cs2001
Introduction Theory of Computation

Lecture 11

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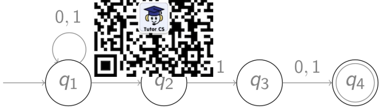
Example: the NFA N_2

For the machine N_2 we get

$$L(N_2) = \{\mathbf{w} \mid \mathbf{w} \text{ contains 1 at the third position from the end}\}$$

Example: the NFA N_2



0101 ✓ 0001 ✗

1111 ✓ 00001 ✗

For the machine N_2 we get

$L(N_2) =$ { all words that has a 1 at the third position from the end }

$= \{ w = w_1 w_2 \ldots w_n \in \Sigma^* \mid w_{n-2} = 1 \}$

Example: the NFA N_3

Let's consider a unary alphabet $\Sigma = \{0\}$.

For the machine N_3 we get

$$L(N_3) = \{\mathbf{w} \mid \text{ The number of 0 symbols in } \mathbf{w} \text{ is divisible by 2 or by 3}\}$$


Example: the NFA
Let's consid
abet $\Sigma = \{0\}$.
q_1
q_3
q_4
q_5
q_6
0
ε
Mouse Select Text Draw Stamp Spotlight Eraser Format Undo Redo Clear Save
Who can see what you share here? Recording On
Talking:
ε ✓
00000 X
00 ✓
$L(N_4) = \{ w \in \{0\}^* \mid$ the length of w is divisible by 2 or by 3 $\}$

Example:

Let's consid[er] ... [alph]abet $\Sigma = \{0\}$.

ε ✓

000000 ✗

00 ✓

is the same word as $\varepsilon 00$

$L(N_1) = \{w \in \{0\}^* \mid$ the length of w is divisible by 2 or by 3 $\}$

$= \{w \in \{0\}^* \mid$ length of w even $\} \cup \{w \in \{0\}^* \mid$ length of w divisible by 3 $\}$

Equivalence of NFAs and DFAs

Definition

We call two finite state machines equivalent if they recognize the same language.

Theorem

For every NFA, there exists an equivalent DFA.

Corrolary

A language L is regular if and only if there exists an NFA that recognizes it.

Equival[illegible] NFA [illegible] DFA

Definition
We call two [illegible] chines equivalent if they recognize the same langua[illegible] ata

Theorem
For every NFA, there exists an equivalent DFA.

For every NFA N, there exist a DFA M with L(M) = L(N).

Corollary
A language L is regular, if and only if there exist NFA that recognizes L.

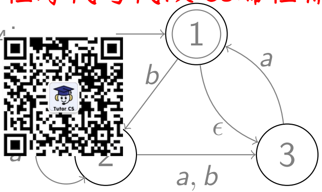
Transforming an NFA into a DFA–example

To prove the theorem from the previous slide, we need to show that for every NFA N, there exists a DFA M with $L(M) = L(N)$.

Before we see the general proof for this, we go through one concrete example. We transform NFA N_4 into an equivalent DFA.

Transforming an NFA into a DFA–example

Consider the NFA N_4:




Transforming an NFA into a DFA–example
Consider the
$\Sigma = \{a, b\}$
1
2
3
a
b
a
ϵ
a, b
Examples of accepted words
ε ✓
a ✓
εa
baba ✓
Examples of words that are not accepted:
bbb ✗
bb ✗
Talking: Ruth Urner


Transforming an NFA into a DFA–example
Mouse Select Text Draw Stamp Spotlight Eraser Format Undo Redo Clear Save
Who can see what you share here?
You are screen sharing
Stop Share
Consider the
Σ = {a,b}
Q = {1,2,3}
1
2
3
a
b
ϵ
a
a, b
Examples of accepted words
ε ✓
a ✓
"
εa
baba ✓
Equivalent DFA
∅
{1}
{2}
{1,2}
{3}
{1,3}
{2,3}
{1,2,3}
a
b
a,b
Examples of words that are not accepted:
bbb X
bb X
Talking: Ruth Urner


Mouse Select Text Draw Stamp Spotlight Eraser Format Undo Redo Clear Save
Who can see what you share here?
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Transforming an NFA into a DFA–example
Consider the
Σ = {a,b}
Q = {1,2,3}
1
2
3
a
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ϵ
a, b
Examples of accepted words
ε ✓
a ✓
εa
baba ✓
Equivalent DFA
a,b
∅
{2}
{3}
{1,3}
{2,3}
{1,2,3}
Examples of words that are not accepted:
bbb X
bb X

Proof

Let $N = (Q, \Sigma, \delta, q_0, F)$ [illegible] A.

We need to construct a [illegible] $Q', \Sigma, \delta', q_0', F')$ with $L(M) = L(N)$.

Assuming that N does n[illegible]ny ϵ-transitions, we construct M as follows:

1. $Q' = \mathcal{P}(Q)$
2. For $R \in Q'$ and $a \in \Sigma$, we let

$$\delta'(R, a) = \{q \in Q \mid q \in \delta(r, a) \text{ for some } r \in R\}$$

3. $q_0' = \{q_0\}$
4. $F' = \{R \in Q' \mid \text{there exists an } r \in F \text{ with } r \in R\}$.

This completes the construction for NFAs that don't contain ϵ-transitions.

General proof of Theorem

Proof

Let $N = (Q, \Sigma$ … e NFA.

We need to c… $= (Q', \Sigma, \delta', q_0', F')$ with $L(M) = L(N)$.

Assuming that … ain any ϵ-transitions, we construct M as follows:

$\in Q' = \mathcal{P}(Q)$

1. $Q' = \mathcal{P}(Q)$

$\subseteq Q$

2. For $R \in Q'$ and $a \in \Sigma$, we let

$$\delta'(R, a) = \{q \in Q \mid q \in \delta(r, a) \text{ for some } r \in R\}$$

3. $q_0' = \{q_0\}$
4. $F' = \{R \in Q' \mid \text{there exists an } r \in F \text{ with } r \in R\}$.

This completes the construction for NFAs that don't contain ϵ-transitions.

Proof continued

If N does contain ϵ-trans[illegible]xtend the above construction as follows:

For a state $R \in Q'$, we let $E(R)$ denote the set of states that can be reached via 0 or more ϵ-transitions from some state $r \in R$ (in N).

Then we modify the transition function of M to be

$$\delta'(R, a) = \{q \in Q \mid q \in E(\delta(r, a)) \text{ for some } r \in R\}$$

and the start state of M to be

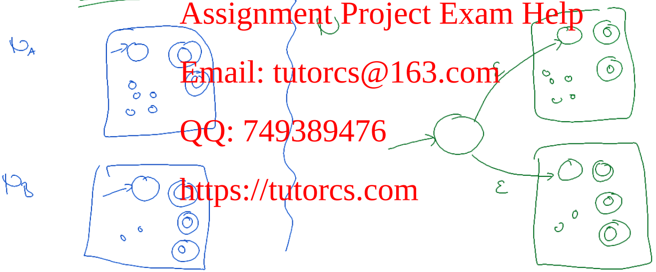
$$q_0' = E(q_0).$$

Theorem

Regular languages ar[illegible]nder unions.

“Proof” by illustrat[illegible]

Given A and B [illegible] NFAs N_A and N_B with $L(N_A) = A$ and $L(N_B) = B$. We need to construct an NFA N that recognizing $A \cup B$.



Regular languages are closed under regular operations

Theorem
Regular languages are closed under unions.

Proof (formal version)

We need to show that if two languages A and B are regular, then so is $A \cup B$. If A and B are regular, then there exist NFAs $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ such that $L(N_1) = A$ and $L(N_2) = B$. We need to show that there exists an NFA M with $L(M) = A \cup B$. The picture on the previous slide illustrates a construction for $M = (Q, \Sigma, \delta, q_0, F)$. Formally, we set:

1. $Q = Q_1 \cup Q_2 \cup \{q_0\}$, where q_0 is a new state, that is $q_0 \notin Q_1 \cup Q_2$ and this state q_0 will then also be the start state of M.
2. $\delta(q, a) = \begin{cases} \{q_1, q_2\} & \text{if} \quad q = q_0 \quad \text{and} \quad a = \epsilon \\ \delta_i(q, a) & \text{if} \quad q \in Q_i \\ \emptyset & \text{else} \end{cases}$
3. $F = F_1 \cup F_2$.

Theorem

Regular languages are closed under concatenations.

"Proof" by illustrat

Theorem

Regular languages are closed under concatenations.

Proof (formal version)

We need to show that if two languages A and B are regular, then so is $A \circ B$. If A and B are regular, then there exist NFAs $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ such that $L(N_1) = A$ and $L(N_2) = B$. We need to show that there exists an NFA M with $L(M) = A \circ B$. The picture on the previous slide illustrates a construction for $M = (Q, \Sigma, \delta, q_0, F)$. Formally, we set:

1. $Q = Q_1 \cup Q_2$
2. $q_0 = q_1$
3. $\delta(q, a) = \begin{cases} \delta_1(q, a) \cup \{q_2\} & \text{if } q \in F_1 \text{ and } a = \epsilon \\ \delta_1(q, a) & \text{if } q \in Q_1 \text{ and } (q \notin F_1 \text{ or } a \neq \epsilon) \\ \delta_2(q, a) & \text{if } q \in Q_2 \end{cases}$
4. $F = F_2$.

Regular languages are closed under regular operations

Theorem

Regular languages ar nder the star-operation.

“Proof” by illustrat

Theorem

Regular languages are closed under the star-operation.

Proof (formal version)

We need to show that if language A is regular, then so is A^*. If A is regular, then there exist an NFA $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ such that $L(N_1) = A$. We need to show that there exists an NFA M with $L(M) = A^*$. The picture on the previous slide illustrates a construction for $M = (Q, \Sigma, \delta, q_0, F)$. We set:

1. $Q = Q_1 \cup \{q_0\}$, where q_0 is a new state, that is $q_0 \notin Q_1$ and this state q_0 will then also be the start state of M.
2. $\delta(q, a) = \begin{cases} \{q_1\} & \text{if } q = q_0 \text{ and } a = \epsilon \\ \delta_1(q, a) \cup \{q_1\} & \text{if } q \in F_1 \text{ and } a = \epsilon \\ \delta_1(q, a) & \text{if } q \in Q_1 \text{ and } (q \notin F_1 \text{ or } a \neq \epsilon) \end{cases}$
3. $F = F_1 \cup \{q_0\}$.

Regular Expressions

Reading: ITC Section 1.3

Regular expression

A regular expression is a compact way of defining a set of words. It is a sequence of symbols that represent a language over some alphabet Σ.

Inductive definition of sets – general pattern

An inductive definition of a set consists of

1. **A universe set** U
2. **A core set** $C \subseteq U$
3. **A finite set** $O = \{o_1, o_2, \ldots, o_k\}$ **of operations from** $o_i : U^{r_i} \to U$ **for some arities** $r_i \in \mathbb{N}$

We define $\mathcal{I}(U, C, O)$ as the set of elements that we obtain by starting with the core set and putting all those elements of U into $\mathcal{I}(U, C, O)$ that one can reach by successively applying the operations in O.

Regular expression–inductive definition

Let Σ be some alpha[illegible] define the set $\mathcal{R}$ of regular expressions over Σ in[illegible] by setting $\mathcal{R}_\Sigma = \mathcal{I}(U, C, O)$, where

1. The universe U is the set of all strings over $\Sigma \cup \{(,), \cup, \circ, ^*, \epsilon, \emptyset\}$.

2. The core set C is the set of all symbols in Σ and $\epsilon, \emptyset$ and two additional symbols: $C = \Sigma \cup \{\epsilon, \emptyset\}$.

3. Three operations:
   - $o_\cup(R_1, R_2) = (R_1 \cup R_2)$,
   - $o_\circ(R_1, R_2) = (R_1 \circ R_2)$,
   - $o_*(R) = (R^*)$.

Regular expression–inductive definition

Let Σ be so… We define the set $\mathcal{R}$ of regular expressions …vely by setting $\mathcal{R}_\Sigma = \mathcal{I}(U, C, O)$, where

1. The universe U is the set of all strings over $\Sigma \cup \{(,), \cup, \circ, ^*, \epsilon, \emptyset\}$.
2. The core set C is the set of al symbols in Σ and $\epsilon, \emptyset$ and two additional symbols: $C = \Sigma \cup \{\epsilon, \emptyset\}$.
3. Three operations:
   - $o_\cup(R_1, R_2) = (R_1 \cup R_2)$,
   - $o_\circ(R_1, R_2) = (R_1 \circ R_2)$,
   - $o_*(R) = (R^*)$.

Examples: $\Sigma = \{0,1\}$

0 ✓

$(1 \cup 0)$ ✓

$(1\ 0\ 1$ ✗

ε ✓

$(\varepsilon\ 0\ 1)$ ✓

$01^*((\Sigma$

member of universe

$)(0\emptyset$

these two are not regular expressions

Exercise: … induction) that the number of "(" always equal to the number of ")" in a regular expression

The language of a regular expression

Each regular expression $R \in \mathcal{R}_\Sigma$ over some alphabet Σ represents a language over Σ. We define the in[illegible] $L(R)$ of a regular expression R according to the inductive definitio[illegible]

Members of the core-s[illegible]

- The expression a for $a \in \Sigma$ represents the language $\{a\}$, that is $L(a) = \{a\}$.

- The expression ϵ represents the language $\{\epsilon\}$, that is $L(\epsilon) = \{\epsilon\}$.

- The expression $\emptyset$ represents the language $\emptyset$,that is $L(\emptyset) = \emptyset$.

Result of operation: For regular expressions R_1, R_2 and R, we define:

- $L((R_1 \cup R_2)) = L(R_1) \cup L(R_2)$

- $L((R_1 \circ R_2)) = L(R_1) \circ L(R_2)$

- $L((R^*)) = L(R)^*$

We call $L(R)$ the language of R.

The language of a regular expression

Each regular e[...] over some alphabet Σ represents a language over Σ. We d[...]ation $L(R)$ of a regular expression R according to the inducti[...]

Members of [...]

- The expression a for $a \in \Sigma$ represents the language $\{a\}$, that is $L(a) = \{a\}$.
- The expression ϵ represents the language $\{\epsilon\}$, that is $L(\epsilon) = \{\epsilon\}$.
- The expression $\emptyset$ represents the language $\emptyset$, that is $L(\emptyset) = \emptyset$.

Result of operation: For regular expressions R_1, R_2 and R, we define:

- $L(\boxed{(R_1 \cup R_2)}) = L(R_1) \cup L(R_2)$

$\Sigma = \{0, 1\}$

$0 \in \mathcal{R}_\Sigma$

$L(0) = \{0\}$

$0, \varepsilon \in \mathcal{R}_\Sigma$

$(0 \cup \varepsilon) \in \mathcal{R}_\Sigma$

$L((0 \cup \varepsilon)) = \{0, \varepsilon\}$

$(0 \cup \emptyset) \in \mathcal{R}_\Sigma$

$L((0 \cup \emptyset)) = \{0\}$

Regular expression–additional notation

For convenience and rea… use the following notational conventions:

1. For an alphabet Σ, …s a regular expression representing all words of length 1 over Σ. And then Σ^* is a regular expression for the set of all words over Σ.

2. We often omit brackets. The order of precedence then is: $*$, $\circ$, $\cup$.

3. The $\circ$-symbol is typically omitted: we use $R_1 R_2$ as shorthand for $R_1 \circ R_2$.

4. We let R^+ be shorthand for RR^*.

5. We let R^k be the k times repeated concatenation of R with itself: $R^k = R \circ R \circ R \circ \cdots \circ R$.

We let $\Sigma = \{0, 1\}$.

Now we can interpre[illegible]wing regular expressions over Σ:

- 0^*10^*
- $\Sigma^*1\Sigma^*$
- $\Sigma^*001\Sigma^*$
- $1^*(01^+)^*$
- $(\Sigma\Sigma)^*$
- $(\Sigma\Sigma\Sigma)^*$

Regular expressions—examples

precedence

We let $\Sigma =$

Now we can ... following regular expressions over Σ:

- 0*10* $= L((0^*)1(0^*))$

$= \{ w \mid w$ contains exactly one $1 \}$

$1 \in L((0^*)1(0^*))$

$0010 \in L((0^*1\,0^*))$

$10000 \in L((0^*1\,0^*))$

$L((0^*)) = \{$set of all words that have only letter $0\}$

↑ this includes ε!

$L((0^*)) = \{\varepsilon, 0, 00, 000, \dots\}$

$L(1) = \{1\}$

Regular expressions—examples

We let $\Sigma =$

Now we can … following regular expressions over Σ:

- 0^*10^*
- $\Sigma^*1\Sigma^*$ $= (\Sigma^*)1(\Sigma^*) = (\Sigma^*) \circ 1 \circ (\Sigma^*)$

$L(\Sigma^*1\Sigma^*) = \{ w \in \Sigma^* \mid w$ contains at least one 1 $\}$

$1 \in L(\Sigma^*1\Sigma^*)$

$\underbrace{01011}_{\in \Sigma^*}1\underbrace{01}_{\in \Sigma^*} \in L(\Sigma^*1\Sigma^*)$

Regular expressions—examples

We let $\Sigma =$

Now we can … following regular expressions over Σ:

- 0^*10^*
- $\Sigma^*1\Sigma^*$
- $\Sigma^*001\Sigma^*$

$L(\Sigma^* 001 \Sigma^*) = \{ w \in \Sigma^* \mid w$ contains 001 as a substring $\}$

Regular expressions—examples

$* \quad \circ \quad \cup$

We let $\Sigma =$

Now we can … following regular expressions over Σ:

- 0^*10^*
- $\Sigma^*1\Sigma^*$
- $\Sigma^*001\Sigma^*$
- $1^*(01^+)^*$

$L(1^*(01^+)^*) = \{w \in \Sigma^* \mid$ every 0 is followed by a 1$\}$

$\in 01^+$

111 01 01101

$\in 1^*$ $\quad \in 01^+ \quad \in 01^+ \quad \in 01^+$