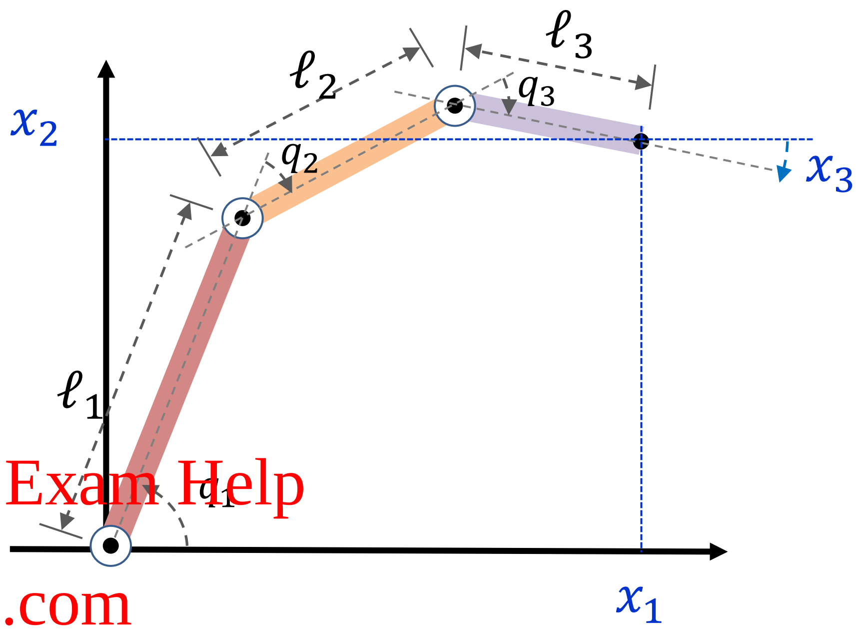
Kinematics of Robotic Manipulators



Forward kinematics $\boldsymbol{x} = \varphi(\boldsymbol{q})$

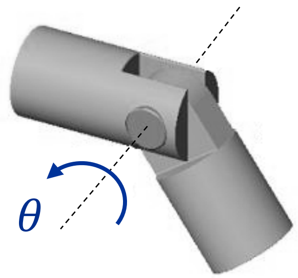
Inverse kinematics $\boldsymbol{q} = \varphi^{-1}(\boldsymbol{x})$

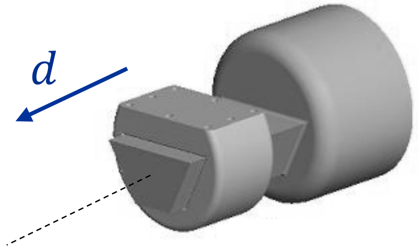
Robot Manipulators

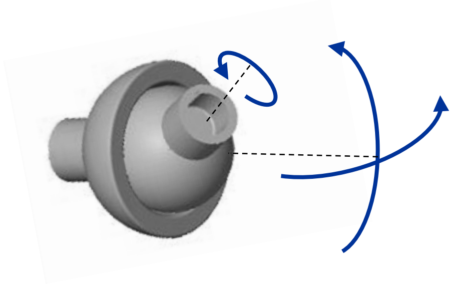
Robot manipulators are composed of **joints** and **links**.

Joints

- **Primary joints**
  - Revolute (rota
    - Allows a relative rotation between two links.
    - Rotation around the axis of rotation is denoted by θ
  - Prismatic (linear)
    - Allows a linear relative motion between two links.
    - Translation along the axis of translation is denoted by d
- **Compound joints**
  - Have more degrees of freedom







Links are connected by joints and form a **kinematic chain**.

Robot Manipulators

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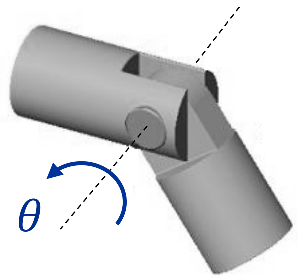
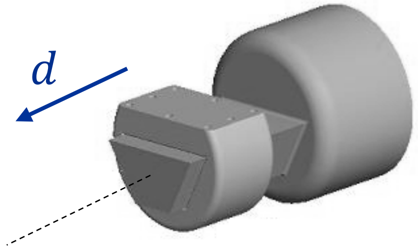
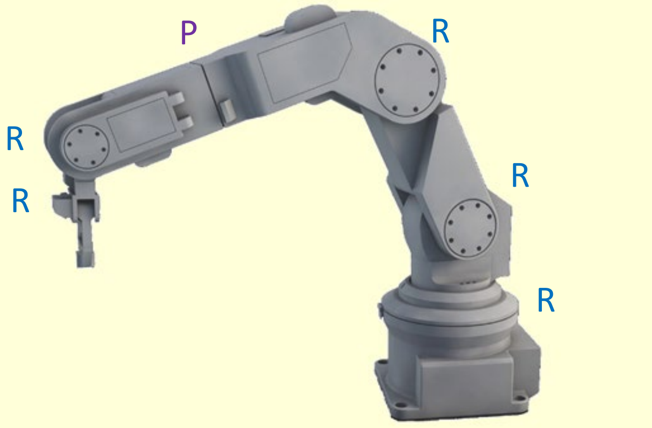
- **Compound joints**





Example:

- 1 prismatic joint
- 5 revolute joints



Robot Configuration

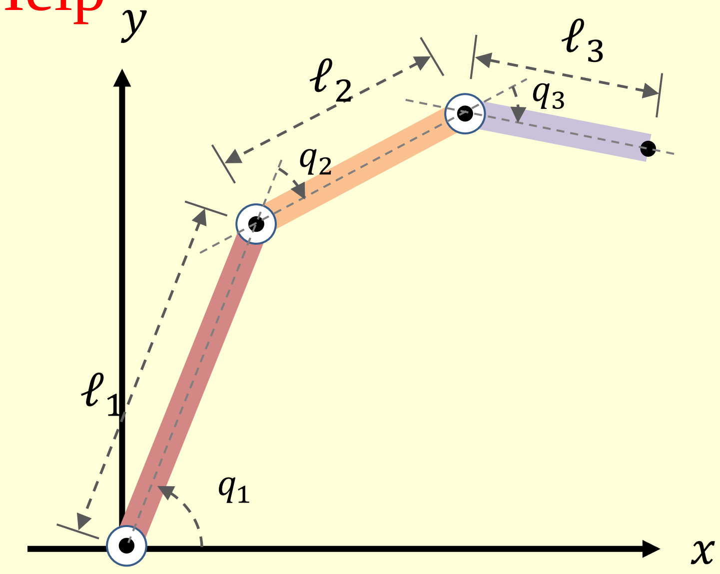
The configuration of a manipulator is a **con**[illegible]**fication** of the location of every point on the manipulator.

The configuration of a rigid manipulator is represented by the **vector of joint variables** denoted by $\boldsymbol{q}$.

- For revolute joints, $q_i = \theta_i$
- For prismatic joints $q_i = d_i$

Example:

Vector of joint variables: $\boldsymbol{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$



Forward & Inverse Kinematics

Forward & Inverse Kinematics

Joint variables ($\boldsymbol{q}$)

Fo… matics $\boldsymbol{x} = \varphi(\boldsymbol{q})$

Inverse kinematics $\boldsymbol{q} = \varphi^{-1}(\boldsymbol{x})$

End effector's configuration ($\boldsymbol{x}$)

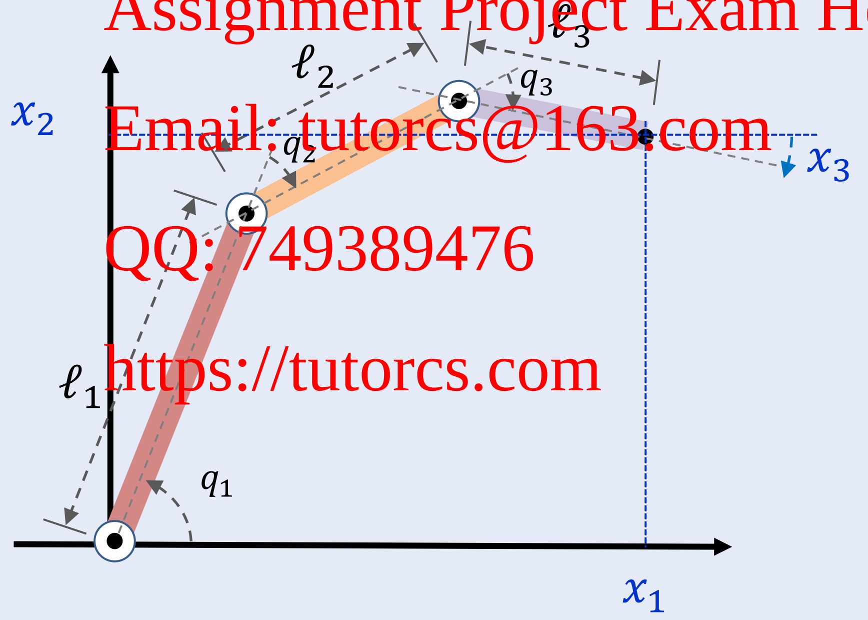
$$\boldsymbol{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Joint space

$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ If only the position of the end-effector is of interest.

$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ If both position and orientation of the end-effector are of interest.

Task space



Forward & Inverse Kinematics

Forward kinematics $\boldsymbol{x} = \varphi(\boldsymbol{q})$

Joint variables ($\boldsymbol{q}$) → End effector's configuration ($\boldsymbol{x}$)

Inverse kinematics $\boldsymbol{q} = \varphi^{-1}(\boldsymbol{x})$

Forward kinematics: Given the joint variables ($\boldsymbol{q}$), find the position and orientation of the end effector ($\boldsymbol{x}$). → A **unique** end-effector configuration can be found.

Inverse kinematics: Given the desired position and orientation of the end effector ($\boldsymbol{x}$), find the joint variables ($\boldsymbol{q}$).

→ There may not exist analytic closed-form solutions.

→ It is useful when we want to command/control the robot to move to a desired location.

Example: Forward kinematics

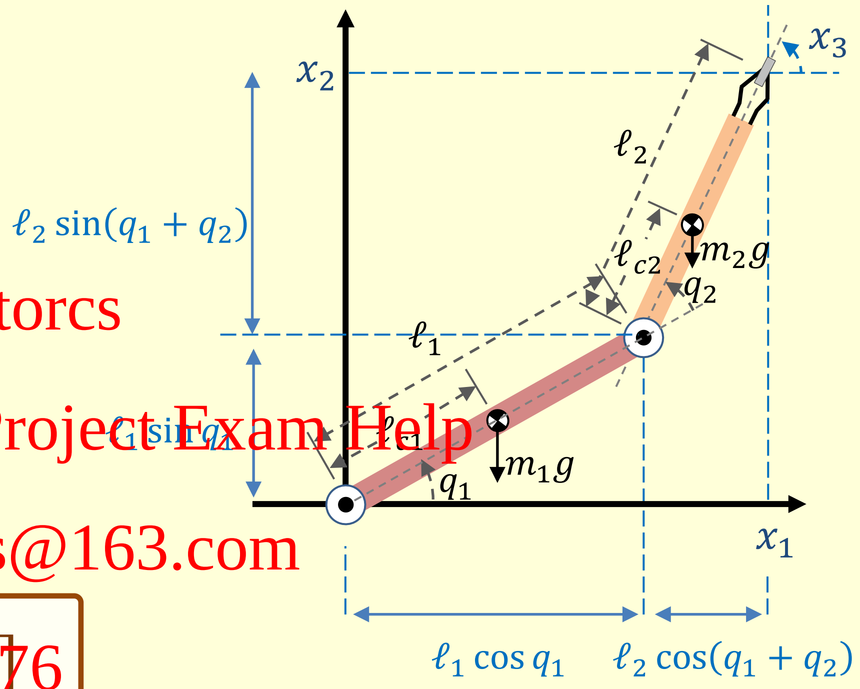
The coordinates of the end-effector:

$$x_1 = \ell_1 \cos q_1 + \ell_2 \cos(q_1 + q_2)$$
$$x_2 = \ell_1 \sin q_1 + \ell_2 \sin(q_1 + q_2)$$

The orientation of the end-effector:

$$x_3 = q_1 + q_2$$

$$\Rightarrow \quad \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \varphi(\boldsymbol{q}) = \begin{bmatrix} \ell_1 \cos q_1 + \ell_2 \cos(q_1 + q_2) \\ \ell_1 \sin q_1 + \ell_2 \sin(q_1 + q_2) \\ q_1 + q_2 \end{bmatrix}}$$

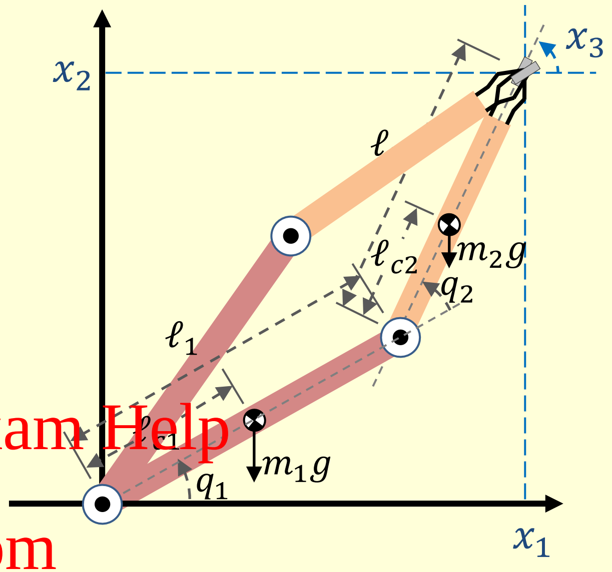


Example: Inverse kinematics

End-effector at (x_1, x_2)

$$\Rightarrow \quad q_1 = \tan^{-1}\left(\frac{x_2}{x_1}\right) \mp \tan^{-1}\left(\frac{\ell_2 \sin q_2}{\ell_1 + \ell_2 \cos q_2}\right)$$

$$q_2 = \pm \cos^{-1}\left(\frac{x_1^2 + x_2^2 - \ell_1^2 - \ell_2^2}{2\ell_1 \ell_2}\right)$$



Note that there are only two possible values for x_3.

Dynamics of Robotic Manipulators

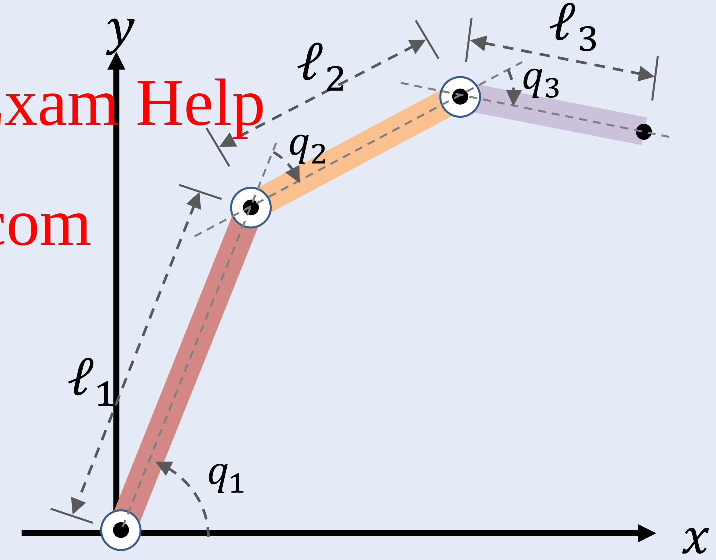
$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

Robot Dynamics

Dynamic equations describe the rela[illegible]tween ***force*** and ***motion***.

Forces: $\boldsymbol{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}$

Motion: $\boldsymbol{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}, \quad \dot{\boldsymbol{q}} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}, \quad \ddot{\boldsymbol{q}} = \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix}$



Robot Dynamics

The main methods to derive the dynamic m[illegible] manipulators:

- **Euler-Lagrange formulation** (or Lagran[illegible]tion)
- **Newton-Euler formulation**

Lagrangian: $L = K - P$

Kinetic energy

Potential energy

$$K(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}\, \dot{\boldsymbol{q}}^T\, \boldsymbol{M}(\boldsymbol{q})\, \dot{\boldsymbol{q}}$$

$$P(\boldsymbol{q})$$

$M(\boldsymbol{q})$ is the $n \times n$ *inertia matrix* which is symmetric and positive definite.

The dynamical model is obtained using **Lagrange equations**:

$$\frac{d}{dt}\left(\frac{\partial L(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial L(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \boldsymbol{q}} = \boldsymbol{\tau}$$

Robot Dynamics

Lagrange equations: $\frac{d}{dt}\left(\frac{\partial L(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial L(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \boldsymbol{q}} = \boldsymbol{\tau}$

$$L(\boldsymbol{q},\dot{\boldsymbol{q}}) = K(\boldsymbol{q},\dot{\boldsymbol{q}}) - P(\boldsymbol{q}) = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

$$\frac{\partial L(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}} = \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

$$\frac{d}{dt}\left(\frac{\partial L(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}}\right) = \frac{d}{dt}(\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}) = \dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}$$

$$\frac{\partial L(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \boldsymbol{q}} = \frac{\partial K(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \boldsymbol{q}} - \frac{\partial P(\boldsymbol{q})}{\partial \boldsymbol{q}} = \frac{1}{2}\frac{\partial}{\partial \boldsymbol{q}}(\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}) - \frac{\partial P(\boldsymbol{q})}{\partial \boldsymbol{q}}$$

$$\frac{d}{dt}\left(\frac{\partial L(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial L(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \boldsymbol{q}} = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \frac{1}{2}\frac{\partial}{\partial \boldsymbol{q}}(\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}) + \frac{\partial P(\boldsymbol{q})}{\partial \boldsymbol{q}} = \boldsymbol{\tau}$$

$$\boxed{\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}}$$

Robot Manipulator Dynamics

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

$\boldsymbol{q} \in \mathbb{R}^n$:	Vector of joint coordinates
$\boldsymbol{M}(\boldsymbol{q}) \in \mathbb{R}^{n\times n}$:	Inertia matrix
$\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} \in \mathbb{R}^n$:	Vector of Coriolis and centrifugal forces
$\boldsymbol{g}(\boldsymbol{q}) \in \mathbb{R}^n$:	Vector of gravity forces
$\boldsymbol{\tau} \in \mathbb{R}^n$:	Vector of joint torques

Summary: Deriving the dynamic model of a robot manipulator

1. Find the kinetic energy $K(\boldsymbol{q}, \dot{\boldsymbol{q}})$ and the potential energy $P(\boldsymbol{q})$
2. The Lagrangian:

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = K(\boldsymbol{q}, \dot{\boldsymbol{q}}) - P(\boldsymbol{q})$$

3. Use Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial L(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial L(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \boldsymbol{q}} = \boldsymbol{\tau}$$

and obtain

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

where

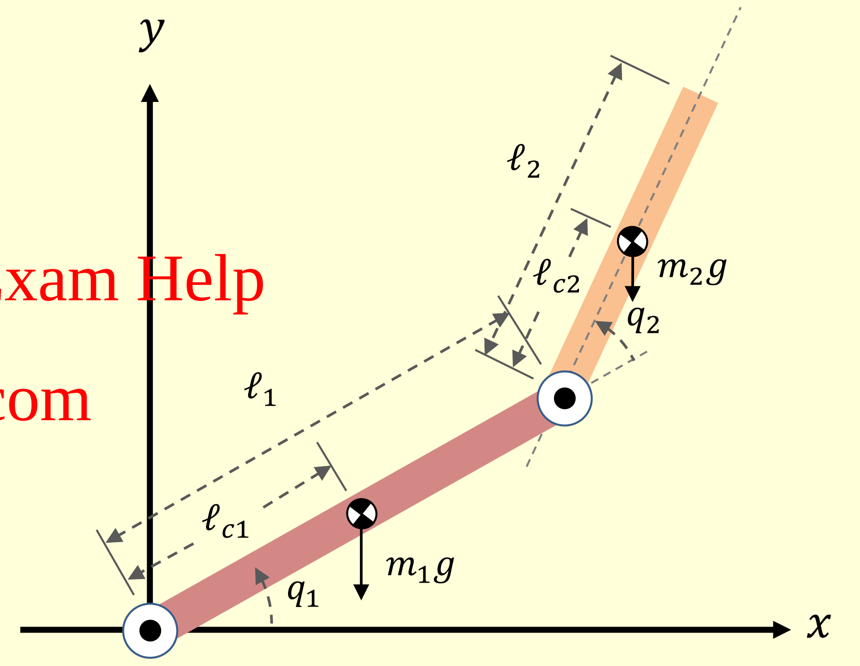
$$\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \frac{1}{2}\frac{\partial}{\partial \boldsymbol{q}}\left(\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}\right)$$

$$\boldsymbol{g}(\boldsymbol{q}) = \frac{\partial P(\boldsymbol{q})}{\partial \boldsymbol{q}}$$

Example: Dynamical model of a 2-DoF robot manipulator

For $i = 1,2$:

- q_i denotes the joint angle
- m_i denotes the mass of link i
- ℓ_i denotes the length of link i
- ℓ_{ci} denotes the distance from the previous joint to the centre of mass of link i
- I_i denotes the moment of inertia of link i about an axis coming out of the screen, passing through the centre of mass of link i.



Calculation of the **kinetic energy**:

The overall kinetic energy of the manipulato s is:

Linear velocity vector of the center of mass

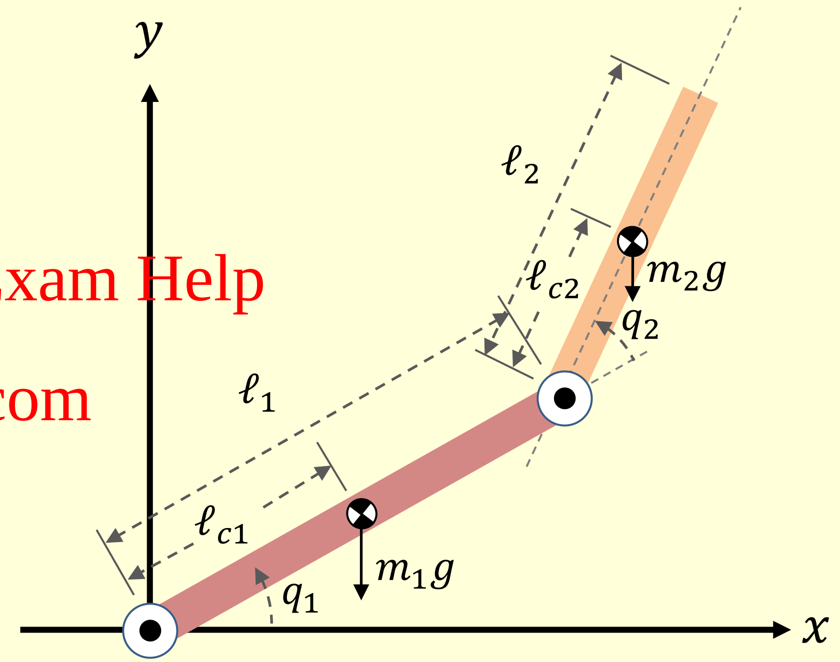
× 3 inertia matrix of link i

Angular velocity vector of link i

$$K = \sum_{i=1}^{n} \underbrace{\frac{1}{2} m_i \, \boldsymbol{v}_{ci}^T \boxed{\boldsymbol{v}_{ci}}} + \underbrace{\frac{1}{2} \boldsymbol{\omega}_i^T \boxed{\boldsymbol{I}_{ci}} \boxed{\boldsymbol{\omega}_i}}$$

kinetic energy due to the linear velocity of the links' centre of mass

kinetic energy due to the angular velocity of the links



We first calculate the ***positions of the centre of mass*** of links 1 & 2:

$$x_{c1} = \ell_{c1} \cos q_1$$
$$y_{c1} = \ell_{c1} \sin q_1$$

$$x_{c2} = \ell_1 \cos q_1 + \ell_{c2} \cos(q_1 + q_2)$$
$$y_{c2} = \ell_1 \sin q_1 + \ell_{c2} \sin(q_1 + q_2)$$

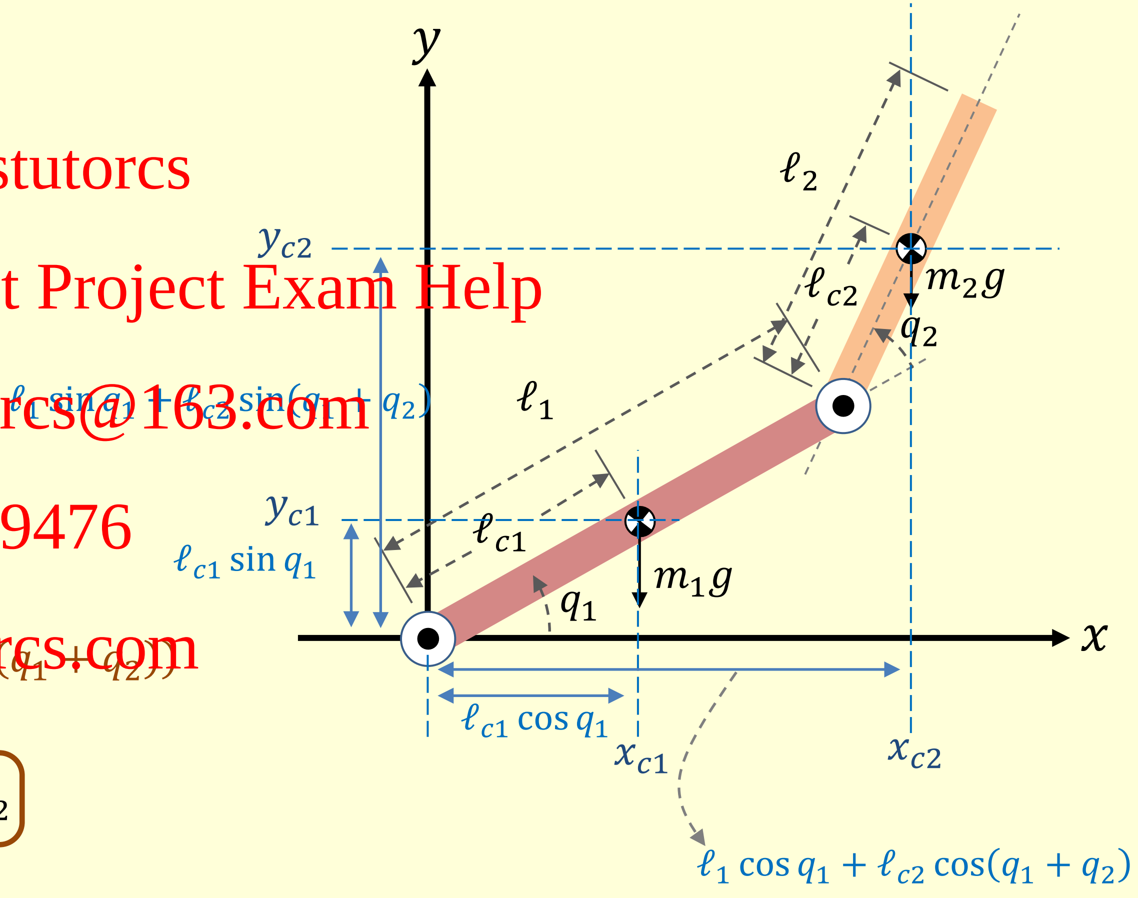
The ***velocities of the centre of mass*** of links 1 & 2:

$$\boldsymbol{v}_{c1} = \begin{bmatrix} \dot{x}_{c1} \\ \dot{y}_{c1} \end{bmatrix} = \begin{bmatrix} -\dot{q}_1 \ell_{c1} \sin q_1 \\ \dot{q}_1 \ell_{c1} \cos q_1 \end{bmatrix} \Longrightarrow \boxed{\boldsymbol{v}_{c1}^T \boldsymbol{v}_{c1} = \dot{q}_1^2 \ell_{c1}^2}$$

$$\boldsymbol{v}_{c2} = \begin{bmatrix} \dot{x}_{c2} \\ \dot{y}_{c2} \end{bmatrix} = \begin{bmatrix} -\dot{q}_1 \ell_1 \sin q_1 - (\dot{q}_1 + \dot{q}_2) \ell_{c2} \sin(q_1 + q_2) \\ \dot{q}_1 \ell_1 \cos q_1 + (\dot{q}_1 + \dot{q}_2) \ell_{c2} \cos(q_1 + q_2) \end{bmatrix}$$

$$\boldsymbol{v}_{c2}^T \boldsymbol{v}_{c2} = \dot{q}_1^2 \ell_1^2 + (\dot{q}_1 + \dot{q}_2)^2 \ell_{c2}^2 + 2\dot{q}_1 (\dot{q}_1 + \dot{q}_2) \ell_1 \ell_{c2} (\sin q_1 \sin(q_1 + q_2) + \cos q_1 \cos(q_1 + q_2))$$

$$\Longrightarrow \boxed{\boldsymbol{v}_{c2}^T \boldsymbol{v}_{c2} = \dot{q}_1^2 \ell_1^2 + (\dot{q}_1 + \dot{q}_2)^2 \ell_{c2}^2 + 2\dot{q}_1 (\dot{q}_1 + \dot{q}_2) \ell_1 \ell_{c2} \cos q_2}$$



$$K_1 = \frac{1}{2}\sum_{i=1}^{n} m_i\, \boldsymbol{v}_{ci}^T\, \boldsymbol{v}_{ci}$$

$$= \underbrace{\frac{1}{2} m_1 \dot{q}_1^2 \ell_{c1}^2}_{\text{Link 1}} + \underbrace{\frac{1}{2} m_2\left(\dot{q}_1^2 \ell_1^2 + (\dot{q}_1 + \dot{q}_2)^2 \ell_{c2}^2 + 2(\dot{q}_1^2 + \dot{q}_1\dot{q}_2)\ell_1 \ell_{c2}\cos q_2\right)}_{\text{Link 2}}$$

$$= \frac{1}{2}\begin{bmatrix}\dot{q}_1 & \dot{q}_2\end{bmatrix}\begin{bmatrix} m_1\ell_{c1}^2 + m_2(\ell_1^2 + \ell_{c2}^2 + 2\ell_1\ell_{c2}\cos q_2) & m_2(\ell_{c2}^2 + \ell_1\ell_{c2}\cos q_2) \\ m_2(\ell_{c2}^2 + \ell_1\ell_{c2}\cos q_2) & m_2\ell_{c2}^2 \end{bmatrix}\begin{bmatrix}\dot{q}_1 \\ \dot{q}_2\end{bmatrix}$$

$$K_2 = \frac{1}{2}\sum_{i=1}^{n} \boldsymbol{\omega}_i^T\, \boldsymbol{I}_{ci}\, \boldsymbol{\omega}_i$$

$$= \underbrace{\frac{1}{2}\dot{q}_1^2 I_1}_{\text{Link 1}} + \underbrace{\frac{1}{2}(\dot{q}_1 + \dot{q}_2)^2 I_2}_{\text{Link 2}}$$

$$= \frac{1}{2}\begin{bmatrix}\dot{q}_1 & \dot{q}_2\end{bmatrix}\begin{bmatrix} I_1 + I_2 & I_2 \\ I_2 & I_2\end{bmatrix}\begin{bmatrix}\dot{q}_1 \\ \dot{q}_2\end{bmatrix}$$

$$\boxed{K(\boldsymbol{q}, \dot{\boldsymbol{q}}) = K_1 + K_2 = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\, \underbrace{\dot{\boldsymbol{q}}}_{\begin{bmatrix}\dot{q}_1 \\ \dot{q}_2\end{bmatrix}}}$$

Note that the kinetic energy is a function of $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$. That is why we wrote is as $K(\boldsymbol{q}, \dot{\boldsymbol{q}})$

$$\boldsymbol{M}(\boldsymbol{q}) := \begin{bmatrix} m_1\ell_{c1}^2 + m_2(\ell_1^2 + \ell_{c2}^2 + 2\ell_1\ell_{c2}\cos q_2) + I_1 + I_2 & m_2(\ell_{c2}^2 + \ell_1\ell_{c2}\cos q_2) + I_2 \\ m_2(\ell_{c2}^2 + \ell_1\ell_{c2}\cos q_2) + I_2 & m_2\ell_{c2}^2 + I_2 \end{bmatrix}$$

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \underbrace{\dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \frac{1}{2}\frac{\partial}{\partial \boldsymbol{q}}(\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}})}_{\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}} \cdots \underbrace{\frac{\partial P(\boldsymbol{q})}{\partial \boldsymbol{q}}}_{\cdots(\boldsymbol{q})} = \cdots$$

$$\boldsymbol{M}(\boldsymbol{q}) := \begin{bmatrix} m_1\ell_{c1}^2 + m_2(\ell_1^2 + \ell_{c2}^2 + 2\ell_1\ell_{c2}\cos q_2) \cdots & \cdots_2(\ell_{c2}^2 + \ell_1\ell_{c2}\cos q_2) + I_2 \\ m_2(\ell_{c2}^2 + \ell_1\ell_{c2}\cos q_2) + I_2 & m_2\ell_{c2}^2 + I_2 \end{bmatrix}$$

$$\dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \begin{bmatrix} -2m_2\ell_1\ell_{c2}\dot{q}_2\sin q_2 & -m_2\ell_1\ell_{c2}\dot{q}_2\sin q_2 \\ -m_2\ell_1\ell_{c2}\dot{q}_2\sin q_2 & 0 \end{bmatrix}\begin{bmatrix}\dot{q}_1 \\ \dot{q}_2\end{bmatrix} = \begin{bmatrix} -m_2\ell_1\ell_{c2}\dot{q}_2(2\dot{q}_1 + \dot{q}_2)\sin q_2 \\ -m_2\ell_1\ell_{c2}\dot{q}_1\dot{q}_2\sin q_2 \end{bmatrix}$$

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial \boldsymbol{q}}(\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}) &= \frac{1}{2}\frac{\partial}{\partial \boldsymbol{q}}\Big((m_1\dot{q}_1^2\ell_{c1}^2) + m_2(\dot{q}_1^2\ell_1^2 + (\dot{q}_1 + \dot{q}_2)^2\ell_{c2}^2 + 2\dot{q}_1(\dot{q}_1 + \dot{q}_2)\ell_1\ell_{c2}\cos q_2) + \dot{q}_1^2 I_1 + (\dot{q}_1 + \dot{q}_2)^2 I_2\Big) \\ &= \frac{1}{2}\begin{bmatrix} 0 \\ -2m_2\ell_1\ell_{c2}\dot{q}_1(\dot{q}_1 + \dot{q}_2)\sin q_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -m_2\ell_1\ell_{c2}\dot{q}_1\sin q_2 & -m_2\ell_1\ell_{c2}\dot{q}_1\sin q_2 \end{bmatrix}\begin{bmatrix}\dot{q}_1 \\ \dot{q}_2\end{bmatrix}\end{aligned}$$

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \overbrace{\boxed{\dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}}} - \boxed{\frac{1}{2}\frac{\partial}{\partial \boldsymbol{q}}\left(\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}\right)}}^{\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}} + \overbrace{\frac{\partial P(\boldsymbol{q})}{\partial \boldsymbol{q}}}^{\boldsymbol{g}(\boldsymbol{q})} = \boldsymbol{\tau}$$

$$\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \begin{bmatrix} -m_2\ell_1\ell_{c2}\dot{q}_2(2\dot{q}_1 + \dot{q}_2)\sin q_2 \\ -m_2\ell_1\ell_{c2}\dot{q}_1\dot{q}_2 \sin q_2 \end{bmatrix} - \begin{bmatrix} 0 \\ -m_2\ell_1\ell_{c2}\dot{q}_1(\dot{q}_1 + \dot{q}_2)\sin q_2 \end{bmatrix}$$

$$= \begin{bmatrix} -m_2\ell_1\ell_{c2}\dot{q}_2(2\dot{q}_1 + \dot{q}_2)\sin q_2 \\ m_2\ell_1\ell_{c2}\dot{q}_1^2 \sin q_2 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -m_2\ell_1\ell_{c2}\dot{q}_2 \sin q_2 & -m_2\ell_1\ell_{c2}(\dot{q}_1 + \dot{q}_2)\sin q_2 \\ m_2\ell_1\ell_{c2}\dot{q}_1 \sin q_2 & 0 \end{bmatrix}}_{\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})} \underbrace{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}}_{\dot{\boldsymbol{q}}}$$

Note that we did not choose $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{bmatrix} -2m_2\ell_1\ell_{c2}\dot{q}_2 \sin q_2 & -m_2\ell_1\ell_{c2}\dot{q}_2 \sin q_2 \\ m_2\ell_1\ell_{c2}\dot{q}_1 \sin q_2 & 0 \end{bmatrix}$.

It is because we want $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})$ to satisfy conditions that will be explained later.

Calculation of the **potential energy**:

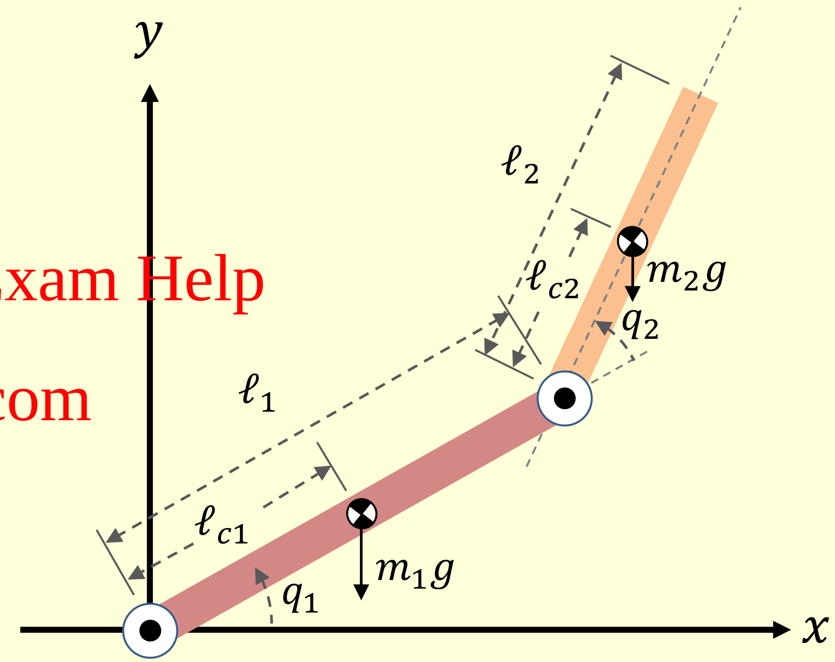
$$P_1 = m_1 g \ell_{c1} \sin q_1$$
$$P_2 = m_2 g(\ell_1 \sin q_1 + \ell_{c2} \sin(q_1 + q_2$$

$$P(\boldsymbol{q}) = P_1 + P_2$$

$$\Rightarrow \boxed{P(\boldsymbol{q}) = (m_1 g \ell_{c1} + m_2 g \ell_1) \sin q_1 + m_2 g \ell_{c2} \sin(q_1 + q_2)}$$

$\Downarrow$

Note that the potential energy is only a function of $\boldsymbol{q}$ (and not $\dot{\boldsymbol{q}}$). That is why we wrote is as $P(\boldsymbol{q})$.



$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \underbrace{\dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \frac{1}{2}\frac{\partial}{\partial \boldsymbol{q}}\left(\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}\right)}_{\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}} + \underbrace{\frac{\partial P(\boldsymbol{q})}{\partial \boldsymbol{q}}}_{\boldsymbol{g}(\boldsymbol{q})} = \boldsymbol{\tau}$$

$$\Rightarrow \boxed{\boldsymbol{g}(\boldsymbol{q}) = \frac{\partial P(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{bmatrix} \dfrac{\partial P(\boldsymbol{q})}{\partial q_1} \\ \dfrac{\partial P(\boldsymbol{q})}{\partial q_2} \end{bmatrix} = \begin{bmatrix} (m_1 g \ell_{c1} + m_2 g \ell_1)\cos q_1 + m_2 g \ell_{c2}\cos(q_1 + q_2) \\ m_2 g \ell_{c2}\cos(q_1 + q_2) \end{bmatrix}}$$

Summary of the 2-DOF robot dynamic equation

Dynamic equation: $\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{bmatrix} m_1\ell_{c1}^2 + m_2(\ell_1^2 + \ell_{c2}^2 + 2\ell_1\ell_{c2}\cos q_2) + I_1 + I_2 & m_2(\ell_{c2}^2 + \ell_1\ell_{c2}\cos q_2) + I_2 \\ m_2(\ell_{c2}^2 + \ell_1\ell_{c2}\cos q_2) + I_2 & m_2\ell_{c2}^2 + I_2 \end{bmatrix}$$

$$\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{bmatrix} -m_2\ell_1\ell_{c2}\dot{q}_2\sin q_2 & -m_2\ell_1\ell_{c2}(\dot{q}_1 + \dot{q}_2)\sin q_2 \\ m_2\ell_1\ell_{c2}\dot{q}_1\sin q_2 & 0 \end{bmatrix}$$

$$\boldsymbol{g}(\boldsymbol{q}) = \begin{bmatrix} (m_1 g\ell_{c1} + m_2 g\ell_1)\cos q_1 + m_2 g\ell_{c2}\cos(q_1 + q_2) \\ m_2 g\ell_{c2}\cos(q_1 + q_2) \end{bmatrix}$$

Properties of the Dynamic Model

Robot Manipulator Dynamics

$$\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

$\boldsymbol{q} \in \mathbb{R}^n$:	Vector of joint coordinates
$\boldsymbol{M}(\boldsymbol{q}) \in \mathbb{R}^{n\times n}$:	Inertia matrix
$\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} \in \mathbb{R}^n$:	Vector of Coriolis and centrifugal forces
$\boldsymbol{g}(\boldsymbol{q}) \in \mathbb{R}^n$:	Vector of gravity forces
$\boldsymbol{\tau} \in \mathbb{R}^n$:	Vector of joint torques

Robot Manipulator Dynamics

$$\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

The properties of robot manipulator dynamics are extensively used in **control design** and **stability analysis**.

Inertia matrix $M(q)$

$M(q)$

- The inertia matrix is **symmet** **itive definite**.

A square matrix is **symmetric** if it is equal to its transpose ($\boldsymbol{A} = \boldsymbol{A}^\top$).

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 3 & 4 \\ -5 & 4 & 2 \end{bmatrix}$$

A square matrix is **positive definite** if all its eigenvalues have positive real parts.

Example of a commonly use Lyapunov function:

$$V(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \underbrace{\boxed{\frac{1}{2}\, \dot{\boldsymbol{q}}^\top \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}}}}_{K(\boldsymbol{q}, \dot{\boldsymbol{q}})} + \cdots$$

Inertia matrix $\boldsymbol{M}(\boldsymbol{q})$

$\boldsymbol{M}(\boldsymbol{q})$

- The inertia matrix is **symmetric and positive definite**.

- Rayleigh-Ritz inequality:

$$\lambda_{min}(\boldsymbol{M}(\boldsymbol{q}))\,\boldsymbol{I} \le \boldsymbol{M}(\boldsymbol{q}) \le \lambda_{max}(\boldsymbol{M}(\boldsymbol{q}))\,\boldsymbol{I}$$

- There exists $\alpha > 0$ and $\beta > 0$ such that

$$\alpha\,\boldsymbol{I} \le \boxed{\boldsymbol{M}(\boldsymbol{q}) \le \beta\,\boldsymbol{I}}$$

Only for robots with revolute joints

Centrifugal and Coriolis Forces Matrix $C(q,\dot{q})$

$C(q,\dot{q})$

- For a given manipulator, the matrix $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})$ may not be unique but the **vector** $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ is unique.

Example:

$$\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \begin{bmatrix} -m_2\ell_1\ell_{c2}\dot{q}_2(2\dot{q}_1+\dot{q}_2)\sin q_2 \\ m_2\ell_1\ell_{c2}\dot{q}_1^2\sin q_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -m_2\ell_1\ell_{c2}\dot{q}_2\sin q_2 & -m_2\ell_1\ell_{c2}(\dot{q}_1+\dot{q}_2)\sin q_2 \\ m_2\ell_1\ell_{c2}\dot{q}_1\sin q_2 & 0 \end{bmatrix}}_{\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})} \underbrace{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}}_{\dot{\boldsymbol{q}}}$$

Another choice for $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})$: $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{bmatrix} -2m_2\ell_1\ell_{c2}\dot{q}_2\sin q_2 & -m_2\ell_1\ell_{c2}\dot{q}_2\sin q_2 \\ m_2\ell_1\ell_{c2}\dot{q}_1\sin q_2 & 0 \end{bmatrix}$

Centrifugal and Coriolis Forces Matrix $C(q, \dot{q})$

$C(q, \dot{q})$

- For a given manipulator, the $, \dot{q})$ may not be unique but the **vector** $C(q, \dot{q})\dot{q}$ is unique.
- The matrix $\dot{M}(q) - 2C(q, \dot{q})$ is **skew symmetric** for a particular choice of $C(q, \dot{q})$ (which is always possible).

A square matrix is **skew symmetric** if it is equal to *the negative of* its transpose ($A = -A^\top$).

$$A = \begin{bmatrix} 0 & 1 & 5 \\ -1 & 0 & -4 \\ -5 & 4 & 0 \end{bmatrix}$$

If $A \in \mathbb{R}^{n \times n}$ is a **skew symmetric** matrix, then

$$z^\top A z = 0$$

for any vector $z \in \mathbb{R}^n$.

Centrifugal and Coriolis Forces Matrix $C(q, \dot{q})$

$C(q, \dot{q})$

- For a given manipulator, the $\ldots, \dot{q})$ may not be unique but the **vector** $C(q, \dot{q})\dot{q}$ is unique.
- The matrix $\dot{M}(q) - 2C(q, \dot{q})$ is **skew symmetric** for a particular choice of $C(q, \dot{q})$ (which is always possible).
- For robots with revolute joints, there exists $k_c > 0$ such that $\|C(q, \dot{q})\dot{q}\| \leq k_c \|\dot{q}\|^2$.

Gravity vector $g(q)$

$\boldsymbol{g(q)}$

For robots that only have revolu

- There exists a constant $k >$ $\|\boldsymbol{g}(\boldsymbol{q})\| \leq k$ for all $\boldsymbol{q} \in \mathbb{R}^n$.
- There exists a constant $k_g > 0$ such that $\|\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{y})\| \leq k_g \|\boldsymbol{x} - \boldsymbol{y}\|$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

Summary

- There exists $\alpha > 0$ and $\beta > 0$ such tha

$$\alpha \boldsymbol{I} \leq \boldsymbol{M}$$

- There exists $k_c > 0$ such that $\|\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}\| \leq k_c \|\dot{\boldsymbol{q}}\|^2$
- There exists a constant $k > 0$ such that $\|\boldsymbol{g}(\boldsymbol{q})\| \leq k$ for all $\boldsymbol{q} \in \mathbb{R}^n$.

Only for robots with revolute joints

- The inertia matrix is **symmetric** and **positive definite**.
- The matrix $\dot{\boldsymbol{M}}(\boldsymbol{q}) - 2\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is **skew symmetric** for a particular choice of $\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ (which is always possible).

- Linearity in the dynamic parameters