X people (Poisson-distributed with mean 10) enter an elevator on the ground floor of a N-floor building ($N \in \mathbb{N}$ fixed). Each passenger is equally likely to get off at any floor, independently from the rest (and from the number of the passengers). Determine the expected number of stops the elevator will make to deliver all passangers.

This writeup here is a very complicated solution to this problem.

Problem

Given

- $N \in \mathbb{N}$: number of floors
- $X \sim \text{Poisson}(10)$: number of people entering the elevator
- $F_i \sim \text{Uniform}(1, N)$: the floor at which person i wants to get off, for $i \in \mathbb{N}$
- X and all F_i are mutually independent

Find the expected number of stops until all passangers leave the elevator.

Solution

Compact

$$\begin{aligned}
\mathbb{E}\left(\#\left\{F_1, \dots, F_X\right\}\right) &= \mathbb{E}\left(\text{stops}_X(\vec{F}_X)\right) \\
&= \sum_{s \in \mathbb{N}} s \mathbb{P}\left(\text{stops}_X(\vec{F}_X) = s\right) \\
&= \sum_{s \in \mathbb{N}} s \sum_{k \in \mathbb{N}} \mathbb{P}\left(\text{stops}_k(\vec{F}_k) = s\right) \mathbb{P}\left(X = k\right) \\
&\stackrel{!}{=} \sum_{s \in \mathbb{N}} s \sum_{k \in \mathbb{N}} \left(\frac{1}{N^k} \binom{N}{s} \sum_{i=0}^{s} (-1)^{s-i} \binom{s}{i} i^k\right) \mathbb{P}\left(X = k\right) \\
&= \sum_{s \in \mathbb{N}} s \binom{N}{s} e^{-\lambda} \left(e^{\lambda/N} - 1\right)^s \\
&= N(1 - e^{-\lambda/N}).
\end{aligned}$$

where in $\stackrel{!}{=}$ we do

$$\begin{aligned}
\mathbb{P}\left(\text{stops}_k(\vec{F}_n) = s\right) &= \mathbb{P}_{\vec{F}_k}\left(\text{stops}_k = s\right) \\
&= \frac{1}{N^k} \#\left(\text{stops}_k = s\right) \\
&= \frac{1}{N^k} \binom{N}{s} \sum_{i=0}^{s} (-1)^{s-i} \binom{s}{i} i^k.
\end{aligned}$$

Goal We're tasked to find the expected size of the (random) set $\{F_1, \ldots, F_X\}$ of the floors each person chose.

For $k \in \mathbb{N}$ (a particular number of passangers), put

$$\vec{F}_k = (F_1, \ldots, F_k)\,,$$

and the number of unique stops among the k choices,

$$\begin{aligned} \mathrm{stops}_k : \{1, \ldots, N\}^k &\to \{1, \ldots, k\} \\ \mathrm{stops}_k(f_1, \ldots, f_k) &\coloneqq \#\{f_1, \ldots, f_k\}. \end{aligned}$$

so that what we seek is the expected size of the random set

$$\mathrm{stops}_X(\vec{F}_X).$$

We keep the k index on both the vector $\vec{F}_k$ and the functionstops$_k$ to remind us that their domains depend on this parameter.

Meaning of independence Independence of all F_i, $i = 1, \ldots, k$, means that $\vec{F}_k$'s law decomposes as

$$\mathbb{P}_{\vec{F}_k} = \mathbb{P}_{\vec{F}_1} \otimes \cdots \otimes \mathbb{P}_{F_k}.$$

Moreover, $F_i \sim \mathrm{Uniform}(1, N)$ means that $\mathbb{P}_{F_i}(S) = \frac{\#S}{N}$ for any $S \subset \{1, \ldots, N\}$, i.e. $\mathbb{P}_{F_i} = \frac{1}{N}\#$, so in fact[1]

$$\mathbb{P}_{\vec{F}_k} = \frac{1}{N}\# \otimes \cdots \otimes \frac{1}{N}\# = \frac{1}{N^k}\#.$$

Calculation of the goal

$$\begin{aligned} \mathbb{E}\left(\#\{F_1, \ldots, F_X\}\right) &= \mathbb{E}\left(\mathrm{stops}_X(\vec{F}_X)\right) \\ &= \sum_{s \in \mathbb{N}} s\mathbb{P}\left(\mathrm{stops}_X(\vec{F}_X) = s\right). \end{aligned}$$

Thanks to my colleague Alex who noticed we can eliminate the random dependence of X here by writing

$$\mathbb{P}\left(\mathrm{stops}_X(\vec{F}_X) = s\right) = \sum_{k \in \mathbb{N}} \mathbb{P}\left(\mathrm{stops}_k(\vec{F}_k) = s\right)\mathbb{P}\left(X = k\right).$$

The second multiple is known, all is left is to compute for $s, n \in \mathbb{N}$

$$\begin{aligned} \mathbb{P}\left(\mathrm{stops}_k(\vec{F}_n) = s\right) &= \mathbb{P}_{\vec{F}_k}\left(\mathrm{stops}_k = s\right) \\ &= \frac{1}{N^k}\#\left(\mathrm{stops}_k = s\right) \end{aligned}$$

[1] the $\#$ in the first equality counts subsets of $\{1, \ldots, N\}$, while the RHS's $\#$ counts subsets of $\{1, \ldots, N\}^n$

The first equality here follows from standard pushforward-measure nonsense that is found at the end of this writeup, and the second equality uses $\mathbb{P}_{\vec{F}_k} = \frac{1}{N^k}\#$ that we proved above.

Finally, we need to find the size of $\text{stops}_n^{-1}\{s\} \subset \{1,\ldots,N\}^n$, which is an entirely combinatorial problem solved at the end below. The result is

$$\#\left(\text{stops}_k = s\right) = \binom{N}{s}\sum_{i=0}^{s}(-1)^{s-i}\binom{s}{i}i^k.$$

Now

$$\begin{aligned}
\mathbb{P}\left(\text{stops}_X(\vec{F}_X) = s\right) &= \sum_{k\in\mathbb{N}} \mathbb{P}\left(\text{stops}_k(\vec{F}_k) = s\right)\mathbb{P}\left(X = k\right) \\
&= \sum_{k\in\mathbb{N}}\left(\frac{1}{N^k}\binom{N}{s}\sum_{i=0}^{s}(-1)^{s-i}\binom{s}{i}i^k\right)\left(\frac{\lambda^k}{k!}e^{-\lambda}\right) \\
\text{(reorder)} \quad &= \binom{N}{s}e^{-\lambda}\sum_{i=0}^{s}(-1)^{s-i}\binom{s}{i}\left(\sum_{k\in\mathbb{N}}\frac{1}{N^k}i^k\frac{\lambda^k}{k!}\right) \\
\text{(Tailor series)} \quad &= \binom{N}{s}e^{-\lambda}\sum_{i=0}^{s}(-1)^{s-i}\binom{s}{i}e^{i\lambda/N} \\
\text{(binomial formula)} \quad &= \binom{N}{s}e^{-\lambda}\left(e^{\lambda/N} - 1\right)^s
\end{aligned}$$

Putting it all together,

$$\begin{aligned}
\mathbb{E}\left(\#\{F_1,\ldots,F_X\}\right) &= e^{-\lambda}\sum_{s\in\mathbb{N}} s\binom{N}{s}\left(e^{\lambda/N} - 1\right)^s \\
&= e^{-\lambda}\left(e^{\lambda/N} - 1\right)\sum_{s\in\mathbb{N}} s\binom{N}{s}\left(e^{\lambda/N} - 1\right)^{s-1} \\
&\overset{!}{=} e^{-\lambda}(e^{\frac{\lambda}{N}} - 1)Ne^{\lambda\frac{N-1}{N}} \\
&= Ne^{-\lambda/N}(e^{\lambda/N} - 1) \\
&= N(1 - e^{-\lambda/N}).
\end{aligned}$$

In $\overset{!}{=}$ the (derivative of the) binomial formula is used.

Justification for the equality $\mathbb{P}\left(\text{stops}_n(\vec{F}_n) = s\right) = \mathbb{P}_{\vec{F}_n}\left(\text{stops}_n = s\right)$:

$$
\begin{aligned}
\mathbb{P}\left(\mathrm{stops}_n(\vec{F}_n) = s\right) &= \mathbb{P}\left(\left(\mathrm{stops}_n \circ \vec{F}_n\right)^{-1}\{s\}\right) \\
&= \mathbb{P}\left(\left(\vec{F}_n^{-1} \circ \mathrm{stops}_n^{-1}\right)\{s\}\right) \\
&= \mathbb{P}\left(\vec{F}_n^{-1}\left(\mathrm{stops}_n^{-1}\{s\}\right)\right) \\
&= \mathbb{P}_{\vec{F}_n}\left(\mathrm{stops}_n^{-1}\{s\}\right) \\
&= \mathbb{P}_{\vec{F}_n}\left(\mathrm{stops}_n = s\right)
\end{aligned}
$$

Combinatorics

$$
\begin{aligned}
\#\left(\mathrm{stops}_k = s\right) &= \#\left\{(f_1 \dots f_k) : \mathrm{stops}_k(f_1 \dots f_k) = s\right\} \\
&= \#\left\{(f_1 \dots f_k) : \#\{f_1 \dots f_k\} = s\right\} \\
&= \# \bigcup_{S \subset \{1,\dots N\}} \left\{(f_1 \dots f_k) : \begin{array}{c} \#\{f_1 \dots f_k\} = s \\ \text{and } \{f_1 \dots f_k\} = S \end{array}\right\} \\
&= \# \bigcup_{\substack{\#S = s \\ S \subset \{1,\dots N\}}} \left\{(f_1 \dots f_k) : \{f_1 \dots f_k\} = S\right\} \\
&\overset{!}{=} \sum_{\substack{\#S = s \\ S \subset \{1,\dots N\}}} \#\left\{(f_1 \dots f_k) : \{f_1 \dots f_k\} = S\right\} \\
&\overset{!!}{=} \sum_{\substack{\#S = s \\ S \subset \{1,\dots N\}}} \sum_{i=0}^{s} (-1)^{s-i} \binom{s}{i} i^k \\
&= \binom{N}{s} \sum_{i=0}^{s} (-1)^{s-i} \binom{s}{i} i^k
\end{aligned}
$$

In $\overset{!}{=}$ we can do the $\# \bigcup = \sum \#$ swap because all the sets are disjoint. In $\overset{!!}{=}$ we apply the inclusion-exclusion principle to get

$$
\#\left\{(f_1 \dots f_k) : \{f_1 \dots f_k\} = S\right\} = \sum_{i=0}^{s} (-1)^{s-i} \binom{s}{i} i^k
$$