Bang-Bang principle for controls in a simplex

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Definition. Let $I \subset \mathbb{R}$ be an interval, $A \in L_1(I, M_{n\times n})$. *Fundamental matrix* of the linear system

$$\dot{x}(t) = A(t)x(t) \tag{1}$$

is any matrix function $\Phi : I \to M_n$, whose columns are linearly independent solutions to the system.

Remark 1. All solutions are expressible in terms of Φ,

$$x(t) = \sum\nolimits_i \alpha_i \Phi_i(t) = \Phi(t)\alpha, \quad \alpha \in \mathbb{R}^n.$$

Provided $\det \Phi(t) \neq 0$ for all $t \in I$, each column of $\Phi(t)$ separately is a solution to (1) iff $\dot{\Phi}(t) = A(t)\Phi(t)$. That is, the fundamental matrices of (1) are determined by

$$\dot{\Phi}(t) = A(t)\Phi(t).$$

Everywhere below we assume $\Phi(t_0) = E$; then the vector α above is actually the initial condition $\alpha = x(t_0)$.

In this sense the matrix $\Phi(t)$ „walks“ the solution *forward* as time passes from t_0 to t.

Remark 2. It is natural to expect $\Phi^{-1}(t)$ to walk the solution *backward,* i.e. to solve the system with $(-A)$ (oppositve velocity $\dot{x}$). Indeed, differentiating $\Phi^{-1}\Phi = E$ we get $\frac{d\Phi^{-1}}{dt}\Phi + \Phi^{-1}\dot{\Phi} = 0$, from where $\frac{d\Phi^{-1}}{dt}\Phi = -\Phi^{-1}A\Phi$ and

$$\frac{d\Phi^{-1}}{dt} = -\Phi^{-1}A. \tag{2}$$

Remark 3. Below at each point in time $t \in I$ the state of the system is a vector $x(t) \in \mathbb{R}^{\boldsymbol{n}}$, and the control is a vector $u(t) \in \mathbb{R}^{\boldsymbol{m}}$.

Problem. Let $A \in L_1(I, M_{n\times n})$ and $B \in L_1(I, M_{n\times m})$. Solve the system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

in terms of the fundamental matrix $\Phi(t)$ of $\dot{x}(t) = A(t)x(t)$.

Solution (wiki books: Control Systems/Linear System Solutions)**.** Rewrite $\dot{x} - Ax = Bu$, multiply by $\Phi^{-1}(t)$, and integrate:

$$\begin{aligned}
\Phi^{-1}\dot{x} + \Phi^{-1}(-A)\,x &= \Phi^{-1}Bu \\
\frac{d}{dt}\left(\Phi^{-1}x\right) &= \Phi^{-1}Bu \quad \text{(by (2))} \\
\Phi^{-1}x &= x(t_0) + \int_{t_0}^{t} \Phi^{-1}(s)B(s)u(s)\,\mathrm{ds},
\end{aligned}$$

from where

$$x(t) = \Phi(t)x(t_0) + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)B(s)u(s)\,\mathrm{ds}\,.$$

Below for convenience we will assume $x(t_0) = 0$, in which case $x(t) = \int_{t_0}^{t} \Phi^{-1}(s)B(s)u(s)\,\mathrm{ds}\,.$

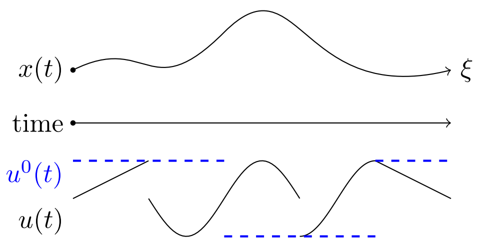

Figure 1: An extremal control u^0 leading to the same ξ as the u control.

Theorem (*The* Bang-Bang *principle*)**.** *Let $C \subset \mathbb{R}^m$ be a convex compact (of allowed control values). For the system*

$$\begin{aligned} \dot{x} &= A(t)x(t) + B(t)u(t) \\ x(0) &= \vec{0} \end{aligned}$$

every point accessible with $\underline{\text{some}}$ control $u : I \to C$ in time t is accessible with an $\underline{\text{extremal}}$ control $u^0 : I \to \operatorname{Ext} C$ as well.

We won't prove it in full generality, but only in the special case when the allowed control values form the unit ball (Holmes) or a *simplex* (Hermes).

Theorem 1 (Holmes)**.** *Let $Y \in L_1^{loc}(I, M_{n\times m})$ have columns Y_i. For each control $u \in \Psi$ where*

$$\Psi := L^\infty\left(I, \{\|x\|_\infty \le 1\}\right) = \left(\Psi'\right)^m$$

$$\Psi' := L^\infty\left(I, [-1, 1]\right),$$

there exists $v \in \Psi$ with $\operatorname{Im} v \subseteq \{-1, 1\}^m$ s.t. $\int_I Y \cdot u = \int_I Y \cdot v$ (here $Y \cdot u$ denotes matrix-vector multiplication).

Proof. We want $\int_I Y \cdot u = \int_I Y \cdot v$, but with$|v_j(t)| \overset{\text{a.e.}}{=} 1$. For $j = 1, \ldots, m$ we are looking for a partition $I = B_j \cup \overline{B}_j$ with $\begin{array}{l} v_j\big|_{B_j} \equiv +1 \\ v_j\big|_{\overline{B}_j} \equiv -1 \end{array}$. Equivalently,

$$\begin{aligned} \int_I u_j Y_j = \int_I v_j Y_j &= \int_{B_j} Y_j - \int_{\overline{B}_j} Y_j \\ &= \int_{B_j} Y_j - \left(\int_I Y_j - \int_{B_j} Y_j \right) \\ &= 2\int_{B_j} Y_j - \int_I Y_j, \end{aligned}$$

that is,

$$\int_{B_j} Y_j = \frac{1}{2}\int_I (u_j + 1)\, Y_j. \tag{3}$$

We want to prove that to each $u_j \in \Psi'$ there corresponds a measurable $B_j \subseteq I$, satisfying the last equality. To express more clearly the relationship between the given u_j and the B_j looked for, we introduce the mappings, defining their part in the respective sides in the equality.

- For B_j this is the vector measure

  $$\vec{\mu}_j(E) := \int_E Y_j, \quad E \in \mathcal{L}_I,$$

  where $\mathcal{L}_I = \{E \subset I : E \text{ is measurable}\}$.

- For u_j this is the affine operator $T_j : \Psi' \to \mathbb{R}^n$,

  $$T_j(y) := \frac{1}{2}\int_I (y(t) + 1)\, \mathrm{d}\vec{\mu}_j, \quad y \in \Psi'.$$

  T_j is $w^\star$-continuous, since for its linear part

  $$L_\infty \ni y \mapsto \int_I yY_j \le \|y\|_\infty\, \vec{\mu}_j(I) \xrightarrow[\|y\|\to\infty]{} 0.$$

Thus the equality (3) can be expressed with the formula $\vec{\mu}_j(B_j) = Tu_j$, and the statement we want to prove becomes the inclusion

$$T_j\Psi' \subset \vec{\mu}_j(\mathcal{L}_I).$$

By Lyapunov's theorem the RHS $\vec{\mu}_j(\mathcal{L}_I)$ is a convex compact, and since Ψ' is obviously convex and T_j is linear, $T_j\Psi'$ is convex as well. Then it suffices to show $T_j\,(\mathrm{Ext}\,\Psi') \subset \vec{\mu}_j(\mathcal{L}_I)$.

Let $u \in \mathrm{Ext}\,\Psi'$, then $u = \chi_E$ for some $E \subset I$ and

$$\begin{aligned} T_j u &= \frac{1}{2}\int_I (\chi_E + 1)\, Y_j \\ &= \int_E Y_j + \frac{1}{2}\int_{\overline{E}} Y_j \\ &= \vec{\mu}_j(E) + \frac{1}{2}\vec{\mu}_j(\overline{E}). \end{aligned}$$

Let $\overline{E}' \subset \overline{E}$ have $\vec{\mu}_j(\overline{E}') = \frac{1}{2}\vec{\mu}_j(\overline{E})$. Then

$$T_j u = \vec{\mu}_j(E \cup \overline{E}').$$

□

Definition 1. From here on we fix a closed bounded interval I, the convex compact

$$\mathcal{C} := \left\{x \in \mathbb{R}^m_{\ge 0} : \textstyle\sum_i x_i = 1\right\},$$

as well as a matrix-valued function $Y \in L_1(I, M_{n\times m})$ (in the system above, $Y(s) := \Phi^{-1}(s)B(s)$).

Remark. $\mathcal{C}$ is actually a **simplex**, since for $f_j \in (\mathbb{R}^m)^\star, j = 0, \ldots, m$,

$$\begin{aligned} f_0(x_1, \ldots, x_m) &= \textstyle\sum_i x_i \\ f_i(x_1, \ldots, x_m) &= x_i, \quad i = 1, \ldots, m \end{aligned}$$

we can express $\mathcal{C} = \bigcap_{i=1}^m f^{-1}[0, \infty) \cap f_0^{-1}(1)$, that is, $\mathcal{C}$ is an intersection of closed half-spaces.

Since $\mathcal{C}$ is a simplex, $\operatorname{Ext}\mathcal{C}$ consists of m **corners** we denote by $r_1, \ldots, r_m$.

Definition. Just to clarify, I will use the term **non-null set** in the sense of a *set of strictly positive measure.*

In the proof of the Bang-bang principle later we'll refer to the following separability proposition:

Lemma 1. *Let $E \subset \mathbb{R}$ be a non-null set and $u(t) \in \mathbb{R}^m \setminus \mathcal{C}$ a.e. in E. Then there exists a vector $\eta \in \mathbb{R}^m$, separating strictly $\mathcal{C}$ from $u(t)$ a.e. in a non-null $E' \subset E$, that is,*

$$\exists \varepsilon, B > 0, \eta \in \mathbb{R}^m : \quad \begin{array}{rl} \langle \eta, x \rangle \geq B & \forall x \in \mathcal{C} \\ \langle \eta, u(t) \rangle \leq B - \varepsilon & \forall t \in E' \end{array}$$

Proof. According to the previous remark we can represent with a *countable* union

$$\begin{aligned} \mathbb{R}^m \setminus \mathcal{C} &= \bigcup_{i=1}^{m+2} g_i^{-1}(-\infty, a_i) \\ &= \bigcup_{i=1}^{m+2} \bigcup_{k \in \mathbb{N}} g_i^{-1}(-\infty, a_i - 1/k], \end{aligned}$$

where$g_i \in (\mathbb{R}^m)^\star$,$a_i \in \mathbb{R}, i = 1, \ldots, m+2$. Then the image under u of some non-null E' must lie in $g_i^{-1}\left(-\infty, a_i - \frac{1}{k}\right]$ for some $k \in \mathbb{N}$. Let $\eta \in \mathbb{R}^m$ be the respective (by Riesz' theorem) vector to the functional g_i (for which $\langle \eta, \cdot \rangle = g_i(\cdot)$). Then

$$\begin{aligned} \forall x \in \mathcal{C} &: \langle \eta, x \rangle = g_i(x) \geq a_i \\ \forall t \in E' &: \langle \eta, u(t) \rangle = g_i(u(t)) \leq a_i - \tfrac{1}{k}. \end{aligned}$$

□

Definition. We denote the sets of the *allowed* and the *extremal* controls, respectively:

$$\begin{aligned} \Psi &:= L_\infty(I, \mathcal{C}) \subset L_\infty(I, \mathbb{R}^m) \\ \Psi_e &:= L_\infty(I, \operatorname{Ext}\mathcal{C}) \subseteq \Psi. \end{aligned}$$

Remark. From here on we'll talk about the space $L_\infty(I, \mathbb{R}^m)$, which, taken as the dual of $L_1(I, \mathbb{R}^m)$ by the action

$$f(\varphi) := \int \langle \varphi(t), f(t) \rangle \, \mathrm{d}t, \quad \text{for all } \varphi \in L_1(I, \mathbb{R}^m), f \in L_\infty(I, \mathbb{R}^m),$$

is equipped with the $w^\star$ topology.

Lemma 2. *$\Psi := L_\infty(I, \mathcal{C})$ is a convex $w^\star$-compact. It is essential that here $\mathcal{C}$ is the simplex defined earlier.*

Proof. Obviously Ψ is convex, since for $u_1, u_2 \in \Psi$, $\lambda u_1 + (1-\lambda)u_2$ is again a L_∞ function, and by that again $\mathcal{C}$-valued due to convexity of $\mathcal{C}$. Moreover, Ψ is bounded (in norm) with

$$\|u\|_\infty = \sup_{t \in I} u(t) \leq \max_{y \in \mathcal{C}} \|y\| \quad \text{for all } u \in \Psi.$$

For Ψ to be a $w^\star$-compact it remains only to check it is $w^\star$-closed, that is[1], any $u^0 \in L_\infty(I, \mathbb{R}^m) \setminus \Psi$ is $w^\star$-separable from Ψ. In order to $w^\star$-separate u^0 from Ψ, we need a suitable functional $\varphi \in L_1 \hookrightarrow (L_\infty)^\star$, with which

$$\varphi\left(u^0\right) \leq B - \varepsilon \quad \text{and} \quad \forall u \in \Psi : \varphi\left(u\right) \geq B.$$

Since $u^0 \notin \Psi$, then in a non-null $E \subseteq I$, $u^0(t) \notin \mathcal{C}$. Then by lemma 1 there exist $\varepsilon > 0, B > 0, \eta \in \mathbb{R}^m$ and a non-null $E' \subset E$ with the property

$$\begin{array}{cl} \langle \eta, x \rangle \geq B & \text{for all } x \in \mathcal{C} \\ \left\langle \eta, u^0(t) \right\rangle \leq B - \varepsilon & \text{for all } t \in E' \end{array}.$$

We set $\varphi(t) = \begin{cases} \frac{1}{\mu(E')}\eta, & t \in E' \\ 0, & \text{else} \end{cases}$, and then $\varphi \in L_1$ and fulfills the required conditions:

$$\begin{aligned} \varphi\left(u^0\right) = \int_I \left\langle \varphi(t), u^0(t) \right\rangle \mathrm{dt} &= \frac{1}{\mu(E')} \int_{E'} \left\langle \eta, u^0(t) \right\rangle \mathrm{dt} \\ &\leq \frac{1}{\mu(E')} \int_{E'} (B - \varepsilon) = B - \varepsilon \end{aligned}$$

and analogously $\varphi(u) = \int_I \langle \varphi, u \rangle = \frac{1}{\mu(E')} \int_{E'} \langle \eta, u \rangle \geq B \quad \text{for all } u \in \Psi.$

Since Ψ is $w^\star$-separated by any $u^0 \notin \Psi$, then Ψ is $w^\star$-closed, and since it is $\|\cdot\|_\infty$-bounded – it is a $w^\star$-compact. □

Remark. For $u \in L_\infty(I, \mathbb{R}^m)$, the product $Y \cdot u$ is a function from $L_1\left(I, \mathbb{R}^m\right)$, because $Y_{ij} \in L_1$ and

$$\left(Y \cdot u\right)_i = \sum_{j=1}^m Y_{ij}(s) u_j(s) \leq \sum_{j=1}^m Y_{ij}(s) \left\|u\right\|_\infty.$$

Then it makes sense to talk about the integral $\int_I Y \cdot u$ that we will look at from here on.

Definition. We say that *the control $u \in \Psi$ leads to the point $\xi = Tu$*, where $T : L_\infty(I, \mathbb{R}^m) \to \mathbb{R}^n$ is the linear operator

$$Tu := \int_I Y(s) \cdot u(s) \, \mathrm{ds} \quad \text{for all } u \in L_\infty(I, \mathbb{R}^m).$$

[1]In Hermes's book the existence of a $u^0 \in \partial\Psi \setminus \Psi$ is assumed, and a contradiction is derived later with the same argument. I find this style unnecessarily complicated, so I just prove opennes of the complement.

Remark. T is continuous (in norm), since for $\varepsilon > 0$, with $\delta := \frac{\varepsilon}{\mu(I)\int_I Y}$ and $\|h\|_\infty < \delta$ we have

$$|T(u+h) - Tu| = \left|\int_I Y(s) \cdot h(s)\,\mathrm{ds}\right| \leq \mu(I)\,\|h\|_\infty \left|\int_I Y\right| < \varepsilon.$$

Moreover, it is $w^\star$-continuous, since its coordinate functionals

$$T_i : L_\infty(I, \mathbb{R}^m) \to \mathbb{R},\, i = 1, \dots n$$

are defined by $T_i(u) = \sum_j \int_I Y_{ij} u_j$ where Y_{ij} are L_1-functions.

Theorem (*Bang-bang* principle for controls in a simplex, Hermes)**.**

$$T(\Psi) = T(\Psi_e).$$

Moreover, the two sets are convex compacts.

Proof. $T\Psi_e \subseteq T\Psi$ is obvious, we need to prove the reverse inclusion. The proof goes in the following steps:

1. According to the last remark, T is $w^\star$-continuous, hence the image $T\Psi \subset \mathbb{R}^n$ of the $w^\star$-compact Ψ is again a convex compact (the second statement in this theorem).

2. T is a continuous map between compacts, then the possible controls $T^{-1}(\xi) \cap \Psi$ leading to any point $\xi \in \mathbb{R}^n$ form a convex $w^\star$-compact, necessarily having an extreme point $u \in T^{-1}(\xi) \subset \Psi$ (Krein-Milman).

3. It turns out that this extremum u is a *bang-bang* control.

   If we assume the opposite, then in some non-null set $E \subset I$ the controls are far from $\operatorname{Ext}\mathcal{C}$ and from the lema below (3) it follows u is a midpoint of a proper line segment of controls $[u-h, u+h] \subset \Psi$, leading to the same ξ, contradicting extremality of u.

□

Lemma 3. *Let $n \in \mathbb{N}$, $Y \in L_1(I, M_{n\times m})$, $E \subset I$ is non-null, and the control $u \in \Psi$ leads to $\xi := Tu$ and satisfies $\operatorname{dist}(u(E), \operatorname{Ext}\mathcal{C}) > \varepsilon$ for some $\varepsilon > 0$.*

Then there exists a displacement $h \in L_\infty(I, \mathbb{R}^m)$ with the following properties:

1. *$u(t) \pm h(t) \in \mathcal{C}$ for all $t \in I$, where $h(t) \neq 0$ for all $t \in E$ (u lies in a proper line segment);*

2. *$\int_I Y \cdot h = 0$ (that is, $u \pm h$ leads again to ξ).*

In other words, $u \in [u-h, u+h] \subset T^{-1}(\xi) \cap \Psi$.

Proof. The proof goes in the following steps

1. For each $t \in E$, according to the next lemma (4), we can embed $u(t)$ in a segment with endpoints $u(t) \pm \tilde{h}(t) \in \mathcal{C}$ so that $\|\tilde{h}(t)\|_\infty > \varepsilon^2$. Then $\|\tilde{h}\|_\infty > \varepsilon^2$ too. Unfortunately, however, $\int_I Y \cdot \tilde{h} \neq 0$ in the general case.

2. From the non-atomicity of the Lebesgue measure we can partition for any $k \in \mathbb{N}^+$ (the suitable k will be determined later) $E = \bigcup_{j=1}^{k} E_k$, so that $E_i \cap E_j = \emptyset$ for all $i \neq j$ and $\mu(E_j) \gneqq 0$ for all j.

3. Now in each $E_j, j = 1, \dots, k$ we will weight the restriction of $\tilde{h}$ with a suitable $\alpha_j \in \mathbb{R}$ so that $\int_I Y \cdot h = 0$:

$$h(t) := \begin{cases} \alpha_j \tilde{h}(t), & t \in E_j \\ 0, & \text{otherwise} \end{cases}.$$

   The condition $\int_I Y \cdot h = 0$ is a homogenous linear system with n equations and k unknowns

$$\int_I Y \cdot h = \sum_{j=1}^{k} \alpha_j \int_{E_j} Y \cdot \tilde{h} = 0.$$

   A non-trivial solution for the coefficients α_j always exists whenever $k \geq n + 1$ – fix one with$|\alpha_j| < 1$ for all j.

4. Let us convince ourselves that h satisfies property 1 (from the lemma statement). Obviously $h(t) \neq 0$ for all $t \in E$, and $u(t) \pm h(t)$ lies in $\mathcal{C}$, since it lies (for all $t \in E$) in the line segment with endpoints $u(t) \pm \tilde{h}(t) \in \mathcal{C}$ (due to $|\alpha_j| < 1$), and $\mathcal{C}$ is convex.

   Therefore $u \pm h \in \Psi = L_\infty(I, \mathcal{C})$ and $u \in [u - h, u + h]$ from convexity of Ψ.

□

Lemma 4. *Let $\xi \in \mathcal{C}$ satisfy $\operatorname{dist}(\xi, \operatorname{Ext} \mathcal{C}) > \varepsilon$ for some $\varepsilon > 0$. Then ξ can be put in a line segment $[\xi - h, \xi + h] \subset \mathcal{C}$ with $\|h\|_\infty \geq \varepsilon^2$.*

Moreover, the mapping $\xi \mapsto h$ is measurable.

(interestingly, we cannot state $\|h\|_\infty \geq \varepsilon$, the largest lower bound is $\frac{\varepsilon^2}{1-\varepsilon}$)

Proof. Geometrically it is natural to choose h *parallel to* $r_j - \xi$, where $r_j \in \operatorname{Ext} \mathcal{C}$ is the closest to ξ corner with smallest index j. We put $h := \beta\,(r_j - \xi)$. Clearly $\xi \pm h \in$ affine span $\mathcal{C}$. We will show that for sufficiently small $\beta > 0$ even $\xi \pm h \in \mathcal{C}$.

$$\xi \pm h = \xi \pm \beta\,(r_j - \xi) = (1 \mp \beta)\,\xi \pm \beta r_j.$$

Since $\xi \in \mathcal{C}$, then $\xi = \sum_i \beta_i r_i$ with some $\beta_i \geq 0$ and $\sum_i \beta_i = 1$. Then

$$\begin{aligned} \xi \pm h &= (1 \mp \beta) \sum_i \beta_i r_i \pm \beta r_j \\ &= \underbrace{(\pm\beta + (1 \mp \beta)\,\beta_j)}_{\beta'_j} r_j + \sum_{i \neq j} \underbrace{(1 \mp \beta)\,\beta_i}_{\beta'_i} r_i \end{aligned}$$

and we have to check that the RHS is a convex combination of $\{r_i\}_{i=1}^m$. $\sum_i \beta'_i = 1$ is equiavelent to $\xi \pm h \in$ affine span $\mathcal{C}$, so it remains to check $0 < \beta'_i$ for all i.

If $|\beta| \leq 1$ then obviously $\beta'_i \geq 0$ for all $i \neq j$, and

$$\beta'_j = \pm\,(1 - \beta_j)\,\beta + \beta_j$$

and hence $\beta'_j \geq 0$ holds always if $\beta \leq \frac{\beta_j}{1-\beta_j}$. We choose namely the maximal $\beta = \beta_j/(1-\beta_j)$.

Since $\operatorname{dist}(\xi, \operatorname{Ext}\mathcal{C}) > \varepsilon$ and r_j is a closest corner to ξ, then $\beta_j > \varepsilon$, $\beta > \frac{\varepsilon}{1-\varepsilon}$ and

$$\|h\|_\infty = \beta \|r_j - \xi\|_\infty > \beta\varepsilon = \frac{\varepsilon^2}{1-\varepsilon} > \varepsilon^2.$$

For measurability of the mapping $\xi \mapsto h$ it suffices to observe that $\xi \mapsto j$ is constant on sets

$$F_j = \left\{ \xi \in \mathcal{C} : \begin{array}{c} \mathrm{d}(\xi, r_j) = \operatorname{dist}(\xi, \operatorname{Ext} C) \\ \text{и } \mathrm{d}(\xi, r_j) > \mathrm{d}(\xi, r_i)\, \forall i < j \end{array} \right\}, \quad j = 1, \ldots, m,$$

which partition $\mathcal{C}$ and are measurable due to continuity of $\mathrm{d}(\cdot, \cdot)$. $\square$

Sources:

- Functional analysis and time optimal control, Volume 56 (Mathematics in Science and Engineering) by Hermes (Editor)
- Geometric Functional Analysis and its Applications Richard B. Holmes