cs2001
Introductio[illegible] Theory of Computation

[illegible]ecture 3

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Basic mathematical notation and terminology

Reading: ITC Section 0.2

By default, we will fo ume that all our sets are subsets of the natural numbers. ve assume a universe $U = \mathbb{N}$. Then we can define:

- The set-difference $A \setminus B$ of two sets A and B

- The intersection $A \cap B$ of two sets A and B

- The union $A \cup B$ of two sets A and B

Set operations

By default, w… ssume that all our sets are subsets of the natural n… we assume a universe $U = \mathbb{N}$. Then we can define…

- The set-difference $\boxed{A \setminus B}$ of two sets A and B

$A \setminus B = \{a \in A \mid a \notin B\}$

$A = \{1, 3, 5, 7\}$, $B = \{1, 2, 3\}$

$A \setminus B = \{5, 7\}$

always: $A \setminus B \subseteq A$

Set operations

By default, w… …ssume that all our sets are subsets of the natural n… … we assume a universe $U = \mathbb{N}$. Then we can define…

- The set-difference $A \setminus B$ of two sets A and B

In general $A - B \neq B - A$

$A = \{1,3,5,7\}$, $B = \{1,2,3\}$

$A - B = \{5,7\}$, $B - A = \{2\}$

Q: Is it possible that $A - B = B - A$ for some sets A and B?

A: Yes: if $A = B = \emptyset$, then $A \setminus B = B \setminus A = \emptyset$

Or if $A = B = \{1,2\}$, then $A - B = B - A = \emptyset$.

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Set operations

By default, w… …ssume that all our sets are subsets of the natural n… …we assume a universe $U = \mathbb{N}$. Then we can define

- ~~The set-difference $A \setminus B$ of two sets A and B~~

$A = \{1, 3, 5, 17\}$, $B = \{1, 2, 3\}$

$A \cap B = \{1, 3\}$

- The intersection $A \cap B$ of two sets A and B

$A \cap B$ is the set of all elements that are members of both A and B.

$A \cap B = \{n \in \mathbb{N} \mid n \in A \text{ and } n \in B\}$

or universe

Set operations

By default, w… …ssume that all our sets are subsets of the natural n… …we assume a universe $U = \mathbb{N}$. Then we can define…

- The set-difference $A \setminus B$ of two sets A and B

- The intersection $A \cap B$ of two sets A and B

$A = \{1,3,5,7\}$, $B = \{1,2,3\}$

$A \cup B = \{1,2,3,5,7\}$

- The union $A \cup B$ of two sets A and B

$A \cup B$ is the set of all elements in A and all elements in B. $A \cup B = \{n \in \mathbb{N} \mid n \in A \text{ or } n \in B\}$

By default, we will fo[illegible]ume that all our sets are subsets of the natural numbers. [illegible] can define:

- The complemen[illegible]et A

- The (cartesian) product of $A \times B$ of two sets A and B

- The set A^k of k-tuples A^k of elements of A

More set operations

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By default, w[illegible] ssume that all our sets are subsets of the natural n[illegible] we can define:

- The com[illegible] set A

is the set of all elements in the universe that are not in A.

$A = \{n \in \mathbb{N} \mid n \text{ divisible by } 2\} = \{\text{even numbers}\}$

$\overline{A} = \{\text{odd numbers}\} = \{n \in \mathbb{N} \mid n \text{ not divisible by } 2\}$

More set operations

By default, w… ssume that all our sets are subsets of the natural nu… e can define:

- The com… set A

- The (cartesian) product of $A \times B$ two sets A and B

is the set of all pairs of elements from A and B.

$A = \{1, 3, 7\}$, $B = \{1, 2, 3\}$

$A \times B = \{(1,1), (1,2), (1,3), (3,1), (3,2), (3,3), (7,1), (7,2), (7,3)\}$

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More set operations

$\mathbb{R}$ reals

$\mathbb{R}^2$ "plane"

$(2.5, 1.3)$

$\mathbb{R}^3$ "space"

$(.5, 1.5, 2)$

By default, w… …ssume that all our sets are subsets of the natural n… …e can define:

- The com… …set A

- The (cartesian) product of $A \times B$ two sets A and B

- The set A^k of k-tuples A^k of elements of A

$A = \{1, 2, 3, 4\}$

$A^3 = \{(1,1,1), (1,1,2), (1,2,1), (1,2,2)$

$(1,1,4) \ldots (4,4,4)\}$

More set operations

- The **power set** $\mathcal{P}(A)$ of a set A

- The **symmetric difference** $A \triangle B$ between two sets A and B

More set operations

- The pow… a set A (alternative notation 2^A) is the … subset of set A.

$A = \{ \ldots \}$

$\mathcal{P}(A) = \{ \emptyset, \{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,4\}, \{1,2,4\} \}$

Notes: $\emptyset$ is a member of the power set of every set; set itself is always a member of its own power set.

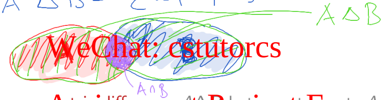
Warning: … $\mathcal{P}(A)$, $\{1\} \notin A$, $\{1\} \in \mathcal{P}(A)$

More set operations

- The pow[illegible] set A

$A = \{ [illegible] 7\}$, $B = \{1, 2, 3\}$

$A \Delta B = \{ [illegible], 5, 7\}$



- The symmetric difference $A \Delta B$ between two sets A and B

is the set of all elements in $A \cup B$ that are not in $A \cap B$

$A \Delta B = (A \cup B) \setminus (A \cap B)$

- Recall: Sets contain each element only once and the order does not matter.

- Multi-sets can contain elements multiple times, but do not contain an order of their elements.

- Sequences or tuples are collections of elements in a fixed order.

Sets, multi-sets, sequences

- Recall: S[...]h element only once and the order does not[...]

- Multi-sets can contain elements multiple times, but do not contain an order of their elements.

by default the '{', '}' brackets indicate sets, not multisets

{1, 1, 3, 5} is not equal to {1, 3, 5} as a multiset, but equal as sets

{1, 1, 3, 5} is equal to {5, 1, 3, 1} as a multiset.

Sets, multi-sets, sequences

- Recall: S[...]h element only once and the order does not [...]
- Multi-sets can contain elements multiple times, but do not contain an order of their elements.
- Sequences or tuples are collections of elements in a fixed order.

(3,1,5) is a 3-tuple

(1,1,1,2,2,3,4) ... an 8-tuple

$(2i)_{i\in\mathbb{N}}$ is the sequence of even numbers $= (0,2,4,6,8,10,\dots)$

Definition:

A relation R between d set B is a subset of $A \times B$.

We sometimes use notation aRb to state that $(a, b) \in R$.

Relations

Definition:
A relation R [redacted]nd set B is a subset of $A \times B$.

"$\leq$" is [redacted] over the natural numbers:

$\{(0,1), (0,0), (1,1), (0,2), (1,2), \ldots\}$

$A = \{1, 3, 5, 7\}$, $B = \{1, 2, 3\}$, then

$R = \{(1,1), (3,3)\}$ is a relation between A and B.

$R' = \{(3,2), (3,1), (5,3), (7,3)\}$

Relations

Definition:
A relation R … nd set B is a subset of $A \times B$.

$2 \leq 4$

$5 \leq$

We sometimes use notation aRb to state that $(a, b) \in R$.

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- If R is a relation between A and itself, that is $R \subseteq A \times A$, we also call R a binary relation, or a relation of arity 2 on A.

- A subset $R \subseteq A^k$ of k-tuples of elements of A, is also called a k-ary relation, or a relation of arity k on A.

Definition:
A binary relation R on some set A is called an equivalence relation if it has the following properties:

- reflexive
- symmetric
- transitive

Equivalence relations

$\leq$ as a ... on $\mathbb{N}$ is ... transitive but not symmetric

$=$ is an equivalence relation

Definition:
A binary relation R on some set A is called an equivalence relation if it has the following properties:

- reflexive — for every $a \in A$, $(a,a) \in R$
- symmetric — if $(a,b) \in R$ for some $a,b \in A$, then $(b,a) \in R$
- transitive — if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$

$=_{mod 5}$ (remainder modulo 5) is an equivalence relation over natural numbers

$6 =_{mod 5} 11$, $6 =_{mod 5} 21$, $11 =_{mod 5} 21$

Definition:
A function f from se[illegible] B is a relation between A and B that satisfies the foll[illegible]perties:

- For every $a \in A$, there exists a $b \in B$ such at $(a, b) \in f$
- If $(a, b) \in f$ and $(a, b') \in f$, then $b = b'$.

Instead of $(a, b) \in f$, we usually use notation $f(a) = b$.
Alternatively, we also use the notation $f : a \mapsto b$.

Functions

$f(x)=|x|$

$g(x)=\sin(x)$

not a function

Definition:

A function f [...]et B is a relation between A and B that satisfies [...]operties:

- For every $a \in A$, there exists a $b \in B$ such at $(a, b) \in f$
- If $(a, b) \in f$ and $(a, b') \in f$ then $b = b'$.

$A = \{1, 3, 5, 7\}$, $B = \{1, 2, 3\}$

i	$f(i)$
1	1
3	1
5	3
7	2

for functions, we can use such tables to define them

$f = R = \{ \ldots \}$

$R = \{(1,1), (5,1), (1,2), (7,2)\}$ is not a function; it violates both properties (input 1 gets mapped to two outputs and input 3 doesn't get mapped to any output)

Instead of $(a, b) \in f$, we usually use notation $f(a) = b$.

Alternatively, we also use the notation $f : a \mapsto b$.

Functions – important properties

We use the notation

$$f : A \to B$$

to state that f is a function from A to B.

We call A the domain of the function f and B the co-domain of the function f.
Note that the textbook uses range for the co-domain.

Further, the image of the funcion f is the following subset of co-domain:

$$\{b \in B \mid \text{there exists an } a \in A \text{ with } f(a) = b\}$$

Functions can be:

- one-to-one:
  We call a function $f : A \to B$ one-to-one if $f(a) = b$ and $f(a') = b$ for some $b \in B$ implies that $a = a'$.
- onto:
  We call a function $f : A \to B$ onto if for all $b \in B$ there exists an $a \in A$ with $f(a) = b$.

Functions – important properties

We use the n $f : A \to B$

to state that from A to B.

We call A the domain of the function f and B the ~~range~~ co-domain of the function f.

We call the set

$\mathrm{Im}(f) = \{ b \in B \mid \text{there is } a \in A \text{ with } b = f(a) \} \subseteq B$

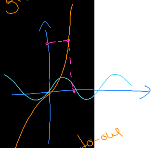
the image of the function f.

Functions – important properties

Let $A = \mathbb{R}$, $B = \mathbb{R}$

$f(x) = \sin(x)$ is neither one-to-one nor onto

$g(x) = x^3$



is both one-to-one and onto

We use the n $f : A \to B$

to state that from A to B.

We call A the domain of the function f and B the range of the function f.

Functions can be:

- one-to-one:
  We call a function $f : A \to B$ one-to-one if $f(a) = b$ and $f(a') = b$ for some $b \in B$ implies that $a = a'$.
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  We call a function $f : A \to B$ onto if for all $b \in B$ there exists an $a \in A$ with $f(a) = b$.

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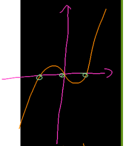
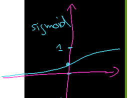


onto, but not one-to-one

sigmoid

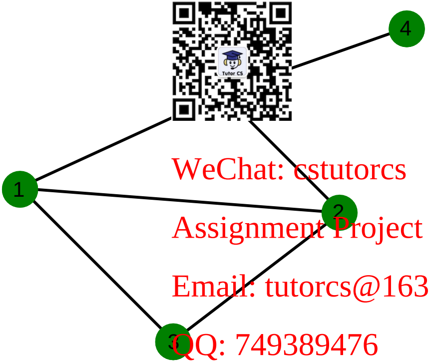


one-to-one but not onto

Graphs

Reading: ITC Section 0.2

We illustrate graphs as points and connecting lines:



An undirected graph (or simply a graph) $G = (V, E)$ is defined as a pair of

- a set V of vertices (also called nodes)
- and a set E of edges,

where each edge in E is a (multi-set) subset of V of size 2.

Graphs

We illustrate gr… connecting lines:

G

Formally:

$G = (\{1,2,3,4,5\},$
$\{\{1,5\},\{5,4\},\{5,2\},$
$\{2,3\},\{1,2\},\{1,3\}\})$

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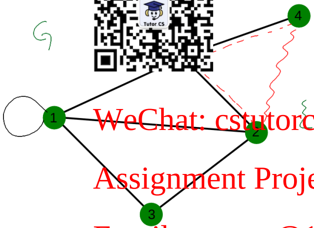
Graphs

Is G (or E)
• reflexive?
no
• transitive?
no
• symmetric?
yes

We illustrate gr[...] connecting lines:

G



Formally:

G = ({1, 2, 3, 4, 5},
{{1, 5}, {5, 4}, {5, 2}, {1, 1},
{2, 3}, {1, 2}, {1, 3}})

transitive
directed
graph

An undirected graph (or simply a graph) $G = (V, E)$ is defined as a pair of

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- and a set E of edges,

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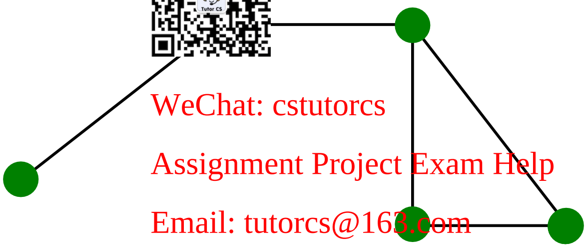
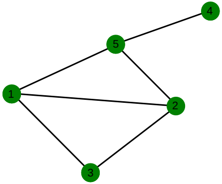
Graphs can be used to il[illegible]tions between objects.



Some definitions and terminology for graphs



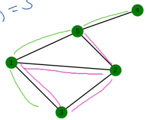
Let $G = (V, E)$ be a g[illegible]define:

- a vertex $v \in V$ is [illegible]o edge $e \in E$ if $v \in e$
- the **degree** $d(v)$ of a vertex $v \in V$ is the number of edges v is adjacent to
- a **path** in a graph is a sequence $p = (v_1, v_2, \ldots, v_n)$ of vertices such that $\{v_i, v_{i+1}\} \in E$ for all $i \in \{1, 2, \ldots, n\}$
- a **simple path** is a path in which no vertex occurs more than once

Some definitions and terminology for graphs

$d(4) = 1$

$d(1) = 3$



Let $G = (V, E)$ ... e define:

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- a simple path is a path in which no vertex occurs more than once

example of a path

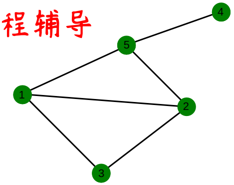
$(5, 2, 3, 1, 2)$

not a simple path!

example of a simple path

$(4, 5, 1, 3)$

Some definitions and terminology for graphs



Let $G = (V, E)$ be a graph. We define:

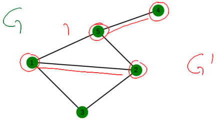
- a subgraph G' of graph G is a a graph $G' = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$

- graph G is connected, if for every two distinct vertices $v, v' \in V$, there is a path from v to v' (that is, a path which has v as it's first and v' as its last vertex)

- we call an edge $e \in E$ a self-loop if $e = \{v, v\}$ for some $v \in V$ (that is, the edge connects some vertex v to itself)

Some definiti[illegible] for graphs

Let $G = (V,$ We define:



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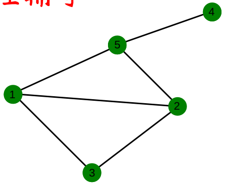
- we call an edge $e \in E$ a self-loop if $e = \{v, v\}$ [illegible] V (that is, the edge connects some vertex v to itself)

Subgraph $G' = (\{1,5,4,2\}, \{\{4,5\},\{1,2\}\})$

G is a connected graph, but G' is not connected

Some definitions and terminology for graphs

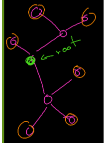


Let $G = (V, E)$ be a graph. We define:

- a cycle in a graph is a path for which the first and last vertex are identical
- a simple cycle in a graph is a cycle that contains at least three different vertices and doesn't contain any vertex more than once (except for the first and last)
- a tree is a graph that is connected and doesn't contain any simple cycles; vertices of degree one in a tree are called leaves; sometimes we declare one vertex in a tree to be special and call it a root

Some definitions and terminology for graphs

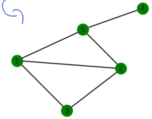
tree



leaves of the tree

Let $G = (V,$ ... We define:

G



- a cycle in a graph is a path for which the the first and last vertex are identical

Cycle in G:

(5, 2, 3, 1, 5)

- a simple cycle in a graph is a cycle that contains at least three different vertices and doesn't contain any vertex more than once (except for the first and last)

(also a simple cycle)

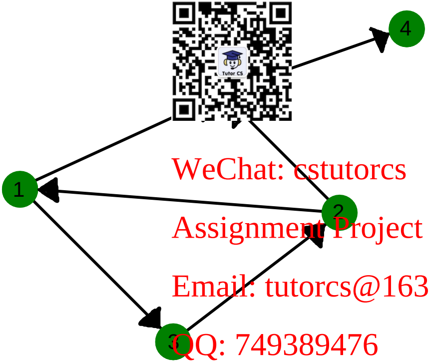
- a tree is a graph that is connected and doesn't contain any simple cycles; vertices of degree one in a tree are called leaves; sometimes we declare one vertex in a tree to be special and call it a root

Cycle that is not simple:

(5, 2, 3, 1, 2, 5)

Directed graphs

We illustrate directed graphs as points and connecting arrows:



A directed graph $G = (V, E)$ is formally defined as a pair of

- a set V of vertices
- and a set E of edges,

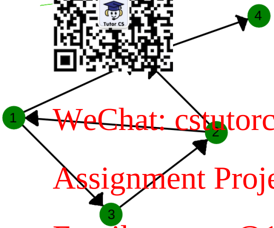
where each edge in E is a 2-tuple (or pair) of members of V.

Directed graphs

For directed graphs the edge set E is a relation over the vertex set V.

In the example graph G, this relation is
- not reflexive
- not transitive
- not symmetric

We illustrate di… points and connecting arrows:



$G = (\{1,2,3,4,5\}, \{(1,5),(5,4),(2,5),(3,2),(2,1),(1,3)\})$

A directed graph $G = (V, E)$ is formally defined as a pair of
- a set V of vertices
- and a set E of edges,

where each edge in E is a 2-tuple (or pair) of members of V.

- For an edge $e = (v_1, v_2)$ in directed graph, we call e an incoming edge t[illegible]an outgoing edge for v_1.

- The in-degree d[illegible] vertex v in a directed graph is the number of its incoming edges. The out-degree $d_{out}(v)$ is the number of outgoing edges from v.

- A directed path in a directed graph is a sequence $p = (v_1, v_2, \ldots, v_n)$ of vertices such that $(v_i, v_{i+1}) \in E$ for all $i \in \{1, 2, \ldots, n\}$.

- A directed graph is strongly connected if every two distinct vertices v and v' in V are connected by a directed path.

- A directed cycle in a directed graph is a directed graph for which the first and last vertex coincide.

Some definitions and terminology for directed graphs

- For an e... in directed graph, we call e an incoming... an outgoing edge for v_1.
- The in-d... a vertex v in a directed graph is the number of its incoming edges. The out-degree $d_{out}(v)$ is the number of outgoing edges from v.
- A directed path in a directed graph is a sequence $p = (v_1, v_2, \ldots, v_n)$ of vertices such that $(v_i, v_{i+1}) \in E$ for all $i \in \{1, 2, \ldots, n\}$
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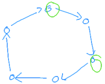

a strongly connected graph

Strings and Languages

Reading: ITC Section 0.2

An alphabet is a non-empty, finite set of symbols.

Examples:

- $\Sigma = \{0, 1\}$.
- $\Gamma = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$

A string or word over some alphabet Σ is a finite sequence of symbols from Σ.

Examples:

- ruth
- 00101

A language is a set of words (set of strings).

Alphabet, words, languages

An **alphabet** is a finite set of **symbols**.

Examples:

- $\Sigma = \{0, 1\}$
- $\Gamma = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$

A **string** or **word** over some alphabet Σ is a **finite sequence of symbols from** Σ.

Examples:

- ruth — word over Γ
- 00101 — word over Σ

a10432 — not a word over Σ or Γ (but a word over $\Sigma \cup \Gamma$)

A **language** is a **set of words** (set of strings).

{ruth}

{ruth, abc, chair, dog}

Alphabet, words, languages

$\Omega = \{$, , $\}$

An alphabet is a finite set of symbols.

Examples:

- $\Sigma = \{0, 1\}$
- $\Gamma = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$

A string or word over some alphabet Σ is a finite sequence of symbols from Σ.

Examples:

- ruth
- 00101

A language is a set of words (set of strings).

Talking: Ruth Urner

Some definitions and terminology for strings and languages

- the length of a string **w** is the number of symbols in **w**
- the empty string ϵ is a string of length 0 (that is, without any symbols)

- the reverse $\mathbf{w}^{\mathcal{R}}$ of a string **w** is the number of symbols in **w**

- the concatenation of strings $\mathbf{w} = s_1 s_2 \ldots s_n$ and $\mathbf{z} = t_1 t_2 \ldots t_m$ is the string $\mathbf{wz} = s_1 s_2 \ldots s_n t_1 t_2 \ldots t_m$

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w = 01011 , z = 1101 , wz = 010111101

Some definitions and terminology for strings and languages

- a substring $\mathbf{z}$ of string $\mathbf{w}$ is a consecutive subsequence of symbols from $\mathbf{w}$
- a prefix $\mathbf{x}$ of string $\mathbf{w}$ is a substring that starts at the first position in $\mathbf{w}$; that is, $\mathbf{x}$ is a prefix of $\mathbf{w}$ if there exists a string $\mathbf{z}$ such that $\mathbf{w} = \mathbf{xz}$
- $\mathbf{x}$ is a proper prefix of $\mathbf{w}$ if $\mathbf{x}$ is a prefix of $\mathbf{w}$ and $\mathbf{w} \neq \mathbf{x}$
- a prefix-free language over some alphabet Σ is a language where no word in the language is a proper prefix of some other word

Some de… strings and languages

- a substri… is a consecutive subsequence of symbols

$w = o$… $= can$ not a substring of w

$= cea$ is a substring of w

Some definitions and terminology for strings and languages

- a substri[…] is a consecutive subsequence of symbols

- a prefix x of string w is a substring that starts at the first position in w; that is, x is a prefix of w if there exists a string z such that w = xz

a proper prefix of w

w = ocean , x = oc is a prefix of w

- x is a proper prefix of w if x is a prefix of w and w ≠ x

w = ocean , x = ocean is a prefix, but not a proper prefix of w

Some definitions and terminology for strings and languages

$\Sigma = \{0,1\}$

$\{00, 11, 01, 0011, 0101\}$

not a prefix-free language since 00 is a prefix of 0011

$\{00, 11, 0101\}$ is a prefix-free language

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- a prefix-free language over some alphabet Σ is a language where no word in the language is a proper prefix of some other word

Given an ordering of the symbols in some alphabet Σ, we can define orderings of the words over Σ:

- Lexicographic order

  Definition: We have $\mathbf{w} \leq_{lex} \mathbf{z}$ in lexicographic order for strings $\mathbf{w} = s_1 s_2 \ldots s_n$ and $\mathbf{z} = t_1 t_2 \ldots t_m$ if $\mathbf{w}$ is a proper prefix of $\mathbf{z}$ or if there exists an index $j < \min\{n, m\}$ such that $s_i = t_i$ for all $i \in \{1, 2, \ldots, j\}$ and $s_{j+1} < t_{j+1}$.

- Short-lex order (or string order)

  Definition: We have $\mathbf{w} \leq_{str} \mathbf{z}$ in string order for strings $\mathbf{w} = s_1 s_2 \ldots s_n$ and $\mathbf{z} = t_1 t_2 \ldots t_m$ if $\mathbf{w}$ is shorter than $\mathbf{z}$ or if $\mathbf{w}$ and $\mathbf{z}$ have the same length and $\mathbf{w} \leq_{lex} \mathbf{z}$ in lexicographic order.

Ordering words

ruth ocean

apple apartment

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cat $\leq_{str}$ dog

ruth $\leq_{str}$ ocean

apple $\leq_{str}$ apartment