Problem 6.1 Spliffy!

(a) Use an integrating factor, $p(x)$, to make $\mathcal{D}_x y = y'' + 2x\, y' + (x^2+1)\, y$ self-adjoint. (b) Show that the resulting differential operator, $\tilde{\mathcal{D}}_x \equiv p(x)\mathcal{D}_x y$, is self-adjoint, $\langle w \mid \tilde{\mathcal{D}}_x y \rangle = \langle \tilde{\mathcal{D}}_x w \mid y \rangle$, by integrating by parts twice. Assume that the Wronskian is $W(w, y) \equiv \begin{vmatrix} w & y \\ w' & y' \end{vmatrix} = wy' - w'y = A/p$. [**Hint:** The goal of integrating by parts is to [illegible]ves of y in the integrals.]

Solution

(a) I have $p_0 = 1$ and [illegible]d the equation is not self-adjoint. The integrating factor is

$$p(x) = \exp\left(\int^x du\, \frac{p_1(u)}{p_0(u)}\right) = \exp\left(\int^x du\, \frac{2u}{1}\right) = \exp\left(x^2\right)$$

so the integrating factor is

$$\frac{p(x)}{p_0(x)} = \frac{\exp(x^2)}{1} = \exp\left(x^2\right). \quad \boxed{\checkmark\ 2}$$

Multiplying $\mathcal{D}_x y$ by this integrating factor,

$$\tilde{\mathcal{D}}_x y = \mathrm{e}^{x^2}\mathcal{D}_x y = \mathrm{e}^{x^2} y'' + 2x\,\mathrm{e}^{x^2} y' + (x^2+1)\,\mathrm{e}^{x^2} y = (\,\mathrm{e}^{x^2} y')' + (x^2+1)\,\mathrm{e}^{x^2} y \quad \boxed{\checkmark\ 2}$$

which is self-adjoint.

(b) Starting with the self-adjoint form

$$\langle w \mid \tilde{\mathcal{D}}_x y \rangle = \int dx\, w(\,\mathrm{e}^{x^2} y')' + \int dx\, w(x^2+1)\,\mathrm{e}^{x^2} y$$

integrate by parts with $u = w$ and $dv = dx\,(\,\mathrm{e}^{x^2} y')'$ so that $du = dx\, w'$ and $v = (\,\mathrm{e}^{x^2} y')$

$$\langle w \mid \tilde{\mathcal{D}}_x y \rangle = \left[w(\,\mathrm{e}^{x^2} y')\right] - \int dx\, w'(\,\mathrm{e}^{x^2} y') + \int dx\, w(x^2+1)\,\mathrm{e}^{x^2} y \quad \boxed{\checkmark\ 2}$$

and again with $u = w'\,\mathrm{e}^{x^2}$ and $dv = dx\, y'$ so that $du = dx\,(\,\mathrm{e}^{x^2} w')'$ and $v = y$

$$\begin{aligned} \langle w \mid \tilde{\mathcal{D}}_x y \rangle &= \left[w\,\mathrm{e}^{x^2} y'\right] - \left[w'\,\mathrm{e}^{x^2} y\right] + \int dx\,(\,\mathrm{e}^{x^2} w')' y + \int dx\, w(x^2+1)\,\mathrm{e}^{x^2} y \\ &= \left[\mathrm{e}^{x^2}(wy' - w'y)\right] + \int dx\,\Big((\,\mathrm{e}^{x^2} w')' + (x^2+1)\,\mathrm{e}^{x^2} w\Big) y \quad \boxed{\checkmark\ 2} \end{aligned}$$

I see that the Wronskian has appeared, $W = (wy' - w'y) = A/p = A/\,\mathrm{e}^{x^2}$, so the boundary term is zero, $\boxed{\checkmark\ 2}$ leaving

$$\langle w \mid \tilde{\mathcal{D}}_x y \rangle = \langle \tilde{\mathcal{D}}_x w \mid y \rangle,$$

so the linear differential operator

$$\tilde{\mathcal{D}}_x \equiv \mathrm{e}^{x^2}\mathcal{D}_x = \mathrm{e}^{x^2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}}{\mathrm{d}x} + (x^2+1)\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\right) + \mathrm{e}^{x^2}(x^2+1)$$

is self-adjoint as required.

I note that

$$
\begin{aligned}
(e^{x^2/2} y)'' &= (e^{x^2/2} y' + x e^{x^2/2} y)' \\
&= e^{x^2/2} y'' + x e^{x^2/2} y' + (e^{x^2/2} + x^2 e^{x^2/2}) y + x e^{x^2/2} y' \\
&= e^{x^2/2} (y'' + xy' + (x^2+1) y) \\
&= e^{x^2/2} \mathcal{D}_x y
\end{aligned}
$$

so I can integrate twice t [illegible] this ODE,

$$(\text{[illegible]} \; e^{-x^2/2}$$

where a and b are constants. ♦

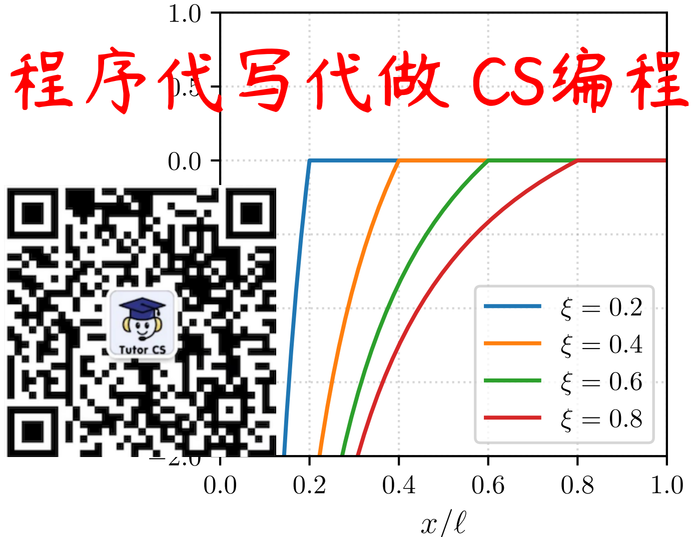

Figure 6.1: A plot of $g(x,\xi)$, Green's function for section 6.2(d). This is the solution for $\left(x^2 g'\right)' = \delta$.

Problem 6.2 Self Adjoint and Green

In this problem you will construct Green's function for the differential operator

$$\mathcal{D}_x\, y(x) = x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} = \frac{d}{dx}\left(x^2 \frac{dy}{dx}\right)$$

with boundary conditions

$$y(\ell) = y'(\ell) = 0.$$

(a) In order to find Green's function for this problem you need to solve

$$\mathcal{D}_x\, g(x,\xi) = \delta(x-\xi).$$

Solve the homogeneous version of this ODE for $x \neq \xi$ for the two cases

$$g(x,\xi) = \begin{cases} g_<(x,\xi), & x < \xi \\ g_>(x,\xi), & x > \xi. \end{cases}$$

For each case you should have two integration constants which are determined by the boundary conditions. Each of these four integration "constants" will be functions of ξ. (b) Apply the four boundary conditions, the usual two given with the original problem (here $y(\ell) = y'(\ell) = 0$) as well as the two at $x = \xi$, to find Green's function, $g(x,\xi)$, for this problem. (c) Show that your Green's function, $g(x,\xi)$, satisfies the boundary conditions. (d) Plot $g(x,\xi)$ as a function of x. Identify the point $x = \xi$ on your plot.

Solution

(a) For $x \neq \xi$ I can solve the homogeneous ODE:

$$\left(x^2 g_{\lessgtr}'(x,\xi)\right)' = 0$$

$$
\begin{aligned}
x^2 g'_{\lessgtr}(x,\xi) &= \alpha_{\lessgtr}(\xi) \\
g'_{\lessgtr}(x,\xi) &= \frac{1}{x^2}\alpha_{\lessgtr}(\xi) \\
g_{\lessgtr}(x,\xi) &= -\frac{1}{x}\alpha_{\lessgtr}(\xi) + \beta_{\lessgtr}(\xi), \quad \boxed{\checkmark\, 2}
\end{aligned}
$$

and I have four int[illegible]hat are functions of ξ (two each of $\alpha_{\lessgtr}$ and $\beta_{\lessgtr}$) as required.

(b) I have $x > \xi$ for bo[illegible]ns,

$$
\begin{aligned}
g_>(\ell,\xi) &= 0 = -\frac{1}{\ell}\alpha_>(\xi) + \beta_>(\xi) \\
g'_>(\ell,\xi) &= 0 = \frac{1}{\ell^2}\alpha_>(\xi) \\
\Rightarrow \alpha_>(\xi) &= \beta_>(\xi) = 0, \quad \boxed{\checkmark\, 2}
\end{aligned}
$$

and $g_>(x,\xi) = 0$.

At $x = \xi$ I have (with $p(x) = x^2$)

$$
\begin{aligned}
\cancelto{0}{g_>(x,x)} - g_<(x,x) &= \frac{1}{x}\alpha_<(x) - \beta_<(x) = 0 \\
\cancelto{0}{g'_>(x,x)} - g'_<(x,x) &= -\frac{1}{x^2}\alpha_<(x) = \frac{-1}{p} = \frac{-1}{x^2} \\
\Rightarrow \quad \alpha_<(x) &= 1 \quad \boxed{\checkmark\, 1} \\
\& \quad \beta_<(x) &= \frac{1}{x}, \quad \boxed{\checkmark\, 1}
\end{aligned}
$$

and I know the four unknown functions.

I have $g_>(x,\xi) = 0$ and

$$
g_<(x,\xi) = -\frac{1}{x} + \frac{1}{\xi}
$$

so

$$
g(x,\xi) = \left(\frac{1}{\xi} - \frac{1}{x}\right) H(\xi - x). \quad \boxed{\checkmark\, 2}
$$

(c) At $x = \ell$ I have

$$
\begin{aligned}
g(\ell,\xi) &= \left(\frac{1}{\xi} - \frac{1}{\ell}\right) \cancelto{0,\ 0<\xi<\ell}{H(\xi - \ell)} \\
&= 0 \quad \boxed{\checkmark\, 1} \\
g'(x,\xi) &= -\left(\frac{1}{\xi} + \frac{1}{x^2}\right)\delta(\xi - x) \\
g'(\ell,\xi) &= -\left(\frac{1}{\xi} + \frac{1}{\ell^2}\right)\cancelto{0,\ 0<\xi<\ell}{\delta(\xi - \ell)} \\
&= 0 \quad \boxed{\checkmark\, 1}
\end{aligned}
$$

so the boundary conditions are satisfied.

(d) This code generates Figure 6.1 ✓ 2

```
import numpy as np

#constants
N = 1000
ell = 1
epsilon = ell / N                  e by zero

x = np.linspace(

# plot the result
from matplotlib.p
fig = figure( )
ax1 = fig.subplot      )

for xi in np.array( [ 0.2, 0.4, 0.6, 0.8 ] ) * ell :
    # Green's function, g(x,ξ)
    g = ( 1 / xi - 1 / x ) * ( x < xi )
    ax1.plot( x / ell, g, label=r'$\xi = {}$'.format( xi ) )

# make the plot presentable
ax1.set_xlabel( r'$ x / \ell $' )
ax1.set_ylabel( r'$ g( x, \xi ) $' )
ax1.axis( [ 0, 1, -2, 1 ] )
ax1.legend( )
fig.savefig( 'Green1' )
```

♦

Problem 6.3 Driven Exponential

In this problem you will construct Green's function, $g(t, t')$, to solve the problem

$$\mathcal{D}_t y(t) = \left(\frac{d^2}{dt^2} - a^2\right) y(t) = f(t),$$

with $f(t) = 0$ for $t < 0$, … initial (boundary) conditions

$$y(0) = \dot{y}(0) = 0.$$

Green's function, $g(t$ …

$$\mathcal{D}_t g(t, t') = \delta(t - t'),$$

where $\delta(t - t')$ is Dirac's delta function. (a) First, solve the ODE for the $t \neq t'$ case

$$\mathcal{D}_t\, g_{\gtrless}(t, t') = 0, \quad t \neq t'$$

where

$$g_{\gtrless}(t, t') = \begin{cases} g_<(t, t'), & t < t' \\ g_>(t, t'), & t > t'. \end{cases}$$

by integration. [**Hint:** This is not an harmonic oscillator but the answer is still exponentially easy. Also, you should have four unknown functions of t'.] (b) Next, apply the boundary conditions,

$$y(0) = \dot{y}(0) = 0.$$

[**Hint:** You should have two conditions for the four unknown functions of t'.] (c) Next, apply the conditions at $t = t'$,

$$g_<(t, t) - g_>(t, t) = 0$$
$$\dot{g}_<(t, t) - \dot{g}_>(t, t) = -1/p(t),$$

where the self-adjoint ODE is

$$\mathcal{D}_t y = \frac{d}{dt}\left(p \frac{dy}{dt}\right) + q y,$$

in which p, q, and y are functions of t. [**Hint:** You should have two further conditions for the four unknown functions of t'. You should now be able to find these four functions.] (d) What is Green's function, $g(t, t')$, for this problem? (e) Find $y(t)$ if

$$f(t) = f_\circ \frac{H(t) - H(t - b)}{b} = \begin{cases} f_\circ / b & 0 < t < b \\ 0, & \text{otherwise} \end{cases}.$$

(f) Plot your analytic solution for $f(t)$ along with a numeric solution. (g) Find Green's function again, but this time using a Laplace transform.

Solution

(a)

$$\frac{d^2}{dt^2} g_{\gtrless}(t,t') = a^2 g_{\gtrless}$$

$$t') = \alpha_{\gtrless}(t')e^{+at} + \beta_{\gtrless}(t')e^{-at} \quad \boxed{\checkmark\ 2}$$

My four unknown $\beta_{\gtrless}$

(b) Both boundary co),

$$, t') = 0 = \alpha_<(t') + \beta_<(t')$$

$$\dot{g}_<(0,t') = 0 = a\alpha_<(t') - a\beta_<(t')$$

$$\Rightarrow \alpha_<(t') = \beta_<(t') = 0 \quad \boxed{\checkmark\ 2}$$

so

$$g_<(t,t') = 0.$$

(c) Now for the $t = t'$ "boundary" conditions,

$$g_>|_{t=t'} - g_<|_{t=t'} = 0$$

$$\Rightarrow \alpha_>(t')\, e^{+at'} + \beta_>(t')\, e^{-at'} = 0 \quad \boxed{\checkmark\ 2}$$

I have $p(t) = 1$ so

$$\frac{d}{dt} g_>|_{t=t'} - \frac{d}{dt}\cancelto{0}{g_<}|_{t=t'} = -1$$

$$-a\alpha_>(t')\, e^{+at'} + a\beta_>(t')\, e^{-at'} = -1$$

$$\alpha_>(t')\, e^{+at'} - \beta_>(t')\, e^{-at'} = 1/a \quad \boxed{\checkmark\ 2}$$

adding and subtracting I get

$$\alpha_>(t') = \frac{1}{2a} e^{-at'}$$

$$\beta_>(t') = -\frac{1}{2a} e^{+at'} \quad \boxed{\checkmark\ 2}$$

(d) I have

$$g_{\gtrless}(t,t') = \alpha_{\gtrless}(t')e^{+at} + \beta_{\gtrless}(t')e^{-at}$$

$$\Rightarrow g(t,t') = \left(\frac{1}{2a}e^{-at'}e^{+at} - \frac{1}{2a}e^{+at'}e^{-at}\right) H(t-t')$$

$$= \frac{1}{2a}\left(e^{+a(t-t')} - e^{-a(t-t')}\right) H(t-t')$$

$$= \frac{1}{a}\sinh\left(a(t-t')\right) H(t-t') \quad \boxed{\checkmark\ 2}$$

(e) The answer is given by the convolution

$$
\begin{aligned}
y(t) = (f * g)(t) &= \int_{-\infty}^{\infty} dt'\, f(t-t')\, g(t') \\
&= \frac{f_o}{ab} \int_{-\infty}^{\infty} dt' \big(H(t-t') - H(t-t'-b)\big) H(t') \sinh\big(at'\big) \\
&= \frac{f_o}{ab} \int_{0}^{\infty} dt' \big(H(t-t') - H(t-t'-b)\big) \sinh\big(at'\big) \quad \boxed{✓\ 2}
\end{aligned}
$$

I can integrate ove[illegible]ions using $\frac{d}{dx} H(x) = \delta(x)$ by first integrating by parts, $\int u\, dv = uv - \int v\, du$, givin[illegible]

$$
\begin{aligned}
u = H(t-t') \quad &\Rightarrow \quad du = -dt'\,\delta(t-t') \\
dv = dt' \sinh\big(at'\big) \quad &\Rightarrow \quad v = \tfrac{1}{a} \cosh\big(at'\big)
\end{aligned}
$$

so that

$$
\begin{aligned}
y(t) =& \frac{f_o}{a^2 b} \left[H(t-t') \cosh\big(at'\big) - H(t-t'-b) \cosh\big(at'\big)\right]_0^{\infty} \\
&- \frac{f_o}{a^2 b} \int_0^{\infty} dt' \cosh\big(at'\big) \big(-\delta(t-t') + \delta(t-t'-b)\big) \\
=& -\frac{f_o}{a^2 b} (H(t) - H(t-b)) - \frac{f_o}{a^2 b} (-H(t) \cosh(at) + H(t-b) \cosh(a(t-b))) \\
=& -\frac{f_o}{a^2 b} H(t)(1 - \cosh(at)) + \frac{f_o}{a^2 b} H(t-b)(1 - \cosh(a(t-b))) \quad \boxed{✓\ 2}
\end{aligned}
$$

(f) See Code 6.1 and Figure 6.2. $\boxed{✓\ 2}$

(g) This is easy.

$$
\begin{aligned}
\ddot{y} - a^2 y &= f \\
\big(s^2 Y - s\cancel{y(0)} - \cancel{\dot{y}(0)}\big) - a^2 Y &= F \\
Y &= \frac{F}{s^2 - a^2} FG \\
G &= \frac{1}{s^2 - a^2}
\end{aligned}
$$

This is on the list of Laplace transforms, **??**, so

$$
g(t) = \tfrac{1}{a} \sinh(at). \quad \boxed{✓\ 2}
$$

Because $g(t) = 0$ for $t < 0$ is implicit for problems solved with Laplace transforms this is the same result in part (d). This hints at the methods that will be examined in the next chapter when I introduce convolutions.

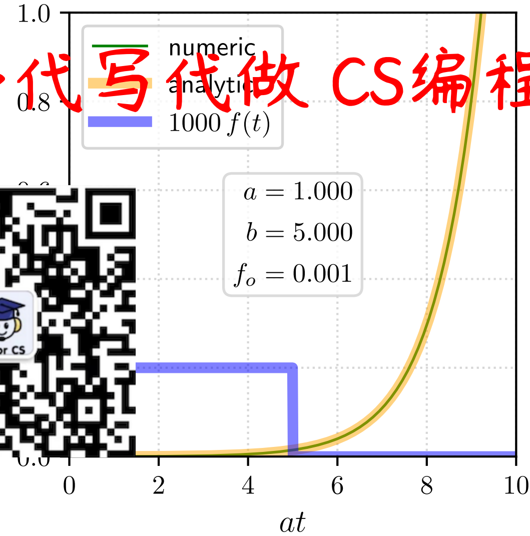

Figure 6.2: As a check on the analytic answer for section 6.3 I have plotted both the analytic and numerical solutions together on the same graph using the code in Code 6.1. Because the response, $y(t)$, grows exponentially I have made the initial impulse, f_o, tiny.

Code 6.1 Green's Function

Code to solve section 6.3 numerically and generate Figure 6.2.


```
import numpy as np
import matplotlib.pyplot as plt
from scipy.constants import pi, golden
from scipy.integrate import odeint  # integrates a system of ODEs

#################### physical constants
a  = 1.             # natural "frequency"
b  = 5.             # impulse duration
fo = .001           # impulse strength
#################### computational constants
N = 1000            # N = number of time steps per period
n = 2               # n = number of oscillations
#################### calculated constants
T = 2*pi / a                    # period
tMax = n * T                    # maximum time
Dt = T / N                      # time step -- the subinterval length
t = np.arange( n*N ) * Dt       # time steps: t = 0, dt, 2dt, ..., nT − dt
#################### state vector
y0, ydot0 = 0, 0                # initial conditions
psi0 = y0, ydot0                # the state vector at t = 0, ψ⃗(0)

################################# shorthand for Heaviside to make code more readable
def H( t ) :
    return np.heaviside( t, 0 )

################################# driving function
```

```
def f( t ) :
    return fo / b * ( H( t ) - H( t - b ) )

################################# this function returns the time derivative of the state-vector
def psiPrime( psi, t ) :          # returns ψ⃗
    y, ydot = psi                 # unpack ψ⃗ = [y, ẏ]
    yddot = + a * a * [illegible]s Newton's second law for this problem, ÿ = F/m
    psiDot = ydot, y[illegible] = [ẏ, ÿ]
    return psiDot     [illegible]me-derivative of the state-vector

####################[illegible]ate
y, ydot     = odeint( [illegible].T

####################[illegible]ic solution
x = -fo / (a*a*b) * ( [illegible]
    H(t) * ( 1 - np.cosh( 1*a*t ) )
    -
    H( t - b ) * ( 1 - np.cosh( 1*a*( t - b ) ) )
)

# create the figure object and the axis child object
fig = plt.figure()
ax1 = fig.add_subplot( 1, 1, 1 )

# and plot the results
ax1.plot( t*a, y, label="numeric", linewidth=1, color='green' )
ax1.plot( t*a, x, label="analytic", alpha=0.5, linewidth=4, color='orange' )
ax1.plot( t*a, 1000*f(t), label='$1000\;f(t)$', alpha=0.5, linewidth=4, color='blue' )

# next, add the parameters used to the figure
parameters = r'\begin{align*} a &= %4.3f \\ b &= %4.3f \\ f_o &= %4.3f \end{align*}' % (a, b, fo)
properties = dict(boxstyle='round', facecolor='white', alpha=0.8, edgecolor='lightgray', pad=.33)
ax1.text(5, .5, parameters, bbox=properties, ha='center', va='center')

# make the plot presentable
ax1.legend( )
ax1.set_xlabel( '$a t$' )
ax1.set_ylabel( '$y$' )
ax1.axis( [ 0, 10, 0, 1 ] )
```

♦