# Extensions of functions on $H^k$

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**Definition.** Let $\Omega \subseteq \Omega'$ and $f \in L_2(\Omega)$. An *extension* of $f$ in $\Omega'$ is called any function $f' \in L_2(\Omega')$, s.t. $f'\big|_\Omega = f$.

**Example.** Every $f \in L_2(\Omega)$ can be extended to $\Omega'$ by setting $f = 0$ outside $\Omega$:

$$f' \coloneqq \begin{cases} f & \text{in } \Omega \\ 0 & \text{in } \Omega' \setminus \Omega \end{cases}.$$

*Remark.* Our goal is to extend the function $f \in H^k(\Omega)$ on $\Omega'$ so that the extension is from $H^k(\Omega')$. In the above definition this is not required — as the example makes clear, extensions can even be discontinuous.

## 1 Extension from a half-cube to a cube

Below, with $\{y_n > 0\}$ we denote the subset of $\mathbb{R}^n$ consisting of vectors $(y_1, \ldots, y_n)$ that satisfy the condition in the brackets.

We introduce the following notations for the hypercube centered around the origin with radius $a$, as well as its "upper" and "lower" halves:

$$\begin{aligned} K_a &\coloneqq \{(y_1, \ldots, y_n) \in \mathbb{R}^n : |y_i| < a \text{ for all } i\} \\ K_a^+ &\coloneqq K_a \cap \{y_n > 0\} \\ K_a^0 &\coloneqq K_a \cap \{y_n = 0\} \\ K_a^- &\coloneqq K_a \cap \{y_n < 0\}. \end{aligned}$$

Moreover, we denote by $y'$ for any vector $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ the vector $y' \coloneqq (y_1, \ldots, y_{n-1})$. For short, for a $y \in \mathbb{R}^n$ we'll denote by $(y', c)$ the vector $(y_1, \ldots, y_{n-1}, c)$.

### 1.1 Of $C^k$ functions

1. Construction

   Let $z(y) \in C^k(\overline{K_a^+})$. We'll extend it to $K_a$. On points $y \in \overline{K_a^-}$ we define the extension by means of of $K_a^+$ points (see figure) of the form $\left(y', -\frac{y_n}{i}\right)$.

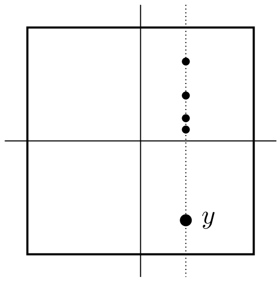



Figure 1: The value of $Z$ at a point $y$ is constructed based on points $\left(y', -\frac{y_n}{i}\right)$.

Let

$$Z(y) := \begin{cases} z(y) & y \in \overline{K_a^+} \\ \sum_{i=1}^m A_i z\left(p_i(y)\right) & y \in \overline{K_a^-} \backslash K_a^0 \end{cases}$$

where $p : \overline{K_a^-} \to \overline{K_a^+}$ is the function

$$p_i(y) := \left(y', -\frac{y_n}{i}\right),$$

while the number $m$ and the coefficients$A_1, \ldots, A_m$ will be determined later so that the resulting extension $Z : \overline{K_a} \to \mathbb{R}$ of $z : \overline{K_a^+} \to \mathbb{R}$ is of class $C^k$.

2. Continuity of the extension

   $Z$ is continuous in $K_a^+$ by definition. We can easily see $Z$ is continuous in $K_a^-$ as well, directly from continuity of $z$ in $K_a^+$. It remains to check continuity on the boundary $\bar{y} \in K_a^0$, when $y$ tends to$\bar{y}$ from $K_a^-$:

$$\begin{aligned} \lim_{K_a^- \ni y \to \bar{y}} Z(y) &= \lim_{K_a^- \ni y \to \bar{y}} \sum_{i=1}^m A_i z\left(p_i(y)\right) \\ &= \sum_{i=1}^m A_i \lim_{y \to \bar{y}} z\left(p_i(y)\right) \\ &= \sum_{i=1}^m A_i z\left(p_i(\bar{y})\right) = \sum_{i=1}^m A_i z\left(\bar{y}\right) = z\left(\bar{y}\right) \sum_{i=1}^m A_i \end{aligned}$$

   To make $Z$ continuous in $\bar{y}$ it is enough that this limit equals equals $z(\bar{y})$, which is the case exactly when

$$\sum_{i=1}^m A_i = 1.$$

3. Smoothness of the extension

Let $\alpha$ be a multiindex with $|\alpha| \leq k$. We will check that the derivative $\partial^\alpha Z(y)$ exists at each $y \in K_a$.

(a) In $K_a^+$ this is just $\partial^\alpha z$.

(b) $Z$ obviously has $\alpha$-derivative in $y \in K_a^-$ as well:

$$\begin{aligned}\partial^\alpha Z(y) &= \sum_{i=1}^m A_i \partial^\alpha \left(z\left(p_i(y)\right)\right) \\ &= \sum_{i=1}^m A_i \partial^\alpha p_i(y) \partial^\alpha z\left(p_i(y)\right). \end{aligned} \tag{1}$$

(c) It remains to see that $Z$ has $\alpha$-derivative in any $y \in K_a^0$, too. We'll do it by induction on $|\alpha|$.

i. Assume$|\alpha| = 1$, that is $\partial^\alpha = \frac{d}{dx_j}$ for some $j \in \{1, \ldots, n\}$. Let's find under what conditions the derivative $\frac{\partial Z}{\partial x_j}$ exists in $y_0 \in K_a^0$. We need the derivative coming "from above", $\lim_{K_a^+ \ni y \to y_0} \partial^\alpha z(y)$, equal to the derivative coming "from below"

$$\begin{aligned}\lim_{K_a^- \ni y \to y_0} \partial^\alpha Z &\overset{(1)}{=} \lim_{K_a^- \ni y \to y_0} \sum_{i=1}^m A_i \partial^\alpha p_i(y) \partial^\alpha z\left(p_i(y)\right) \\ &= \sum_{i=1}^m A_i \partial^\alpha p_i(y_0) \partial^\alpha z\left(p_i(y_0)\right) \\ &= \sum_{i=1}^m A_i \partial^\alpha p_i(y_0) \partial^\alpha z\left(y_0\right) \\ &= \partial^\alpha z\left(y_0\right) \sum_{i=1}^m A_i \partial^\alpha p_i(y_0).\end{aligned}$$

The derivatives agree when the right multiple does not play:

$$\sum_{i=1}^m \partial^\alpha p_i(y_0) A_i = 1.$$

ii. Analoguously, for $1 < |\alpha| \leq k$, calculating the derivative as above we get the same condition

$$\sum_{i=1}^m \partial^\alpha p_i(y_0) A_i = 1.$$

In particular, if we take $p_i(y) = \left(y_1, \ldots, y_{n-1}, -\frac{1}{i} y_n\right)$, we get the system

$$\sum_{i=1}^m \left(-\frac{1}{i}\right)^l A_i = 1, \quad \text{for all } i = 0, \ldots k,$$

that, being a Vandermonde system, has a unique solution when $m = k+1$.

4. Bound of $\|Z\|_{H^k(K_a)}$ via $\|z\|_{H^k(K_a^+)}$
By Minkowski inequality:

$$
\begin{aligned}
|\partial^\alpha Z(y)|^2 &= \left|\sum_{i=1}^m A_i \left(-\frac{1}{i}\right)^{\alpha_n} \partial^\alpha z\left(p_i(y)\right)\right|^2 \\
&= \left(\sum_{i=1}^m \left(A_i \left(-\frac{1}{i}\right)^{\alpha_n}\right) \left(\partial^\alpha z\left(p_i(y)\right)\right)\right)^2 \\
&\le \left(\sum_{i=1}^m A_i^2 \left(-\frac{1}{i}\right)^{2\alpha_n}\right) \left(\sum_{i=1}^m \left(\partial^\alpha z\left(p_i(y)\right)\right)^2\right) \\
&= C_\alpha \sum_{i=1}^m \left(\partial^\alpha z\left(p_i(y)\right)\right)^2
\end{aligned}
$$

and after integrating over $K_a^-$:

$$
\begin{aligned}
\int_{K_a^-} |\partial^\alpha Z(y)|^2 &\le C_\alpha \sum_{i=1}^{k+1} \int_{K_a^-} \left(\partial^\alpha z\left(p_i(y)\right)\right)^2 \\
&= C_\alpha \sum_{i=1}^{k+1} i \int\limits_{K_a^{\le a/i}} \left(\partial^\alpha z\left(y\right)\right)^2 \quad (\text{put } x = p_i(y)) \\
&\le C_\alpha \sum_{i=1}^{k+1} (k+1) \int\limits_{K_a^{\le a/i}} \left(\partial^\alpha z\left(y\right)\right)^2 \\
&= \underbrace{C_\alpha (k+1)^2}_{C'_\alpha} \int\limits_{K_a^{\le a/i}} \left(\partial^\alpha z\left(y\right)\right)^2,
\end{aligned}
$$

where $K_a^{\le a/i} \coloneqq K_a^+ \cap \left\{y_n < \frac{a}{i}\right\}$. Then, over the entire $K_a$ we have

$$
\begin{aligned}
\int_{K_a} |\partial^\alpha Z(y)|^2 &= \int_{K_a^+} |\partial^\alpha z(y)|^2 + \int_{K_a^-} |\partial^\alpha Z(y)|^2 \\
&\le \int_{K_a^+} |\partial^\alpha z(y)|^2 + C'_\alpha \int_{K_a^+} |\partial^\alpha z(y)|^2 \\
&\le C''_\alpha \int_{K_a^+} |\partial^\alpha z(y)|^2,
\end{aligned}
$$

and after summing over $|\alpha| \le k$ we get the bound

$$
\|Z\|_{H^k(K_a)} \le C \|z\|_{H^k\left(K_a^+\right)}, \tag{2}
$$

where $C$ is a constant, depending solely on $k$.

## 1.2 Of functions in $H^k$

Let $z \in H^k(K_a^+)$. We'll find an extension $Z \in H^k(K_a)$, based on the construction above for $C^k$ functions. We'll prove the following lemma in a bit:

**Lemma.** *The set $C^\infty\left(\overline{K_a}\right)$ is dense in $H^k\left(K_a\right)$.*

According to it, there is some sequence $\{z_s\}_s \subset C^k(\overline{K_a^+})$ tending to $z$ in the $H^k(K_a^+)$ norm. We extend its elements as in the previous section to $\overline{K_a}$ and get a sequence $\{Z_s\}_s \subset C^k\left(\overline{K_a}\right)$. It remains to check the the newly constructed sequence has a limit $Z$ in $H^k\left(K_a\right)$.

Since $H^k$ is complete, it is enough to show $\{Z_s\}$ is Cauchy. Observe that the difference $Z_s - Z_p$ is an extension (by above construction) of the difference $z_s - z_p$. Then the bound $\|Z\|_{H^k(K_a)} \le C\|z\|_{H^k\left(K_a^+\right)}$ holds here as well, that is

$$\|Z_s - Z_p\|_{H^k(K_a)} \le C\|z_s - z_p\|_{H^k\left(K_a^+\right)}.$$

Now, since $\{z_s\}_s$ is Cauchy, $\{Z_s\}_s$ is Cauchy, too, and the limit $Z \in H^k\left(K_a\right)$ exists and is an extension of $z \in H^k(K_a^+)$.

Finally, in the limit $p \to \infty$ in

$$\|Z_p\|_{H^k(K_a)} \le C\|z_p\|_{H^k\left(K_a^+\right)}$$

we see that the bound (2) holds for extensions in $H^k$ too. Thus we proved

**Proposition.** *Every $z \in H^k(K_a^+)$ has an extension $Z \in H^k(K_a)$, satisfying*

$$\|Z\|_{H^k(K_a)} \le C\|z\|_{H^k\left(K_a^+\right)}.$$

It's time to get back to the proof of the lemma.

*Proof ($C^\infty(\overline{K_a})$ is dense in $H^k(K_a)$).* Let $f \in H^k\left(K_a\right)$ and $\varepsilon > 0$. We need to find $F \in C^\infty\left(\overline{K_a}\right)$ with $\|f - F\| < \varepsilon$.

**Idea** The main (actually the only one we have seen so far) technique for obtaining $C^\infty$ functions from $L_2$ functions comes from the averaging operator: we know that if $f \in L_2(\mathbb{R}^n)$, then the average $f_h$ is infinitely smooth for every $h > 0$. We will average the function $f \in H^k(K_a)$ in a slightly narrower region with a sufficiently small averaging radius so that the $H^k$-norm of the average does not deviate significantly from that of $f$. In this way, we will obtain an infinitely smooth function that is sufficiently close (in $H^k$) to $f$.

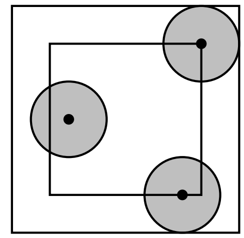
Before we start constructing the average, we will take approximations of the derivatives, since for the average to be close to $f$ (in $H^k$), the derivatives also need to be close (in $L_2$). As elements of $L_2\left(K_a\right)$, the derivatives $\partial^\alpha f$ are within an $\varepsilon$-neighborhood in the $L_2$-norm of some functions $\varphi_\alpha \in C\left(\overline{K_a}\right)$, that is

$$\forall \varepsilon > 0\ \forall \alpha : |\alpha| \le k\ \exists \varphi_\alpha \in C\left(K_a\right) : \|\partial^\alpha f - \varphi_\alpha\|_{L_2} < \varepsilon.$$

Instead of narrowing $K_a$ and averaging in the narrowed region, equivalently (but more conveniently for calculations), we will expand $K_a$ by a factor $\sigma > 1$ to

$$K_{\sigma a} \coloneqq \{x \in \mathbb{R}^n : |x_i| < \sigma a\},$$

expanding simultaneously the function $f$ to $F_\sigma \in H^k(K_{\sigma a})$,



$$F_\sigma(x) = f\left(\frac{x}{\sigma}\right),$$

and after that we'll average the expanded function in the narrower domain $K_a \subset K_{\sigma a}$.

**Bound on the expansion** $\|f - F_\sigma\|_{H^k(K_a)}$ To evaluate the $H^k$-norm, we first evaluate the $L_2$-norms of each derivative separately. We have

$$\begin{aligned}
\|\partial^\alpha F_\sigma(x) - \varphi_\alpha(x)\|_{L_2(K_a)} &= \left\|\partial^\alpha F_\sigma(x) - \varphi_a\left(\frac{x}{\sigma}\right) + \varphi_a\left(\frac{x}{\sigma}\right) - \varphi_\alpha(x)\right\|_{L_2(K_a)} \\
&\overset{\triangle\text{-inequality}}{\leq} \left\|\partial^\alpha F_\sigma(x) - \varphi_a\left(\frac{x}{\sigma}\right)\right\|_{L_2(K_a)} + \left\|\varphi_a\left(\frac{x}{\sigma}\right) - \varphi_\alpha(x)\right\|_{L_2(K_a)} \\
&\leq \left\|\partial^\alpha F_\sigma(x) - \varphi_a\left(\frac{x}{\sigma}\right)\right\|_{L_2(K_{\sigma a})} + \left\|\varphi_a\left(\frac{x}{\sigma}\right) - \varphi_\alpha(x)\right\|_{L_2(K_a)}.
\end{aligned}$$

For the right summand we have

$$\begin{aligned}
\left\|\partial^\alpha F_\sigma(x) - \varphi_a\left(\frac{x}{\sigma}\right)\right\|_{L_2(K_{\sigma a})} &\overset{\text{def. of } F_\sigma}{=} \left\|\frac{1}{\sigma^{|\alpha|}}\partial^\alpha f\left(\frac{x}{\sigma}\right) - \varphi_a\left(\frac{x}{\sigma}\right)\right\|_{L_2(K_{\sigma a})} \\
&\overset{\triangle\text{-inequality}}{\leq} \left\|\left(\frac{1}{\sigma^{|\alpha|}} - 1\right)\partial^\alpha f\left(\frac{x}{\sigma}\right)\right\|_{L_2(K_{\sigma a})} + \left\|+1\partial^\alpha f\left(\frac{x}{\sigma}\right) - \varphi_a\left(\frac{x}{\sigma}\right)\right\|_{L_2(K_{\sigma a})} \\
&= \sigma^{n/2}\left(1 - \frac{1}{\sigma^{|\alpha|}}\right)\|\partial^\alpha f(x)\|_{L_2(K_a)} + \sigma^{n/2}\|\partial^\alpha f(x) - \varphi_a(x)\|_{L_2(K_a)} \\
&\leq \sigma^{n/2}\left(1 - \frac{1}{\sigma^{|\alpha|}}\right)\|\partial^\alpha f\|_{L_2(K_a)} + \sigma^{n/2}\varepsilon
\end{aligned}$$

and substituting above, we get

$$\begin{aligned}
\|\partial^\alpha F_\sigma(x) - \varphi_\alpha(x)\|_{L_2(K_a)} \leq \sigma^{n/2}&\left(1 - \frac{1}{\sigma^{|\alpha|}}\right)\|\partial^\alpha f\|_{L_2(K_a)} + \sigma^{n/2}\varepsilon \\
&+ \left\|\varphi_a\left(\frac{x}{\sigma}\right) - \varphi_\alpha(x)\right\|_{L_2(K_a)}.
\end{aligned}$$

Now

$$\begin{aligned}
\|\partial^\alpha f - \partial^\alpha F_\sigma\|_{L_2(K_a)} &\leq \underbrace{\|\partial^\alpha f - \varphi_\alpha\|_{L_2(K_a)}}_{\leq\varepsilon} + \|\partial^\alpha F_\sigma - \varphi_\alpha\|_{L_2(K_a)} \\
&\leq \left(1 + \sigma^{n/2}\right)\varepsilon + \sigma^{n/2}\left(1 - \frac{1}{\sigma^{|\alpha|}}\right)\|\partial^\alpha f\|_{L_2(K_a)} + \left\|\varphi_a\left(\frac{x}{\sigma}\right) - \varphi_\alpha(x)\right\|_{L_2(K_a)}.
\end{aligned}$$

It remains to evaluate the second and the third summand

- For the second summand we have

$$\sigma^{n/2}\left(1-\frac{1}{\sigma^{|\alpha|}}\right)\xrightarrow[\sigma\to 1^+]{}0,$$

  therefore for some $\sigma>1$, the second summand is smaller than $\varepsilon$.

- Since $\varphi_a$ is uniformly continuous, it follows $\varphi_\alpha\left(\frac{x}{\sigma}\right)\xrightarrow[\sigma\to 1]{}\varphi_\alpha(x)$, hence for some $\sigma>1$,

$$\left\|\varphi_a\left(\frac{x}{\sigma}\right)-\varphi_\alpha(x)\right\|_{L_2(K_a)}\leq\varepsilon.$$

From the two bounds it follows that for some $\sigma_0>1$ (WLOG $\sigma_0<2$ (?? TODO check typo??) )

$$\begin{aligned}\|\partial^\alpha f-\partial^\alpha F_{\sigma_0}\|_{L_2(K_a)}&\leq\left(1+\sigma_0^{n/2}\right)\varepsilon+\varepsilon+\varepsilon\\&\leq 4\varepsilon.\end{aligned}$$

Summing over all $\alpha$ with $|\alpha|\leq k$ we get an analoguous bound of for the $H^k$-norm:

$$\|f-F_{\sigma_0}\|_{H^k(K_a)}\leq 4n^k\varepsilon.$$

**The Averaging** For sufficiently small $h>0$ ($h<a(\sigma-1)$) we look at the averaging $(F_{\sigma_0})_h\in C^\infty(K_a)$ of $F_{\sigma_0}$. In the narrower domain $K_a$ we have

$$\|(F_{\sigma_0})_h-F_{\sigma_0}\|_{H^k(K_a)}\xrightarrow[h\to 0]{}0,$$

therefore for sufficiently small $h=h_0$,

$$\left\|(F_{\sigma_0})_{h_0}-F_{\sigma_0}\right\|\leq\varepsilon.$$

It remains to observe that

$$\begin{aligned}\left\|(F_{\sigma_0})_{h_0}-f\right\|_{H^k(K_a)}&\leq\underbrace{\left\|(F_{\sigma_0})_{h_0}-F_{\sigma_0}\right\|_{H^k(K_a)}}_{\leq\varepsilon}+\underbrace{\|f-F_{\sigma_0}\|}_{\leq 4n^k\varepsilon}\\&\leq\left(1+4n^k\right)\varepsilon,\end{aligned}$$

which means we can make the averaging close enough to $f$. □

## 2 Change of variables in $H^k$

**Lemma.** *Let $\bar{Q}'\subset Q$, $f\in H^k(Q)$ and $|\alpha|\leq k$. Then for $h\leq dist(Q',\partial Q)$, the derivative of the averaging $f_h\in C^\infty$ is the averaging of the derivative, that is*

$$\partial^\alpha f_h=(\partial^\alpha f)_h\,.$$

*Proof.* We'll check it for $k = 1$. Let $1 \le i \le n$. We have to prove

$$\frac{\partial}{\partial x_i} \int\limits_Q \omega_{x,h}(y) \cdot f(y)\,\mathrm{dy} = \int\limits_Q \omega_{x,h}(y) \cdot \frac{\partial f}{\partial y_i}(y)\,\mathrm{dy}\,.$$

We write out the definition of the generalized derivative $\partial^\alpha f$ in the RHS with a test function $\omega_{x,\varepsilon}$:

$$\int\limits_Q \omega_{x,h}(y) \cdot \frac{\partial f}{\partial y_i}(y)\,\mathrm{dy} = -\int\limits_Q \frac{\partial \omega_{x,h}}{\partial y_i}(y) \cdot f(y)\,\mathrm{dy}\,.$$

Therefore it remains to convince us that

$$\frac{\partial}{\partial x_i} \int\limits_Q \omega_{x,h}(y) \cdot f(y)\,\mathrm{dy} = -\int\limits_Q \frac{\partial \omega_{x,h}}{\partial y_i}(y) \cdot f(y)\,\mathrm{dy}\,.$$

We write out the derivative

$$\frac{\partial \omega_{x,h}(y)}{\partial y_i} = \frac{\partial \omega\left(\frac{x-y}{\varepsilon}\right)}{\partial y_i} = \frac{1}{\varepsilon}\omega'\left(\frac{x-y}{\varepsilon}\right) = \omega\left(\frac{x-y}{\varepsilon}\right)\frac{1}{\left(1-|x|^2\right)^2}\,(2\,|x|)$$

□

**Proposition.** *Let $y : \bar{Q} \to \bar{\Omega}$ and $x : \bar{\Omega} \to \bar{Q}$ be inverse to each other functions in $C^k$.*

*Then, $f \in H^k(\Omega)$ if and only if $F = f \circ y \in H^k(Q)$, and in that case the derivatives of $F$ are calculated by the usual chain rule.*

*Moreover, for these $x$ and $y$ there exist such constants $C_1, C_2$ that*

$$C_1 \|f\|_{H^k(\Omega)} \le \|F\|_{H^k(Q)} \le C_2 \|f\|_{H^k(\Omega)} \quad \textit{for all } f \in H^k(\Omega).$$

*Proof.* Let $f \in H^k$. First, it is clear that $F \in L_2$, because $F = f \circ y$ has an integrable square by the change of variables formula in the Lebesgue integral $\int_Q f(y(x))\mathrm{dx}$.

**Existence of the first derivatives** As in the previous lemma, we will approximate (the almost everywhere defined) $f \in H^k$ with an averaging in $C^\infty$, and then we will see that the derivatives of the average approximate the derivatives of $f$.

To perform averaging at all, we need it to be defined "a little outside," so we narrow the considered region to $\bar{\Omega}' \subset \Omega$. We set $Q' = x(\Omega)$, and let $f_h \in C^\infty(\Omega')$ be the average of $f$ in $\Omega'$ with a sufficiently small radius $h \le \operatorname{dist}(\Omega', \partial\Omega)$. Then, from the property

$$\|f_h - f\|_{L_2(Q)} \xrightarrow[h \to 0]{} 0$$

we get (again by change of variables in the integral)

$$\|F_h - F\|_{L_2(Q')} = \|f_h \circ y - f \circ y\| = \|(f_h - f) \circ y\| \xrightarrow[h\to 0]{} 0.$$

We need an analoguous bound for the derivative of $F$, too. From

$$\left\|\frac{\partial f_h}{\partial y_i} - \frac{\partial f}{\partial y_i}\right\|_{L_2(\Omega')} \xrightarrow[h\to 0]{} 0$$

it follows

$$\begin{aligned}
\left\|\frac{\partial}{\partial x_i}(f_h \circ y) - \frac{\partial}{\partial x_i}(f \circ y)\right\|_{L_2(\Omega')} &= \left\|\frac{\partial}{\partial x_i}\left((f_h - f) \circ y\right)\right\|_{L_2(\Omega')} \\
&\leq \left\|\frac{\partial}{\partial x_i}\left(f_h - f\right)\right\| \left\|\frac{\partial}{\partial x_i} y\right\| \\
&= \tilde{C}\left\|\frac{\partial f_h}{\partial x_i} - \frac{\partial f}{\partial x_i}\right\| \\
&= \tilde{C}\left\|\left(\frac{\partial f}{\partial x_i}\right)_h - \frac{\partial f}{\partial x_i}\right\| \quad (\text{сменяме } \frac{\partial}{\partial x_i} \text{ и } (\cdot)_h) \\
&\xrightarrow[h\to 0]{} 0.
\end{aligned}$$

From these estimations it follows that the function sequences$\left\{F_{h=1/n}\right\}_{n=1}^{\infty}$ and $\left\{\partial^{x_i} F_{h=1/n}\right\}_{n=1}^{\infty}$ are bounded by $F + L$ and $\partial^{x_i} F + L$, respectively, where $L$ is a constant. Then in the definition for a generalized derivative of $F_h = f_h \circ y$

$$\int\limits_{Q'} F_h \frac{\partial g}{\partial x_i} \mathrm{dx} = -\int\limits_{Q'} \frac{\partial F_h}{\partial x_i} g \mathrm{dx} \quad \text{for all } g \in C^1(\bar{Q})$$

we can perform on both sides dominated convergence witj $h \to 0$ and get

$$\int\limits_{Q} \underbrace{\left(\lim_{h\to 0} F_h\right)}_{F} \frac{\partial g}{\partial x_i} \mathrm{dx} = -\int\limits_{Q} \left(\lim_{h\to 0} \frac{\partial F_h}{\partial x_i}\right) g \mathrm{dx} \quad \text{for all } g \in C^1(\bar{Q}),$$

and now we have the first generalized derivatives of $F$:

$$\frac{\partial F}{\partial x_i} = \lim_{h\to 0} \frac{\partial F_h}{\partial x_i}.$$

To ensure $F \in H^1$, it remains only to verify that the derivatives are square-integrable. By the chain rule

$$\frac{\partial F}{\partial x_i}(x) = \sum_j \frac{\partial f}{\partial y_j}(y(x)) \frac{\partial y_j}{\partial x_i}(x),$$

we immediately get the bound

$$\begin{aligned}\left\|\frac{\partial F}{\partial x_i}\right\| &\leq \sum_j \left\|\frac{\partial f}{\partial y_j}\right\|_{L_2(\Omega)} \left\|\frac{\partial y_j}{\partial x_i}\right\|_{L_2(Q)} \\ &\leq C\,\|f\|_{H^1}\,,\end{aligned}$$

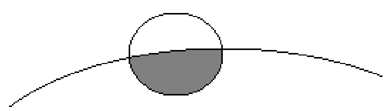
where the constant $C$ depends only on $y$.

The derivatives of order $\geq 2$ we get after repeated differentiation in the chain rule. $\square$

# 3 Extension outside $Q$ with smooth boundary

## 3.1 Locally

Denote by $B_r(\xi)$ the ball $\{x \in \mathbb{R}^n : \|x-\xi\| < r\}$.



In the next lemma, we will ensure that if we have an extension of $f$ in a neighborhood of each point ("individually") of the boundary, then we can construct an extension of $f$ in any arbitrary extension of the considered region.

**Lemma.** *Let $Q \subset \mathbb{R}^n$ be a bounded domain, $f \in H^k(Q)$ and in each point on the boundary of $Q$ there is an extension of $f$ in some ball around that point , with „uniformly smaller“ $H^k$-norm than $f$, that is*

$$\begin{aligned}&\forall \xi \in \partial Q \quad \exists r_\xi > 0 \quad \exists F_\xi \in H^k\left(B_r(\xi)\right): \\ &\qquad\qquad F_\xi\big|_Q \equiv f \ \text{and} \ \|F_\xi\|_{H^k(B_r(\xi))} \leq C\,\|f\|_{H^k(Q)}\,,\end{aligned}$$

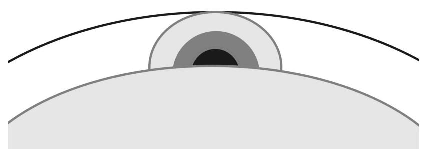
*where $C$ is a constant.*

*Then in every superset $Q_\rho := \{x : dist(x,Q) < \rho\}$ of $Q$ there exists an extension $F \in H^k(Q_\rho)$ of $f$, vanishing outside $Q_{\rho/2}$ with „uniformly smaller“ norm than $f$, that is*

$$\begin{aligned}\forall \rho > 0 \quad \exists F \in H^k(Q_\rho) : F\big|_Q &\equiv f, \\ \text{and } F\big|_{Q_\rho \setminus Q_{\rho/2}} &\equiv 0 \\ \text{and } \|F\|_{H^k(Q_\rho)} &\leq C'\,\|f\|_{H^k(Q)}\end{aligned}$$

*where the constant $C$ does not depend on $f$.*

*Proof.* With the given $F_\xi$ we'll build extensions $f_i$ of $f$ in open supersets $Q_i \supset Q$, for which $\bigcup_i Q_i \supset \bar{Q}$. Then we'll smooth them out to zero outside of $Q_{\rho/2}$ and with suitable weights we'll set their sum to be the extension wanted.



In the colored regions, we use the extension by assumption or the original values of $f$, while in the dark gray region $\left(\frac{\rho}{2}\right)$, we will smooth the jump that otherwise occurs between the light gray and the white regions.

**1. A cover and rought extensions** Fix $\rho > 0$. We can assume the radii $r_\xi$ smaller than $\rho$.

First we'll extend the $r_\xi$ to be defined not only for $\xi \in \partial Q$, but over the whole $\bar{Q}$. By lemma assumption, around each point $\xi \in \bar{Q}$ there is a ball$B_r(\xi), r = r(\xi)$, where either $f$ is defined, or a smooth extension of $f$.

It is clear that

$$Q \subset \bigcup_{\xi \in \bar{Q}} B_{r_\xi/3}(\xi).$$

We use only half a radius $\frac{1}{2} r_\xi$ in order to have space for the cut later. Since $Q$ is bounded, $\bar{Q}$ is a compact, hence we can choose from the cover $\bigcup_{\xi \in \bar{Q}} B_{r_\xi/2}(\xi)$ a finite subcover $B_{r_1/2}(\xi_1), \dots, B_{r_N/2}(\xi_N)$.

Now we define the functions $f_i \in L_2(\mathbb{R}^n), i = 1, \dots, N$ in the way described above – (rough) extensions of $f$ over the entire$\mathbb{R}^n$, supported in $Q \cup B_{r_i}(x_i)$:

$$f_i(x) := \begin{cases} F_{x_i}(x) & \text{, if } x \in B_{r_i}(x_i) \\ f(x) & \text{, if } x \in Q \\ 0 & \text{, else} \end{cases}$$

Let's bound their norm

$$\begin{aligned} \|f_i\|_{H^k(Q \cup B_{r_i}(x_i))} &\leq \|F_{\xi_i}\| + \|f\| \\ &\leq C\|f\| + \|f\| = \tilde{C}\|f\|. \end{aligned}$$

It remains to glue them with suitable $\gamma_i$, to define a now smooth extension

$$F = \sum_i \gamma_i f_i.$$

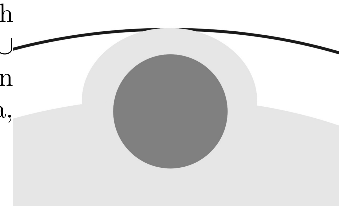
**2. Glueing by partition of unity** For $F$ to be an extension of $f$ at all, we need $F\big|_Q \equiv f$. For $x \in Q$ we have defined $f_i(x) = f(x)$, therefore

$$F\big|_Q = \sum_i \gamma_i f_i = \sum_i \gamma_i f = f \sum_i \gamma_i.$$

Since we want $F\big|_Q \equiv f$, we need to have $\sum_i \gamma_i\big|_Q \equiv 1$. This reminds us to use partition of unity.

Let $\{\gamma_i\}_{i=1}^N$ be a partition of unity subordinate to the cover $\left\{B_{r_i/2}\right\}_{i=1}^N$ and consisting of compactly supported in $Q_{\rho/2}$ functions, and $F = \sum_i \gamma_i f_i$. According to what we just calculated and the property $\sum_i \gamma_i\big|_Q = 1$, $F$ is truly an *extension* of $f$. It remains to check it satisfies the needed properties:

1. $F \in H^k(Q_\rho)$, because every $\gamma_i$ smooths the edge of the respective $f_i$, since every $f_i$ can be non-smooth only on the boundary of the light-gray area $Q \cup B_{r_i}(x_i)$, a neighbourhood of which is contained in $Q_{\rho/2} \setminus B_{r_i/2}$ – the complement of the dark area, where $\gamma_i$ vanishes.



2. $F\big|_{Q_\rho \setminus Q_{\rho/2}} \equiv 0$, since we required $\gamma_i$ to be compactly supported in the respective set.

3. $\|F\|_{H^k(Q_\rho)} \leq C' \|f\|_{H^k(Q)}$, since

$$\begin{aligned}\|F\|_{H^k(Q_\rho)} &= \left\| \sum_i \gamma_i f_i \right\|_{H^k(Q_\rho)} \\ &\leq \sum_i \|\gamma_i\| \|f_i\| \\ &\leq \sum_i \|\gamma_i\| \tilde{C} \|f\| \\ &\leq C' \|f\|,\end{aligned}$$

where in the last inequality we used that $\gamma_i$ depend not on $f$, but only on the domain — so in our context, their norm turns out to be a constant.

□

## 3.2 Globally

**Theorem** (for the extension). *Let $Q$ and $Q'$ be bounded domains in $\mathbb{R}^n$, s.t. $\bar{Q} \subset Q'$ and $\partial Q \in C^k$.*

*Then every $f \in H^k(Q)$ has a compactly supported (vanishing in a neighbourhood of $\partial Q'$) extension $F \in H^k(Q')$, for which the bound holds*

$$\|F\|_{H^k(Q')} \leq C \|f\|_{H^k(Q)}$$

*with a constant $C$, independent of $f$.*

In the bound of $F$ in the theorem we can also instead of the$H^k$-norm take any norm$H^s$ with $s \leq k$.

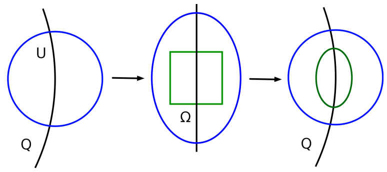
*Proof.* We'll build an extension of $f$ in a neighbourhood of a given point on the boundary of the domain, after which we'll apply the previous lemma.

Let $\xi \in \partial Q$. We will construct an extension in a neighbourhood $U_\xi$ of $\xi$ in 3 steps:

1. We will flatten (bijectively) the boundary around $\xi$ to the hypersurface$\{y_n = 0\}$;

2. We will apply the lemma for extension from $\overline{K_a^+}$ to $\overline{K_a}$ from the previous section;

3. We will send the extension from $K_a^-$ back behind the non-flat boundary (?? TODO rephrase ??).

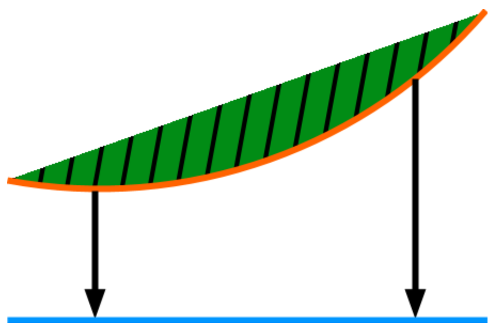



Since $\partial Q \in C^k$, in a neighbourhood $U_\xi$ of $\xi$ we can, up to reordering variables, represent the equations of $\partial Q$ in the form

$$x_n = \varphi(x_1, \ldots, x_{n-1}),$$

where $\varphi \in C^k(D)$ in a domain $D \subset \mathbb{R}^{n-1}$. Moreover we'll require the domain $Q$ in $U_\xi$ to lie „above", that is

$$x_n > \varphi(x_1, \ldots, x_{n-1}) \quad \text{for all } x \in U_\xi \cap Q.$$



We change the variables so that $\xi$ is in the origin of of the coordinate system $y_1, \ldots, y_n$, and the hypersurface$\{y_n = 0\}$ fits $\partial Q \cap U_\xi$, that is

$$\begin{aligned} y' &\coloneqq x' - \xi' \\ y_n &\coloneqq x_n - \varphi(x'). \end{aligned}$$

Then $U_\xi$ is mapped bijectively to a neighbourhood $\Omega$ of 0, and the function $f\big|_{Q \cap U_\xi}$ – to the function

$$z(y) = f\left(y' + \xi', y_n + \varphi(y' + \xi')\right),$$

which, according to an earlier proposition is again in $H^k$. It is time to apply the lemma from the first section.

Let $K_a$ be cube with a sufficiently small side to fit $K_a \subset \Omega$. Then $z$, as a function from $H^k(\overline{K_a^+})$, has an extension $Z \in H^k\left(\overline{K_a}\right)$, which after an inverse change of variables gives us an extension $F_\xi$ of $f$ in the set $y^{-1}(K_a)$.

$y^{-1}(K_a)$ is open and contains $\xi$, hence we have an extension of $f$ in some ball$B_r(\xi)$. To apply the previous lemma, though, we need one more condition.

From the property for change of variables (?? TODO revisit wording ??) for the function $F_\xi$ we have

$$\|F_\xi\|_{H^k(B_r(\xi))} \leq \|F_\xi\|_{H^k(y^{-1}(\Omega))} \leq C_3 \|Z\|_{H^k(K_a)},$$

and for the function $f$ –

$$\|z\|_{H^k(K_a^+)} \leq C_4 \|f\|_{H^k(Q \cap y^{-1}(\Omega))} \leq C_4 \|f\|_{H^k(Q)},$$

where the constants $C_3, C_4$ depend only on the change $y = y(x)$, that is – in our case – depend on $\partial Q$ and are universal for all functions we may want to extend. With this we are allowed to apply the last lemma with $\rho < \operatorname{dist}(Q, Q')$ and get an extension of $f$ over the entire $Q$. $\square$

# 4 Extension "inward" in $Q$ for $\partial Q \in C^k$

**Theorem.** *Let $Q$ have a smooth boundary $\partial Q \in C^k$ for some $k \geq 1$. Then every function $f \in C^k(\partial Q)$ has an extension $F \in C^k(\bar{Q})$, satisfying the bound*

$$\|F\|_{C^k(\bar{Q})} \leq C \|f\|_{C^k(\partial Q)},$$

*where the constant $C$ depends only on $\partial Q$.*

Without proof.