n \text{ even}\} \subsetneq \mathbb{N}$

- $A = B$ "A and B are equal as sets"

  $A \subseteq B$ and $B \subseteq A$.

  $\{1,2,4,2\} = \{4,2,1\}$

Set operations

$\notin$ ← symbols used to indicate that A is a proper subset of B

$A \subset B$

$A \subseteq B$

By default, w ssume that all our sets are subsets of the natural n we assume a universe $U = \mathbb{N}$. Then we can define

How to define sets

Here are some tools [illegible]g sets:

1. Make a list of th[illegible]ts
   - Set of all students in this class
   - Set of odd natural numbers smaller than 10: $\{1, 3, 5, 7, 9\}$

   Problem: this technique fails for large or infinite sets

2. Identify by a common characteristic
   - Odd natural numbers $\{n \in \mathbb{N} \mid n \text{ is not divisible by } 2\}$

   Problem: sometimes we don't know a precise defining characteristic

3. Inductive definition
   - We'll see how to do this later

Warning:
When we define sets, [illegible]s need to specify from which universe (that is a possibly much larger ground set, for example the natural numbers) the elements of our set should be taken from!

Example:

- Odd natural numbers $\{n \in \mathbb{N} \mid n \text{ is not divisible by } 2\}$
- Interval on the real line $\{x \in \mathbb{R} \mid 2 \leq x \leq 4\}$

Otherwise we can fall into Russell's paradox...!

Russel's paradox

Consider the following definition of a set:

$$\{r | r \notin r\}$$

That is, the set R contains all those sets that do not contain themselves as an element.

Question: Is R an element of R?

Now, $R \in R$ implies that $R \notin R$, and vice versa ($R \notin R$ implies that $R \in R$)– a contradiction.

Russel's paradox

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By default, we will fo[illegible]ume that all our sets are subsets of the natural numbers. [illegible]we assume a universe $U = \mathbb{N}$. Then we can define:

- The set-difference $A \setminus B$ of two sets A and B

- The intersection $A \cap B$ of two sets A and B

- The union $A \cup B$ of two sets A and B

Set operations

By default, w[…]ssume that all our sets are subsets of the natural n[…] we assume a universe $U = \mathbb{N}$. Then we can define[…]

- The set-difference $A \setminus B$ of two sets A and B

$A \setminus B = \{a \in A \mid a \notin B\}$

$A = \{1, 3, 5, 7\}$, $B = \{1, 2, 3\}$

$A \setminus B = \{5, 7\}$

always: $A \setminus B \subseteq A$

Set operations

By default, w[...]ssume that all our sets are subsets of the natural n[...] we assume a universe $U = \mathbb{N}$. Then we can define[...]

- The set-difference $A \setminus B$ of two sets A and B

In general $A \setminus B \neq B \setminus A$

$A = \{1,3,5,7\}$, $B = \{1,2,3\}$

$A \setminus B = \{5,7\}$, $B \setminus A = \{2\}$

Q: Is it possible that $A \setminus B = B \setminus A$ for some sets A and B?

A: Yes: if $A = B = \emptyset$, then $A \setminus B = B \setminus A = \emptyset$

Or, if $A = B = \{1,2\}$, then $A \setminus B = B \setminus A = \emptyset$.

Talking: Ruth Urner

Set operations

By default, w[...] assume that all our sets are subsets of the natural n[...] we assume a universe $U = \mathbb{N}$. Then we can define

- ~~The set-difference $A \setminus B$ of two sets A and B~~

$A = \{1,3,5,17\}$ $B = \{1,2,3\}$

$A \cap B = \{1,3\}$

- The intersection $A \cap B$ of two sets A and B

$A \cap B$ is the set of all elements that are members of both A and B.

$A \cap B = \{n \in \mathbb{N} \mid n \in A$ and $n \in B\}$

or universe

Set operations

By default, w[...] assume that all our sets are subsets of the natural n[...] we assume a universe $U = \mathbb{N}$. Then we can define

- The set-difference $A \setminus B$ of two sets A and B

- The intersection $A \cap B$ of two sets A and B

$A = \{1,3,5,7\}$, $B = \{1,2,3\}$

$A \cup B = \{1,2,3,5,7\}$

- The union $A \cup B$ of two sets A and B

$A \cup B$ is the set of all elements in A and all elements in B. $A \cup B = \{n \in \mathbb{N} \mid n \in A \text{ or } n \in B\}$

By default, we will fo[…]ume that all our sets are subsets of the natural numbers. […] can define:

- The complemen[…]et A

- The (cartesian) product $A \times B$ of two sets A and B

- The set A^k of k-tuples A^k of elements of A

More set operations

By default, w… …ssume that all our sets are subsets of the natural n… …e can define:

- The com… …set A

is the set of all elements in the universe that are not in A.

$A = \{n \in \mathbb{N} \mid n \text{ divisible by } 2\} = \{\text{even numbers}\}$

$\overline{A} = \{\text{odd numbers}\} = \{n \in \mathbb{N} \mid n \text{ not divisible by } 2\}$

More set operations

By default, w[…] ssume that all our sets are subsets of the natural nu[…] ve can define:

- The com[…] set A

- The (cartesian) product of $A \times B$ two sets A and B

is the set of all pairs of elements from A and B.

$A = \{1, 3, 7\}, B = \{1, 2, 3\}$

$A \times B = \{(1,1), (1,2), (1,3), (3,1), (3,2), (3,3), (7,1), (7,2), (7,3)\}$

Talking: Ruth Urner

More set operations

ℝ reals

$\mathbb{R}^2$ "plane"

(2.5, 1.3)

$\mathbb{R}^3$ "space"

(.5, 1.5, 2)

By default, w[...] ssume that all our sets are subsets of the natural n[...] e can define:

- The com[...] set A

- The (cartesian) product of $A \times B$ two sets A and B

- The set A^k of k-tuples A^k of elements of A

$A = \{1, 2, 3, 4\}$

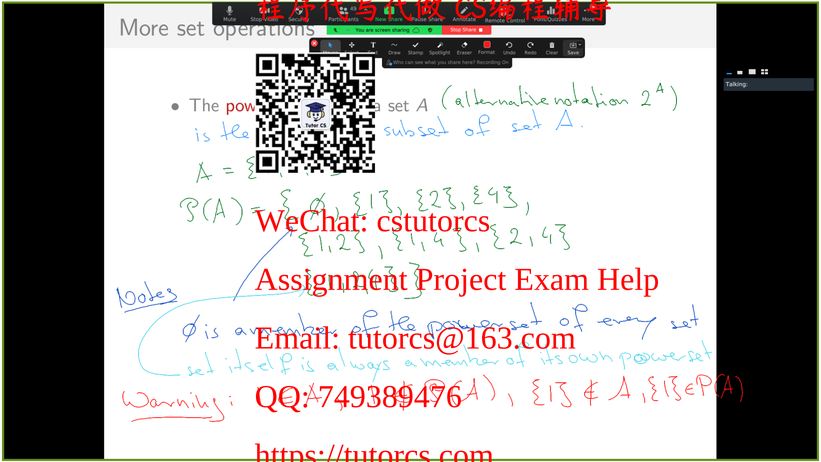
$A^3 = \{(1,1,1), (1,1,2), (1,2,1), (1,2,2)$

$(1,4,4) \ldots \quad (4,4,4)\}$

More set operations

- The power set $\mathcal{P}(A)$ of a set A

- The symmetric difference $A \Delta B$ between two sets A and B

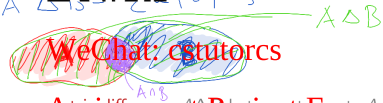

Mute
Stop Video
Participants
New Share
Pause Share
Annotate
Remote Control
Polls/Quizzes
More
You are screen sharing
Stop Share
Draw
Stamp
Spotlight
Eraser
Format
Undo
Redo
Clear
Save
Who can see what you share here? Recording On
Talking:
More set operations
The pow
set A (alternative notation 2^A)
is the
subset of set A.
A = {
P(A) = { ∅, {1}, {2}, {4},
{1,2}, {1,4}, {2,4}
Notes
∅ is a member of the power set of every set
set itself is always a member of its own power set
Warning:
P(A), {1} ∉ A, {1} ∈ P(A)

More set operations

- The pow... a set A

$A = \{\ldots 7\}$, $B = \{1, 2, 3\}$

$A \Delta B = \{\ldots, 5, 7\}$



- The symmetric difference $A \Delta B$ between two sets A and B is the set of all elements in $A \cup B$ that are not in $A \cap B$

$$A \Delta B = (A \cup B) \setminus (A \cap B)$$

Sets, multi-sets, sequences

- Recall: Sets contain each element only once and the order does not matter.

- Multi-sets can contain elements multiple times, but do not contain an order of their elements.

- Sequences or tuples are collections of elements in a fixed order.

Sets, multi-sets, sequences

- Recall: S… h element only once and the order does not…

- Multi-sets can contain elements multiple times, but do not contain an order of their elements.

{1, 1, 3, 5} is not equal to {1, 3, 5} as a mutiset, but equal as sets

{1, 1, 3, 5} equal {5, 1, 3, 1} as a multiset.

by default the '{', '}' brackets indicate sets, not multisets

Sets, multi-sets, sequences

- Recall: S… h element only once and the order does not …
- Multi-sets can contain elements multiple times, but do not contain an order of their elements.
- Sequences or tuples are collections of elements in a fixed order.

(3,1,5) is a 3-tuple

(1,1,1,2,2,3,4,…) is an 8-tuple

$(2i)_{i\in\mathbb{N}}$ is the sequence of even numbers

$= (0,2,4,6,8,10,\ldots)$

Definition:
A relation R between [illegible]d set B is a subset of $A \times B$.

We sometimes use notation aRb to state that $(a, b) \in R$.

Relations

Definition:
A relation R ... nd set B is a subset of $A \times B$.

"$\leq$" is ... over the natural numbers:

$\{(0,1),(0,0),(1,1),(0,2),(1,2),\ldots\}$

$A = \{1,3,5,7\}$, $B = \{1,2,3\}$, then

$R = \{(1,1),(3,3)\}$ is a relation between A and B.

$R' = \{(3,2),(3,1),(5,3),(7,3)\}$

Relations

Definition:

A relation R [illegible] nd set B is a subset of $A \times B$.

$2 \leq 4$

$5 \leq$ [illegible]

We sometimes use notation aRb to state that $(a, b) \in R$.

- If R is a relation between A and itself, that is $R \subseteq A \times A$, we also call R a binary relation, or a relation of arity 2 on A.

- A subset $R \subseteq A^k$ of k-tuples of elements of A, is also called a k-ary relation, or a relation of arity k on A.

Equivalence relations

Definition:
A binary relation R on some set A is called an equivalence relation if it has the following properties:

- reflexive
- symmetric
- transitive

Equivalence relations

$\leq$ on $\mathbb{N}$

is reflexive and transitive but not symmetric

$=$ is an equivalence relation

Definition:
A binary relation R on some set A is called an equivalence relation if it has the following properties:

- reflexive — for every $a \in A$, $(a,a) \in R$
- symmetric — if $(a,b) \in R$ for some $a,b \in A$, then $(b,a) \in R$
- transitive — if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$

$\equiv_{\text{mod } 5}$ (remainder modulo 5) is an equivalence relation over natural numbers

$6 \equiv_{\text{mod } 5} 11$, $6 \equiv_{\text{mod } 5} 21$, $11 \equiv_{\text{mod } 5} 21$

Definition:
A function f from se[illegible] B is a relation between A and B that satisfies the foll[illegible]perties:

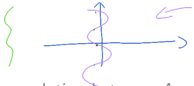
- For every $a \in A$, there exists a $b \in B$ such at $(a, b) \in f$
- If $(a, b) \in f$ and $(a, b') \in f$, then $b = b'$.

Instead of $(a, b) \in f$, we usually use notation $f(a) = b$.
Alternatively, we also use the notation $f : a \mapsto b$.

Functions

$f(x)=|x|$

$g(x)=\sin(x)$



not a function

i	$f(i)$
1	1
3	1
5	3
7	2

for functions, we can use such tables to define them

Definition:

A function f … et B is a relation between A and B that satisfies … operties:

- For every $a \in A$, there exists a $b \in B$ such at $(a, b) \in f$
- If $(a, b) \in f$ and $(a, b') \in f$, then $b = b'$.

$A = \{1,3,5,7\}$, $B = \{1,2,3\}$

$f: = R = \{$ …

$R = \{(1,1),(5,1),(1,2),(7,2)\}$ is not a function; it violates both properties (input 1 gets mapped to two outputs and input 3 doesn't get mapped to any output)

Instead of $(a, b) \in f$, we usually use notation $f(a) = b$.
Alternatively, we also use the notation $f : a \mapsto b$.

Functions – important properties

We use the notation

$$f : A \to B$$

to state that f is a function from A to B.

We call A the domain of the function f and B the range of the function f.

Functions can be:

- one-to-one:
  We call a function $f : A \to B$ one-to-one if $f(a) = b$ and $f(a') = b$ for some $b \in B$ implies that $a = a'$.
- onto:
  We call a function $f : A \to B$ onto if for all $b \in B$ there exists an $a \in A$ with $f(a) = b$.