Additional OSF Materials for "Cubic Response Surface Analysis: Investigating Asymmetric and Level-Dependent Congruence Effects With Third-Order Polynomial Models"

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Contents

A OSF Material A: Mathematical Proofs of Claims About the Strict Simple Congruence Model

A.1 Basics and Notation

Having estimated the regression coefficients of the strict simple congruence model, we write the estimated model equation as a function $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto \hat{z}$:

$$\begin{aligned} f(x,y) = \hat{z} &= \hat{c_0} + \hat{c_1}(x-y)^2 \\ &= \hat{c_0} + 0x + 0y + \hat{c_1}x^2 - 2\hat{c_1}xy + \hat{c_1}y^2 \end{aligned} \tag{1}$$

A.2 The Strict Simple Congruence Model is in Line With a *Congruence Effect*, Reflected in $\hat{c_1}$

We will now show that the strict simple congruence model reflects a congruence effect as defined in Table 2 in the paper: "For two people, the person whose x and y values are closer to one another has the higher (or lower) value of z." Furthermore, we will show that the sign of the estimated coefficient $\hat{c_1}$ affects the direction of the congruence effect. More specifically, we make the following claim about the function f that is defined in Equation 1:

Claim. Let $(x_i, y_i), (x_j, y_j) \in \mathbb{R}^2$ satisfy $|x_i - y_i| < |x_j - y_j|$ (i.e., x_i and y_i are closer to one another than x_j and y_j). Then the following inequalities hold:

$$\begin{cases} f(x_i, y_i) > f(x_j, y_j), & \text{if } \hat{c_1} < 0 \\ f(x_i, y_i) < f(x_j, y_j), & \text{if } \hat{c_1} > 0. \end{cases} \tag{2}$$

Proof. Let $(x_i, y_i), (x_j, y_j) \in \mathbb{R}^2$ satisfy $|x_i - y_i| < |x_j - y_j|$. We have

$$\begin{aligned} |x_i - y_i| &< |x_j - y_j| \\ \Leftrightarrow (x_i - y_i)^2 &< (x_j - y_j)^2 \end{aligned} \tag{3}$$

where this equivalence relation holds because the function $q : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, x \mapsto x^2$ is strictly monotonically increasing and $|x_i - y_i|, |x_j - y_j| \in \mathbb{R}_{\geq 0}$. We now prove the claim separately for the two cases that are distinguished in Equation 2.

Case 1: $\boldsymbol{\hat{c}_1 < 0}$***.*** We have

$$f(x_i, y_i) = \hat{c}_0 + \underbrace{\hat{c}_1}_{<0} \underbrace{(x_i - y_i)^2}_{<(x_j - y_j)^2} > \hat{c}_0 + \hat{c}_1 (x_j - y_j)^2 = f(x_j, y_j). \tag{4}$$

Case 2: $\boldsymbol{c_1 > 0}$***.*** We have

$$f(x_i, y_i) = \hat{c}_0 + \underbrace{\hat{c}_1}_{>0} \underbrace{(x_i - y_i)^2}_{<(x_j - y_j)^2} < \hat{c}_0 + \hat{c}_1 (x_j - y_j)^2 = f(x_j, y_j). \tag{5}$$

□

B OSF Material B: Mathematical Proofs of Claims About the Strict Asymmetric Congruence Model

B.1 Basics and Notation

Having estimated the regression coefficients of the strict asymmetric congruence model, we write the estimated model equation as a function $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto \hat{z}$:

$$\begin{aligned} f(x,y) = \hat{z} &= \hat{c_0} + \hat{c_1}(x-y)^2 + \hat{c_2}(x-y)^3 \\ &= \hat{c_0} + 0x + 0y + \hat{c_1}x^2 - 2\hat{c_1}xy + \hat{c_1}y^2 + \hat{c_2}x^3 - 3\hat{c_2}x^2y + 3\hat{c_2}xy^2 - \hat{c_2}y^3 \end{aligned} \tag{6}$$

Now consider the predictor combinations (x, y) that lie on the line g_k that lies in the xy plane and is perpendicular to the LOC and runs through the point (k, k). Formally, g_k is defined as a linear equation $g_k : \mathbb{R} \to \mathbb{R}, x \mapsto -x + 2k$.

The outcome values that the asymmetric congruence model (Equation 6) predicts for predictor combinations that lie on g_k are thus given by (for any $k \in \mathbb{R}$):

$$\begin{aligned} f_{|g_k}(x) &= f(x, g_k(x)) = f(x, -x+2k) = \hat{c_0} + \hat{c_1}(x-(-x+2k))^2 + \hat{c_2}(x-(-x+2k))^3 \\ &= (\hat{c_0} + 4\hat{c_1}k^2 - 8\hat{c_2}k^3) + (-8\hat{c_1}k + 24\hat{c_2}k^2)x + (4\hat{c_1} - 24\hat{c_2}k)x^2 + 8\hat{c_2}x^3 \end{aligned} \tag{7}$$

The first derivative of $f_{|g_k}$ is:

$$\frac{\partial f_{|g_k}(x)}{\partial x} = -8\hat{c_1}k + 24\hat{c_2}k^2 + (8\hat{c_1} - 48\hat{c_2}k)x + 24\hat{c_2}x^2 \tag{8}$$

The first derivative $\frac{\partial f_{|g_k}(x)}{\partial x}$ is zero for $x_1 = k$ and for $x_2 = k - \frac{\hat{c_1}}{3\hat{c_2}}$. Thus, the extremum values of $f_{|g_k}$ are:

$$\begin{aligned} (x_1, g_k(x_1)) &= (k, k) \\ &\text{and} \\ (x_2, g_k(x_2)) &= \left(k - \frac{\hat{c_1}}{3\hat{c_2}}, k + \frac{\hat{c_1}}{3\hat{c_2}}\right). \end{aligned} \tag{9}$$

Finally, the second derivative of $f_{|g_k}$ is given by:

$$\frac{\partial^2 f_{|g_k}(x)}{\partial^2 x} = 8\hat{c_1} - 48\hat{c_2}k + 48\hat{c_2}x \tag{10}$$

B.2 Formula of E_2 as a Linear Equation

We claim that, as a linear equation, the second extremum line E_2 is given by $y = x + \frac{2\hat{c_1}}{3\hat{c_2}}$.

By definition, E_2 consists of all predictor combinations
$(x_{k2}, y_{k2}) = (k - \frac{\hat{c_1}}{3\hat{c_2}}, -(k - \frac{\hat{c_1}}{3\hat{c_2}}) + 2k) = (k - \frac{\hat{c_1}}{3\hat{c_2}}, k + \frac{\hat{c_1}}{3\hat{c_2}}), k \in \mathbb{R}$.

To write E_2 as a linear equation, we need to determine the slope m and intercept b of the linear function $E_2 : \mathbb{R} \to \mathbb{R}, x \mapsto y = mx + b$ which satisfies $E_2(k - \frac{\hat{c_1}}{3\hat{c_2}}) = k + \frac{\hat{c_1}}{3\hat{c_2}}$ for all $k \in \mathbb{R}$. That is:

$$\begin{aligned}
& E_2(k - \tfrac{\hat{c_1}}{3\hat{c_2}}) &= k + \tfrac{\hat{c_1}}{3\hat{c_2}} \\
\Leftrightarrow\quad & m(k - \tfrac{\hat{c_1}}{3\hat{c_2}}) + b &= k + \tfrac{\hat{c_1}}{3\hat{c_2}} \\
\Leftrightarrow\quad & mk - m\tfrac{\hat{c_1}}{3\hat{c_2}} + b &= k + \tfrac{\hat{c_1}}{3\hat{c_2}} \\
\overset{m=1}{\Leftrightarrow}\quad & k - \tfrac{\hat{c_1}}{3\hat{c_2}} + b &= k + \tfrac{\hat{c_1}}{3\hat{c_2}} \\
\Leftrightarrow\quad & b &= \tfrac{2\hat{c_1}}{3\hat{c_2}}
\end{aligned} \tag{11}$$

where $m = 1$ follows because the statement in the third row of Equation 11 is true for all $k \in \mathbb{R}$. Taken together, E_2 is given by the linear equation $E_2 : \mathbb{R} \to \mathbb{R}$,
$x \mapsto y = mx + b = 1x + \frac{2\hat{c_1}}{3\hat{c_2}}$.

□

B.3 All Predictor Combinations on E_2 Have the Same Predicted Outcome Value and Standard Error

We claim that the strict asymmetric congruence model (Equation 6) makes the same outcome prediction $\hat{z}$ for all (x, y) on E_2, and that the standard error of $\hat{z}$ is also the same for all of these points.

We will even prove a more general version of this claim: Given *any* arbitrary line h that is parallel to the LOC (e.g., E_2), the strict asymmetric congruence model predicts the same outcome value for all points (x, y) on h, and the estimated standard error of $\hat{z} = f(x, y)$ is also the same for all of these points.

To prove this, let h be an arbitrary line that is parallel to the LOC, that is, $h : \mathbb{R} \to \mathbb{R}, x \mapsto x + \eta$ for some $\eta \in \mathbb{R}$. For any point $(x, y) = (x, x + \eta)$ on h, the asymmetric congruence model (Equation 6) predicts the following outcome value:

$$\begin{aligned} f(x, x+\eta) &= \hat{c_0} + \hat{c_1}(x - (x+\eta))^2 + \hat{c_2}(x - (x+\eta))^3 \\ &= \hat{c_0} + \hat{c_1}\eta^2 - \hat{c_2}\eta^3 \end{aligned} \tag{12}$$

Because $f(x, x+\eta) = \hat{c_0} + \hat{c_1}\eta^2 - \hat{c_2}\eta^3$ does not depend on the value of x, the outcome prediction of the strict asymmetric congruence model is constant for all points $(x, y) = (x, x + \eta)$ on h.

Turning to the standard error of the predicted outcome values, the estimated standard error of $f(x, y)$ with $(x, y) = (x, x + \eta)$ any point on h can be computed as

$$se(f(x, x+\eta)) = \sqrt{\mathbf{x}_{\eta}{}^t \; \Sigma_{\hat{b}} \; \mathbf{x}_{\eta}} \tag{13}$$

Here, $\Sigma_{\hat{b}}$ is the covariance matrix of the vector $\hat{b} = (\hat{b_0}, \ldots, \hat{b}_9)$ of the estimated regression coefficients (in the notation of the full third-order polynomial model). In the strict asymmetric congruence model (Equation 6), $\hat{b}$ simplifies to

$$\hat{b} = (\hat{c_0},\ 0,\ 0,\ \hat{c_1},\ -2\hat{c_1},\ \hat{c_1},\ \hat{c_2},\ -3\hat{c_2},\ 3\hat{c_2},\ -\hat{c_2}). \tag{14}$$

The vector $\mathbf{x}_{\eta}$ in Equation 13 is the vector of predictor variables at the point $(x, x + \eta)$:

$$\mathbf{x}_{\eta} = (1,\ x,\ x+\eta,\ x^2,\ x(x+\eta),\ (x+\eta)^2,\ x^3,\ x^2(x+\eta),\ x(x+\eta)^2,\ (x+\eta)^3). \tag{15}$$

One can now use Equations 13, 14, and 15 to compute the estimated standard error of $f(x, x + \eta)$ for any $x \in \mathbb{R}$, which yields:

$$\begin{aligned} &se(f(x, x+\eta)) \\ &= \sqrt{\mathrm{Var}(\hat{c_0}) \ + 2\,\mathrm{Cov}(\hat{c_0}, \hat{c_1})\eta^2 \ - 2\,\mathrm{Cov}(\hat{c_0}, \hat{c_2})\eta^3 \ + \mathrm{Var}(\hat{c_1})\eta^4 \ - 2\,\mathrm{Cov}(\hat{c_1}, \hat{c_2})\eta^5 \ + \mathrm{Var}(\hat{c_2})\eta^6}. \end{aligned} \tag{16}$$

That is, the standard error of $f(x, x + \eta)$ does not depend on the value of x; it takes the same value for all points $(x, y) = (x, x + \eta)$ on h.

To sum up, for all points $(x, y) = (x, x + \eta)$, the predicted outcome value and the standard error of the outcome prediction are constant, respectively. Because this is also true for $\eta = \frac{2\hat{c_1}}{3\hat{c_2}}$, the claim particularly holds for all points (x, y) on $E_2 : \mathbb{R} \to \mathbb{R}$, $x \mapsto y = x + \frac{2\hat{c_1}}{3\hat{c_2}}$. $\square$

B.4 The Strict Asymmetric Congruence Model is in Line With a *Congruence Effect*, Reflected in $\hat{c_1}$

Consider the strict asymmetric congruence model as defined in Equation 6, and let $\hat{c_1} \neq 0$ and $\hat{c_2} \neq 0$. For the proof of model conformity with a congruence effect, we restrict the domain of predictor combinations to the combinations (x, y) that lie "in front of" the line E_2. More specifically, we define the domain $D(\hat{c_1}, \hat{c_2})$ of predictor combinations "in front of" E_2 as follows, depending on the constellation of $\hat{c_1}$ and $\hat{c_2}$:

$$D(\hat{c_1}, \hat{c_2}) := \begin{cases} \{(x, y) \in \mathbb{R}^2 \mid y < x + \frac{2\hat{c_1}}{3\hat{c_2}}\}, & \text{if } (\hat{c_1} < 0 \text{ and } \hat{c_2} < 0) \text{ or } (\hat{c_1} > 0 \text{ and } \hat{c_2} > 0), \\ \{(x, y) \in \mathbb{R}^2 \mid y > x + \frac{2\hat{c_1}}{3\hat{c_2}}\}, & \text{if } (\hat{c_1} < 0 \text{ and } \hat{c_2} > 0) \text{ or } (\hat{c_1} > 0 \text{ and } \hat{c_2} < 0). \end{cases} \tag{17}$$

We will now show that the predictions of f (Equation 6) for predictor combinations $(x, y) \in D(\hat{c_1}, \hat{c_2})$ are in line with a congruence effect as defined in Table 2 in the paper: "For two people who either both have $x > y$ or both have $x < y$, the person whose x and y values are closer to one another has the higher (or lower) value of z." Furthermore, we will show that the sign of the estimated coefficient $\hat{c_1}$ affects the direction of the congruence effect. Specifically, we claim the following:

Claim. Let $(x_i, y_i), (x_j, y_j) \in D(\hat{c_1}, \hat{c_2})$ with the following two properties (hereafter referred to as "Property (1)" and "Property (2)"):

(1) $(x_i > y_i$ and $x_j > y_j)$ or $(x_i < y_i$ and $x_j < y_j)$,

(2) $|x_i - y_i| < |x_j - y_j|$.

Then the following inequalities hold:

$$\begin{cases} f(x_i, y_i) > f(x_j, y_j), & \text{if } \hat{c_1} < 0 \\ f(x_i, y_i) < f(x_j, y_j), & \text{if } \hat{c_1} > 0. \end{cases} \tag{18}$$

Proof. Let $(x_i, y_i), (x_j, y_j) \in D(\hat{c_1}, \hat{c_2})$ satisfy Properties (1) and (2). We show the inequalities in Equation 18 by distinguishing between four cases, determined by the constellations of the signs of $\hat{c_1}$ and $\hat{c_2}$.

***Case 1:* $\hat{c_1} < 0$ *and* $\hat{c_2} < 0$.** We need to show that $f(x_i, y_i) > f(x_j, y_j)$.

Consider the auxiliary function

$$h : \mathbb{R} \to \mathbb{R}, d \mapsto \hat{c_0} + \hat{c_1}d^2 + \hat{c_2}d^3. \tag{19}$$

Note that $f(x, y) = h(x - y)$; this will be of use at a later point in the proof. The first derivative of h is $h'(d) = 2\hat{c_1}d + 3\hat{c_2}d^2 = (2\hat{c_1} + 3\hat{c_2}d)d$, so the extrema of h are located at $d_1 = 0$ and $d_2 = -\frac{2\hat{c_1}}{3\hat{c_2}}$, where $d_2 < 0$ because $\hat{c_1} < 0$ and $\hat{c_2} < 0$. Furthermore, because $\hat{c_2} < 0$, we have $\lim_{d\to-\infty} h(d) = \infty$ and $\lim_{d\to\infty} h(d) = -\infty$. Figure 1 shows an example graph of h that can be used to trace back these observations. Overall, we know that

$$h \text{ is } \begin{cases} \text{strictly monotonically decreasing on the inverval } (-\infty, -\frac{2\hat{c_1}}{3\hat{c_2}}], \\ \text{strictly monotonically increasing on the inverval } [-\frac{2\hat{c_1}}{3\hat{c_2}}, 0], \\ \text{strictly monotonically decreasing on the inverval } [0, \infty). \end{cases} \tag{20}$$

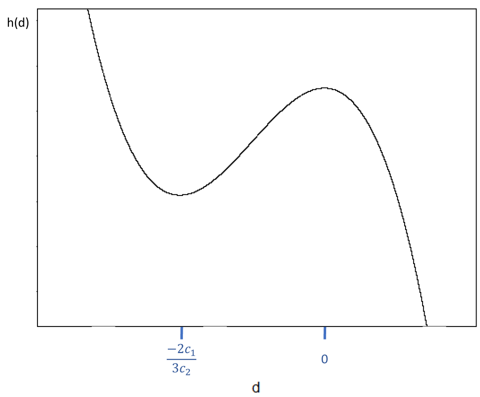


Figure 1. Prototypical graph of $h(d) = \hat{c_0} + \hat{c_1}d^2 + \hat{c_2}d^3$ for $\hat{c_1} < 0$ and $\hat{c_2} < 0$.

Now consider the two points (x_i, y_i) and (x_j, y_j). Because $(x_i, y_i), (x_j, y_j) \in D(\hat{c_1}, \hat{c_2}) = \{(x, y) \in \mathbb{R}^2 \mid y < x + \frac{2\hat{c_1}}{3\hat{c_2}}\}$, we know that $x_i - y_i > -\frac{2\hat{c_1}}{3\hat{c_2}}$ and $x_j - y_j > -\frac{2\hat{c_1}}{3\hat{c_2}}$. From Property (1), it thus follows that either (Case A) $(x_i - y_i), (x_j - y_j) \in [-\frac{2\hat{c_1}}{3\hat{c_2}}, 0]$ or (Case B) $(x_i - y_i), (x_j - y_j) \in [0, \infty)$. We treat these two cases separately.

Case A: $\boldsymbol{(x_i - y_i), (x_j - y_j) \in [-\frac{2\hat{c_1}}{3\hat{c_2}}, 0]}$. Because $(x_i - y_i), (x_j - y_j) < 0$ and $|x_i - y_i| < |x_j - y_j|$ (Property (2)), we have

$$x_i - y_i = -|x_i - y_i| > -|x_j - y_j| = x_j - y_j. \tag{21}$$

Because h is strictly monotonically increasing on $[-\frac{2\hat{c_1}}{3\hat{c_2}}, 0]$, we can thus conclude that

$$f(x_i, y_i) = h(x_i - y_i) > h(x_j - y_j) = f(x_j, y_j). \tag{22}$$

Case B: $\boldsymbol{(x_i - y_i), (x_j - y_j) \in [0, \infty)}$*.* Because of $(x_i - y_i), (x_j - y_j) > 0$ and Property (2), we have

$$x_i - y_i = |x_i - y_i| < |x_j - y_j| = x_j - y_j. \tag{23}$$

Because h is strictly monotonically decreasing on $[0, \infty)$, it again follows that

$$f(x_i, y_i) = h(x_i - y_i) > h(x_j - y_j) = f(x_j, y_j). \tag{24}$$

Case 2: $\boldsymbol{\hat{c_1} < 0}$ ***and*** $\boldsymbol{\hat{c_2} > 0}$***.*** We need to show that $f(x_i, y_i) > f(x_j, y_j)$.

The proof for this case follows the same idea as for Case 1. Consider again the auxiliary function h defined as in Case 1. Because $\hat{c_1} < 0$ and $\hat{c_2} > 0$, the d value of the second extremum of h is positive: $d_2 = -\frac{2\hat{c_1}}{3\hat{c_2}} > 0$. Moreover, because $\hat{c_2} > 0$, we have $\lim_{d \to -\infty} h(d) = -\infty$ and $\lim_{d \to \infty} h(d) = \infty$. Overall, we thus know that

$$h \text{ is } \begin{cases} \text{strictly monotonically increasing on the inverval } (-\infty, 0], \\ \text{strictly monotonically decreasing on the inverval } [0, -\frac{2\hat{c_1}}{3\hat{c_2}}], \\ \text{strictly monotonically increasing on the inverval } [-\frac{2\hat{c_1}}{3\hat{c_2}}, \infty). \end{cases} \tag{25}$$

We have $(x_i, y_i), (x_j, y_j) \in D(\hat{c_1}, \hat{c_2}) = \{(x, y) \in \mathbb{R}^2 \mid y > x + \frac{2\hat{c_1}}{3\hat{c_2}}\}$, so $x_i - y_i < -\frac{2\hat{c_1}}{3\hat{c_2}}$ and $x_j - y_j < -\frac{2\hat{c_1}}{3\hat{c_2}}$. From Property (1), it follows that either (Case A) $(x_i - y_i), (x_j - y_j) \in (-\infty, 0]$ or (Case B) $(x_i - y_i), (x_j - y_j) \in [0, -\frac{2\hat{c_1}}{3\hat{c_2}}]$.

Case A: $\boldsymbol{(x_i - y_i), (x_j - y_j) \in (-\infty, 0]}$*.* Because of $(x_i - y_i), (x_j - y_j) < 0$ and Property (2), we have $x_i - y_i > x_j - y_j$. Because h is strictly monotonically increasing on $(-\infty, 0]$, we find

$$f(x_i, y_i) = h(x_i - y_i) > h(x_j - y_j) = f(x_j, y_j). \tag{26}$$

Case B: $\boldsymbol{(x_i - y_i), (x_j - y_j) \in [0, -\frac{2\hat{c_1}}{3\hat{c_2}}]}$. Because of $(x_i - y_i), (x_j - y_j) > 0$ and Property (2), we have $x_i - y_i < x_j - y_j$. h is strictly monotonically decreasing on $[0, -\frac{2\hat{c_1}}{3\hat{c_2}}]$, so

$$f(x_i, y_i) = h(x_i - y_i) > h(x_j - y_j) = f(x_j, y_j). \tag{27}$$

Case 3: $\boldsymbol{\hat{c_1} > 0}$ ***and*** $\boldsymbol{\hat{c_2} < 0}$***.*** We need to show that $f(x_i, y_i) < f(x_j, y_j)$.

Again, the proof follows the same idea as for Case 1, using the auxiliary function h as defined in Case 1. Because $\hat{c_1} > 0$ and $\hat{c_2} < 0$, the d value of the second extremum of h is positive: $d_2 = -\frac{2\hat{c_1}}{3\hat{c_2}} > 0$. Moreover, because $\hat{c_2} < 0$, we have $\lim_{d \to -\infty} h(d) = \infty$ and $\lim_{d \to \infty} h(d) = -\infty$. Overall, we thus know that

$$h \text{ is } \begin{cases} \text{strictly monotonically decreasing on the inverval } (-\infty, 0], \\ \text{strictly monotonically increasing on the inverval } [0, -\frac{2\hat{c_1}}{3\hat{c_2}}], \\ \text{strictly monotonically decreasing on the inverval } [-\frac{2\hat{c_1}}{3\hat{c_2}}, \infty). \end{cases} \tag{28}$$

We have $(x_i, y_i), (x_j, y_j) \in D(\hat{c_1}, \hat{c_2}) = \{(x, y) \in \mathbb{R}^2 \mid y > x + \frac{2\hat{c_1}}{3\hat{c_2}}\}$, so $x_i - y_i < -\frac{2\hat{c_1}}{3\hat{c_2}}$ and $x_j - y_j < -\frac{2\hat{c_1}}{3\hat{c_2}}$. From Property (1), it follows that either (Case A) $(x_i - y_i), (x_j - y_j) \in (-\infty, 0]$ or (Case B) $(x_i - y_i), (x_j - y_j) \in [0, -\frac{2\hat{c_1}}{3\hat{c_2}}]$.

Case A: $\boldsymbol{(x_i - y_i), (x_j - y_j) \in (-\infty, 0]}$. Because of $(x_i - y_i), (x_j - y_j) < 0$ and Property (2), we have $x_i - y_i > x_j - y_j$. Because h is strictly monotonically decreasing on $(-\infty, 0]$, we find

$$f(x_i, y_i) = h(x_i - y_i) < h(x_j - y_j) = f(x_j, y_j). \tag{29}$$

Case B: $\boldsymbol{(x_i - y_i), (x_j - y_j) \in [0, -\frac{2\hat{c_1}}{3\hat{c_2}}]}$. Because of $(x_i - y_i), (x_j - y_j) > 0$ and Property (2), we have $x_i - y_i < x_j - y_j$. h is strictly monotonically increasing on $[0, -\frac{2\hat{c_1}}{3\hat{c_2}}]$, so

$$f(x_i, y_i) = h(x_i - y_i) < h(x_j - y_j) = f(x_j, y_j). \tag{30}$$

Case 4: $\boldsymbol{\hat{c_1} > 0}$ ***and*** $\boldsymbol{\hat{c_2} > 0}$***.*** We need to show that $f(x_i, y_i) < f(x_j, y_j)$.

Again, the proof follows the same idea as for Case 1, using the auxiliary function h as defined in Case 1. Because $\hat{c_1} > 0$ and $\hat{c_2} > 0$, the d value of the second extremum of h is negative: $d_2 = -\frac{2\hat{c_1}}{3\hat{c_2}} < 0$. Moreover, because $\hat{c_2} > 0$, we have $\lim_{d \to -\infty} h(d) = -\infty$ and

$\lim_{d\to\infty} h(d) = \infty$. Overall, we thus know that

$$h \text{ is } \begin{cases} \text{strictly monotonically increasing on the inverval } (-\infty, -\frac{2\hat{c_1}}{3\hat{c_2}}], \\ \text{strictly monotonically decreasing on the inverval } [-\frac{2\hat{c_1}}{3\hat{c_2}}, 0], \\ \text{strictly monotonically increasing on the inverval } [0, \infty). \end{cases} \tag{31}$$

We have $(x_i, y_i), (x_j, y_j) \in D(\hat{c_1}, \hat{c_2}) = \{(x, y) \in \mathbb{R}^2 \mid y < x + \frac{2\hat{c_1}}{3\hat{c_2}}\}$, so $x_i - y_i > -\frac{2\hat{c_1}}{3\hat{c_2}}$ and $x_j - y_j > -\frac{2\hat{c_1}}{3\hat{c_2}}$. From Property (1), it follows that either (Case A) $(x_i - y_i), (x_j - y_j) \in [-\frac{2\hat{c_1}}{3\hat{c_2}}, 0]$ or (Case B) $(x_i - y_i), (x_j - y_j) \in [0, \infty)$.

Case A: $\boldsymbol{(x_i - y_i), (x_j - y_j) \in [-\frac{2\hat{c_1}}{3\hat{c_2}}, 0]}$. Because of $(x_i - y_i), (x_j - y_j) < 0$ and Property (2), we have $x_i - y_i > x_j - y_j$. Because h is strictly monotonically decreasing on $[-\frac{2\hat{c_1}}{3\hat{c_2}}, 0]$, we find

$$f(x_i, y_i) = h(x_i - y_i) < h(x_j - y_j) = f(x_j, y_j). \tag{32}$$

Case B: $\boldsymbol{(x_i - y_i), (x_j - y_j) \in [0, \infty)}$. Because of $(x_i - y_i), (x_j - y_j) > 0$ and Property (2), we have $x_i - y_i < x_j - y_j$. h is strictly monotonically increasing on $[0, \infty)$, so

$$f(x_i, y_i) = h(x_i - y_i) < h(x_j - y_j) = f(x_j, y_j). \tag{33}$$

$\square$

B.5 The Strict Asymmetric Congruence Model is in Line With an *Asymmetry Effect*, Reflected in $\hat{c_2}$

We will now show that the predictions of the strict asymmetric congruence model (Equation 6) are in line with an asymmetry effect as defined in Table 2 in the paper: "For two people whose discrepancy values $x - y$ are equal in magnitude but opposite in sign, the person with $x < y$ has the higher (or lower) value of z." Furthermore, we will show that the sign of the estimated coefficient $\hat{c_2}$ affects the direction of the asymmetry effect. Specifically, we consider the following claim:

Claim. Let $(x_i, y_i), (x_j, y_j) \in \mathbb{R}^2$ satisfy $x_i - y_i = -(x_j - y_j)$. Without loss of generality, let $x_i < y_i$ (so that $x_i - y_i < 0$ and $x_j - y_j > 0$). Then the following inequalities hold:

$$\begin{cases} f(x_i, y_i) > f(x_j, y_j), & \text{if } \hat{c_2} < 0 \\ f(x_i, y_i) < f(x_j, y_j), & \text{if } \hat{c_2} > 0. \end{cases} \tag{34}$$

Proof. Let $(x_i, y_i), (x_j, y_j) \in \mathbb{R}^2$ satisfy $x_i - y_i = -(x_j - y_j)$ and $x_i < y_i$.

Case 1: $\boldsymbol{\hat{c_2} < 0}$. We have

$$\begin{aligned} f(x_i, y_i) &= \hat{c_0} + \hat{c_1}(x_i - y_i)^2 + \hat{c_2}(x_i - y_i)^3 \\ &= \hat{c_0} + \hat{c_1}(x_j - y_j)^2 - \underbrace{\hat{c_2}}_{<0} \underbrace{(x_j - y_j)^3}_{>0} \\ &> \hat{c_0} + \hat{c_1}(x_j - y_j)^2 + \hat{c_2}(x_j - y_j)^3 \\ &= f(x_j, y_j) \end{aligned} \tag{35}$$

where in the second row, $(x_j - y_j)^3 > 0$ holds because $x_i < y_i \Rightarrow x_i - y_i < 0$, so $x_j - y_j = -(x_i - y_i) > 0$.

Case 2: $\boldsymbol{\hat{c_2} > 0}$. We have

$$\begin{aligned} f(x_i, y_i) &= \hat{c_0} + \hat{c_1}(x_i - y_i)^2 + \hat{c_2}(x_i - y_i)^3 \\ &= \hat{c_0} + \hat{c_1}(x_j - y_j)^2 - \underbrace{\hat{c_2}}_{>0} \underbrace{(x_j - y_j)^3}_{>0} \\ &< \hat{c_0} + \hat{c_1}(x_j - y_j)^2 + \hat{c_2}(x_j - y_j)^3 \\ &= f(x_j, y_j) \end{aligned} \tag{36}$$

□

B.6 The Strict Asymmetric Congruence Model Must not Contain the Linear Term $(x - y)$

We now explain why, although it might seem unintuitive at first sight, the strict asymmetric congruence model (Equation 6) must not include the linear term $(x - y)$, because the model predictions would systematically contradict a congruence effect if this term were included. More specifically, let $\hat{c_0}, \hat{c_m}, \hat{c_1}$, and $\hat{c_2} \in \mathbb{R}$ and consider the function that extends the strict asymmetric congruence model by the term $(x - y)$:

$$F(x, y) = \hat{c_0} + \hat{c_m}(x - y) + \hat{c_1}(x - y)^2 + \hat{c_2}(x - y)^3 \tag{37}$$

As in Section B.4, let $(x_i, y_i), (x_j, y_j) \in \mathbb{R}$ satisfy the following two properties:

(1) $(x_i > y_i$ and $x_j > y_j)$ or $(x_i < y_i$ and $x_j < y_j)$,

(2) $|x_i - y_i| < |x_j - y_j|$.

Without loss of generality, consider a situation where $\hat{c_1} < 0$ and $\hat{c_2} < 0$. From the proof in Section B.4, we know that when $\hat{c_m} = 0$ (i.e., when the function in Equation 37 reduces to the function in Equation 6), then the inequality $F(x_i, y_i) > F(x_j, y_j)$ holds, in line with the congruence effect.

We now show that in case that $\hat{c_m} \neq 0$, the inequality $F(x_i, y_i) > F(x_j, y_j)$ is no longer true for all predictor combinations that satisfy Properties (1) and (2), so the congruence effect would be violated in this case. This observation justifies the exclusion of the linear term $(x - y)$ in the strict asymmetric congruence model.

More specifically, we make the following claim:

Claim. Let $\hat{c_1} < 0$, $\hat{c_2} < 0$, and $\hat{c_m} \neq 0$. Let the function $F : \mathbb{R}^2 \to \mathbb{R}$ be defined as in Equation 37. Then there exist $(x_i, y_i), (x_j, y_j) \in \mathbb{R}^2$ which satisfy Properties (1) and (2), such that $F(x_i, y_i) < F(x_j, y_j)$.

Proof. Without loss of generality, let $\hat{c_m} > 0$. To prove the claim, we need to find two elements $(x_i, y_i), (x_j, y_j)$ that satisfy Property (1), Property (2), and $F(x_i, y_i) < F(x_j, y_j)$. We restrict our search for these points to predictor combinations that lie on the line g_0, that is, to elements (x, y) with $y = -x$.

The outcome values that the function in Equation 37 predicts for predictor combinations that lie on g_0 are given by:

$$\begin{aligned} F_{|g_0}(x) &= F(x, -x) = \hat{c_0} + \hat{c_m}(x - (-x)) + \hat{c_1}(x - (-x))^2 + \hat{c_2}(x - (-x))^3 \\ &= \hat{c_0} + 2\hat{c_m}x + 4\hat{c_1}x^2 + 8\hat{c_2}x^3 \end{aligned} \tag{38}$$

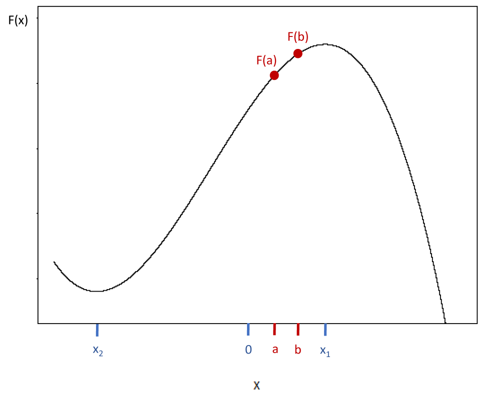


Figure 2. Prototypical graph of $F_{|g0}$ (Equation 38), when $\hat{c_1} < 0$, $\hat{c_2} < 0$, and $\hat{c_m} > 0$.

Figure 2 shows a prototypical graph of $F_{|g0}$. The extremum values of $F_{|g0}$ are given by the two roots of $\frac{\partial F_{|g0}(x)}{\partial x} = 2\hat{c_m} + 8\hat{c_1}x + 24\hat{c_2}x^2$. Let x_1 denote the larger and x_2 denote the smaller root (see Figure 2). Without loss of generality, let $x_2 < 0 < x_1$ (otherwise, in what follows, replace the interval $[0, x_1]$ by $[x_2, x_1]$ if $x_1, x_2 > 0$ and by $[x_1, 0]$ if $x_1, x_2 < 0$).

Because $\hat{c_2} < 0$, we have $\lim_{x\to-\infty} F_{|g0}(x) = \infty$ and $\lim_{x\to\infty} F_{|g0}(x) = -\infty$, so $F_{|g0}$ is strictly monotonically increasing on the interval $[x_2, x_1]$.

Now choose two arbitrary values $a, b \in [0, x_1]$ such that $a < b$ (see Figure 2). Define $(x_i, y_i) := (a, -a)$ and $(x_j, y_j) := (b, -b)$. Then (x_i, y_i) and (x_j, y_j) satisfy Property (1) because $a > 0 > -a$ and $b > 0 > -b$. They also satisfy Property (2) because

$$|x_i - y_i| = |2a| < |2b| = |x_j - y_j| \tag{39}$$

Moreover, we have

$$F(x_i, y_i) = F(a, -a) = F_{|g0}(a) < F_{|g0}(b) = F(b, -b) = F(x_j, y_j), \tag{40}$$

where the inequality in the middle of Equation 40 follows because $F_{|g0}$ is strictly monotonically increasing on $[0, x_1] \subset [x_2, x_1]$. $\square$

C OSF Material C: Mathematical Proofs of Claims About the Strict Level-Dependent Congruence Model

C.1 Basics and Notation

Having estimated the regression coefficients of the strict level-dependent congruence model, we write the estimated model equation as a function $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto \hat{z}$:

$$\begin{aligned} f(x,y) = \hat{z} &= \hat{c_0} + \hat{c_1}(x-y)^2 + \hat{c_3}(x+y)(x-y)^2 \\ &= \hat{c_0} + 0x + 0y + \hat{c_1}x^2 - 2\hat{c_1}xy + \hat{c_1}y^2 + \hat{c_3}x^3 - \hat{c_3}x^2y - \hat{c_3}xy^2 + \hat{c_3}y^3 \end{aligned} \tag{41}$$

Now consider the line $g_k : \mathbb{R} \to \mathbb{R}, x \mapsto -x + 2k$. The outcome values that the level-dependent congruence model (Equation 41) predicts for predictor combinations that lie on g_k are given by (for any $k \in \mathbb{R}$):

$$\begin{aligned} f_{|g_k}(x) &= f(x, g_k(x)) = f(x, -x+2k) \\ &= \hat{c_0} + \hat{c_1}(x-(-x+2k))^2 + \hat{c_3}(x+(-x+2k))(x-(-x+2k))^2 \\ &= (\hat{c_0} + 4\hat{c_1}k^2 + 8\hat{c_3}k^3) + (-8\hat{c_1}k - 16\hat{c_3}k^2)x + (4\hat{c_1} + 8\hat{c_3}k)x^2 \end{aligned} \tag{42}$$

The first derivative of $f_{|g_k}$ is:

$$\frac{\partial f_{|g_k}(x)}{\partial x} = -8\hat{c_1}k - 16\hat{c_3}k^2 + (8\hat{c_1} + 16\hat{c_3}k)x \tag{43}$$

The first derivative $\frac{\partial f_{|g_k}(x)}{\partial x}$ is zero for $x_1 = k$. Thus, the extremum value of $f_{|g_k}$ is $(x_1, g_k(x_1)) = (k, k)$.

Finally, the second derivative of $f_{|g_k}$ is given by:

$$\frac{\partial^2 f_{|g_k}(x)}{\partial^2 x} = 8\hat{c_1} + 16\hat{c_3}k \tag{44}$$

C.2 The Coefficient $\hat{c_3}$ Reflects the Level-Dependency Effect.

We claim that when $\hat{c_3}$ is negative, then an increase in the mean predictor level k causes a decrease in the curvature (i.e., in the second derivative) of the surface above g_k. By contrast, when $\hat{c_3}$ is positive, the curvature of the surface above g_k increases as the predictor level k increases.

Using Equation 44, we know that the second derivative of the surface above g_k (at any value x) is given by:

$$\frac{\partial^2 f_{|g_k}(x)}{\partial^2 x} = 8\hat{c_1} + 16\hat{c_3}k. \tag{45}$$

Now we increase k by 1 and consider the surface above g_{k+1}. The curvature above this line is:

$$\frac{\partial^2 f_{|g_{k+1}}(x)}{\partial^2 x} = 8\hat{c_1} + 16\hat{c_3}k + 16\hat{c_3}. \tag{46}$$

That is, an increase of 1 in the mean predictor level k causes a change of $\frac{\partial^2 f_{|g_{k+1}}(x)}{\partial^2 x} - \frac{\partial^2 f_{|g_k}(x)}{\partial^2 x} = 16\hat{c_3}$ in the curvature of the surface above g_k. When $\hat{c_3}$ is negative, this change means a decrease in the curvature. When $\hat{c_3}$ is positive, the curvature increases as k increases. □

C.3 The Strict Level-Dependent Congruence Model is in Line With a *Congruence Effect*

Consider the strict level-dependent congruence model as defined in Equation 41, and let $\hat{c_1} \neq 0$ and $\hat{c_3} \neq 0$. We will now show that the predictions of the strict level-dependent congruence model are in line with a congruence effect as defined in Table 2 in the paper: "For two people with the same mean predictor level $(x + y)/2$, the person whose x and y values are closer to one another has the higher (or lower) value of z."

For the proof, we need to distinguish between the U-shaped and the inverse U-shaped parts of the respective regression surface. For example, the claim that more congruent predictors are associated with higher outcome levels only holds when we consider the inverse U-shaped parts of the surface, but not the parts where the curvature changed its direction due to the level-dependency effect. The cubic RSA strategy that we describe in the paper ensures that we only restrict the interpretation to either the U-shaped or the inverse U-shaped parts of the surface when it is indeed justified to do so; we suggest to consider the regions of significance to this aim.

When we consider the underlying mathematical model without taking random noise into account, the situation is a bit easier: With given values of $\hat{c_1}$ and $\hat{c_3}$, there is one line in the xy plane that separates the U-shaped part from the inverse U-shaped part of the surface. This line is perpendicular to the LOC (i.e., it is given by g_k for a specific value of k) and defined by the property that the surface above this line has a curvature of zero. We refer to this line as the *turn-over line*, because it is the line where the surface turns from a U-shape into an inverse U-shape (or vice versa). By setting $\frac{\partial^2 f_{|g_k}(x)}{\partial^2 x} = 0$ in Equation 44 and solving for k, we find that $k = -\frac{\hat{c_1}}{2\hat{c_3}}$, so the turn-over line is given by all predictor combinations (x, y) with $y = -x + 2k = -x - \frac{\hat{c_1}}{\hat{c_3}}$, that is, with $x + y = -\frac{\hat{c_1}}{\hat{c_3}}$.

To show that the strict level-dependent congruence model is in line with a congruence effect, we need to show that when the level-dependency effect is positive ($\hat{c_3} > 0$), then more congruence is associated with higher outcome values for predictor combinations that lie "in front of" the turn-over line, which is the inverse U-shaped part of the surface in this case (for a visual representation, see Figure 6b in the paper). For predictor combinations that lie "behind" the turn-over line (U-shaped part), more congruent combinations are predicted to have lower outcome values (Figure 6d). When $\hat{c_3} < 0$, the situation is reversed: In front of the turn-over line (U-shaped part), more congruence is associated with lower outcome values (Figure 6c). Behind the turn-over line (inverse U-shaped part), more congruent predictor combinations have higher outcome values (Figure 6a).

In more technical terms, we make the following claim:

Claim. Let $(x_i, y_i), (x_j, y_j) \in \mathbb{R}^2$ satisfy the following two properties (hereafter referred to as "Property (1)" and "Property (2)"):

(1) $x_i + y_i = x_j + y_j$,

(2) $|x_i - y_i| < |x_j - y_j|$.

Then the following inequalities hold:

$$\begin{cases} f(x_i, y_i) > f(x_j, y_j), & \text{if } (\hat{c_3} > 0 \text{ and } x_i + y_i < -\frac{\hat{c_1}}{\hat{c_3}}) \\ & \quad \text{or } (\hat{c_3} < 0 \text{ and } x_i + y_i > -\frac{\hat{c_1}}{\hat{c_3}}), \\ f(x_i, y_i) < f(x_j, y_j), & \text{if } (\hat{c_3} > 0 \text{ and } x_i + y_i > -\frac{\hat{c_1}}{\hat{c_3}}) \\ & \quad \text{or } (\hat{c_3} < 0 \text{ and } x_i + y_i < -\frac{\hat{c_1}}{\hat{c_3}}). \end{cases} \tag{47}$$

Proof. Let $(x_i, y_i), (x_j, y_j) \in \mathbb{R}^2$ satisfy Properties (1) and (2). We show the inequalities in Equation 47 separately for the four cases that are distinguished in it.

Case 1: $\hat{c_3} > 0$ and $x_i + y_i < -\frac{\hat{c_1}}{\hat{c_3}}$. We have to show that $f(x_i, y_i) > f(x_j, y_j)$.

We have

$$\begin{aligned} f(x_i, y_i) &= \hat{c_0} + \hat{c_1}(x_i - y_i)^2 + \hat{c_3}\underbrace{(x_i + y_i)}_{=x_j+y_j}(x_i - y_i)^2 \\ &= \hat{c_0} + \underbrace{(\hat{c_1} + \hat{c_3}(x_j + y_j))}_{<0 \text{ (see below)}}\underbrace{(x_i - y_i)^2}_{<(x_j-y_j)^2} \\ &> \hat{c_0} + (\hat{c_1} + \hat{c_3}(x_j + y_j))(x_j - y_j)^2 \\ &= f(x_j, y_j) \end{aligned} \tag{48}$$

where in the second row, the inequality $\hat{c_1} + \hat{c_3}(x_j + y_j) < 0$ holds because

$$\begin{aligned} x_j + y_j = x_i + y_i &< -\tfrac{\hat{c_1}}{\hat{c_3}} \\ \overset{\hat{c_3}>0}{\Leftrightarrow} \qquad \hat{c_3}(x_j + y_j) &< -\hat{c_1} \\ \Leftrightarrow \quad \hat{c_1} + \hat{c_3}(x_j + y_j) &< 0. \end{aligned} \tag{49}$$

Case 2: $\boldsymbol{\hat{c_3} < 0}$ ***and*** $\boldsymbol{x_i + y_i > -\frac{\hat{c_1}}{\hat{c_3}}}$***.*** We have to show that $f(x_i, y_i) > f(x_j, y_j)$.

We have

$$\begin{aligned} f(x_i, y_i) &= \hat{c_0} + \hat{c_1}(x_i - y_i)^2 + \hat{c_3}\underbrace{(x_i + y_i)}_{=x_j+y_j}(x_i - y_i)^2 \\ &= \hat{c_0} + \underbrace{(\hat{c_1} + \hat{c_3}(x_j + y_j))}_{<0 \text{ (see below)}}\underbrace{(x_i - y_i)^2}_{<(x_j-y_j)^2} \\ &> \hat{c_0} + (\hat{c_1} + \hat{c_3}(x_j + y_j))(x_j - y_j)^2 \\ &= f(x_j, y_j) \end{aligned} \tag{50}$$

where in the second row, the inequality $\hat{c_1} + \hat{c_3}(x_j + y_j) < 0$ holds because

$$\begin{aligned} x_j + y_j = x_i + y_i &> -\tfrac{\hat{c_1}}{\hat{c_3}} \\ \overset{\hat{c_3}<0}{\Leftrightarrow} \qquad \hat{c_3}(x_j + y_j) &< -\hat{c_1} \\ \Leftrightarrow \quad \hat{c_1} + \hat{c_3}(x_j + y_j) &< 0. \end{aligned} \tag{51}$$

Case 3: $\boldsymbol{\hat{c_3} > 0}$ ***and*** $\boldsymbol{x_i + y_i > -\frac{\hat{c_1}}{\hat{c_3}}}$***.*** We have to show that $f(x_i, y_i) < f(x_j, y_j)$.

We have

$$\begin{aligned} f(x_i, y_i) &= \hat{c}_0 + \hat{c}_1(x_i - y_i)^2 + \hat{c}_3 \underbrace{(x_i + y_i)}_{=x_j+y_j}(x_i - y_i)^2 \\ &= \hat{c}_0 + \underbrace{\left(\hat{c}_1 + \hat{c}_3(x_j + y_j)\right)}_{>0 \text{ (see below)}} \underbrace{(x_i - y_i)^2}_{<(x_j-y_j)^2} \\ &< \hat{c}_0 + \left(\hat{c}_1 + \hat{c}_3(x_j + y_j)\right)(x_j - y_j)^2 \\ &= f(x_j, y_j) \end{aligned} \tag{52}$$

where in the second row, the inequality $\hat{c}_1 + \hat{c}_3(x_j + y_j) > 0$ holds because

$$\begin{aligned} x_j + y_j = x_i + y_i &> -\tfrac{\hat{c}_1}{\hat{c}_3} \\ \stackrel{\hat{c}_3>0}{\Leftrightarrow} \qquad \hat{c}_3(x_j + y_j) &> -\hat{c}_1 \\ \Leftrightarrow \quad \hat{c}_1 + \hat{c}_3(x_j + y_j) &> 0. \end{aligned} \tag{53}$$

Case 4: $\boldsymbol{\hat{c}_3 < 0}$ ***and*** $\boldsymbol{x_i + y_i < -\frac{\hat{c}_1}{\hat{c}_3}}$***.*** We have to show that $f(x_i, y_i) < f(x_j, y_j)$.

We have

$$\begin{aligned} f(x_i, y_i) &= \hat{c}_0 + \hat{c}_1(x_i - y_i)^2 + \hat{c}_3 \underbrace{(x_i + y_i)}_{=x_j+y_j}(x_i - y_i)^2 \\ &= \hat{c}_0 + \underbrace{\left(\hat{c}_1 + \hat{c}_3(x_j + y_j)\right)}_{>0 \text{ (see below)}} \underbrace{(x_i - y_i)^2}_{<(x_j-y_j)^2} \\ &< \hat{c}_0 + \left(\hat{c}_1 + \hat{c}_3(x_j + y_j)\right)(x_j - y_j)^2 \\ &= f(x_j, y_j) \end{aligned} \tag{54}$$

where in the second row, the inequality $\hat{c}_1 + \hat{c}_3(x_j + y_j) > 0$ holds because

$$\begin{aligned} x_j + y_j = x_i + y_i &< -\tfrac{\hat{c}_1}{\hat{c}_3} \\ \stackrel{\hat{c}_3<0}{\Leftrightarrow} \qquad \hat{c}_3(x_j + y_j) &> -\hat{c}_1 \\ \Leftrightarrow \quad \hat{c}_1 + \hat{c}_3(x_j + y_j) &> 0. \end{aligned} \tag{55}$$

$\square$

C.4 The Strict Level-Dependent Congruence Model is in Line With a *Level-Dependency Effect*

Consider again the strict level-dependent congruence model as defined in Equation 41, and let $\hat{c_1} \neq 0$ and $\hat{c_3} \neq 0$. We will now show that the predictions of the strict level-dependent congruence model are in line with a level-dependency effect as defined in Table 2 in the paper: "For four people A, B, C, and D of which A and B, as well as C and D, respectively, have equal mean predictor levels $(x + y)/2$, where A and B have a lower mean predictor level than C and D, and of which A and C, as well as B and D, respectively, have equal discrepancy values $x - y$, the absolute difference between A's and B's z values is larger (or smaller) than the absolute difference between C's and D's z values."

Specifically, we consider the following claim:

Claim. Let $(x_i, y_i), (x_j, y_j), (x_m, y_m), (x_n, y_n) \in \mathbb{R}^2$ satisfy either $[x_i - y_i, x_j - y_j, x_m - y_m, x_n - y_n < -\frac{\hat{c_1}}{\hat{c_3}}]$ or $[x_i - y_i, x_j - y_j, x_m - y_m, x_n - y_n > -\frac{\hat{c_1}}{\hat{c_3}}]$ (i.e., the four points lie on the same side of the turn-over line). Furthermore, let the following properties hold (hereafter referred to as "Property (1)" to "Property (3)"):

(1) $x_i + y_i = x_j + y_j$, $x_m + y_m = x_n + y_n$

(2) $x_i + y_i < x_m + y_m$,

(3) $x_i - y_i = x_m - y_m$, $x_j - y_j = x_n - y_n$.

Then the following inequalities hold:

$$\begin{cases} |f(x_i, y_i) - f(x_j, y_j)| > |f(x_m, y_m) - f(x_n, y_n)|, & \text{if } (\hat{c_3} > 0 \text{ and } x_i + y_i < -\frac{\hat{c_1}}{\hat{c_3}}) \\ & \text{or } (\hat{c_3} < 0 \text{ and } x_i + y_i < -\frac{\hat{c_1}}{\hat{c_3}}), \\ |f(x_i, y_i) - f(x_j, y_j)| < |f(x_m, y_m) - f(x_n, y_n)|, & \text{if } (\hat{c_3} > 0 \text{ and } x_i + y_i > -\frac{\hat{c_1}}{\hat{c_3}}) \\ & \text{or } (\hat{c_3} < 0 \text{ and } x_i + y_i > -\frac{\hat{c_1}}{\hat{c_3}}). \end{cases} \tag{56}$$

Proof. Let $(x_i, y_i), (x_j, y_j), (x_m, y_m), (x_n, y_n) \in \mathbb{R}^2$ satisfy all properties stated in the claim. We show the inequalities in Equation 56 separately for the four cases that are distinguished in it.

Case 1: $\boldsymbol{\hat{c_3} > 0}$ ***and*** $\boldsymbol{x_i + y_i < -\frac{\hat{c_1}}{\hat{c_3}}}$**.** We have to show that $|f(x_i, y_i) - f(x_j, y_j)| > |f(x_m, y_m) - f(x_n, y_n)|$.

Without loss of generality, let $|x_i - y_i| < |x_j - y_j|$, which also implies $|x_m - y_m| < |x_n - y_n|$ because of Property (3). Then, according to the claim in Section C.3 and using Property (1), $f(x_i, y_i) > f(x_j, y_j)$ and $f(x_m, y_m) > f(x_n, y_n)$. That is, the inequality that is to show can be rewritten as

$$\text{To show: } f(x_i,y_i)-f(x_j,y_j) = |f(x_i,y_i)-f(x_j,y_j)| > |f(x_m,y_m)-f(x_n,y_n)| = f(x_m,y_m)-f(x_n,y_n). \tag{57}$$

That is, it is to show that $f(x_i, y_i) - f(x_j, y_j) > f(x_m, y_m) - f(x_n, y_n)$ and this inequality holds because

$$\begin{aligned}
f(x_i,y_i) - f(x_j,y_j) &= \hat{c_1}(x_i-y_i)^2 + \hat{c_3}(x_i+y_i)(x_i-y_i)^2 - \hat{c_1}(x_j-y_j)^2 - \hat{c_3}\underbrace{(x_j+y_j)}_{=(x_i+y_i)}(x_j-y_j)^2 \\
&= \underbrace{(\hat{c_1} + \underbrace{\hat{c_3}}_{>0}\underbrace{(x_i+y_i)}_{<(x_m+y_m)})}_{<\hat{c_3}(x_m+y_m)}\underbrace{(\underbrace{(x_i-y_i)^2}_{=(x_m-y_m)^2} - \underbrace{(x_j-y_j)^2}_{=(x_n-y_n)^2})}_{<0 \text{ because } |x_i-y_i|<|x_j-y_j|} \\
&> (\hat{c_1} + \hat{c_3}(x_m+y_m))((x_m-y_m)^2 - (x_n-y_n)^2) \\
&= f(x_m,y_m) - f(x_n,y_n)
\end{aligned} \tag{58}$$

Case 2: $\boldsymbol{\hat{c_3} < 0}$ ***and*** $\boldsymbol{x_i + y_i < -\frac{\hat{c_1}}{\hat{c_3}}}$**.** We have to show that $|f(x_i, y_i) - f(x_j, y_j)| > |f(x_m, y_m) - f(x_n, y_n)|$.

Again, let $|x_i - y_i| < |x_j - y_j|$ and thus $|x_m - y_m| < |x_n - y_n|$ without loss of generality. Then $f(x_i, y_i) < f(x_j, y_j)$ and $f(x_m, y_m) < f(x_n, y_n)$, so the inequality that is to show can be rewritten as

$$\text{To show: } f(x_i,y_i)-f(x_j,y_j) = -|f(x_i,y_i)-f(x_j,y_j)| < -|f(x_m,y_m)-f(x_n,y_n)| = f(x_m,y_m)-f(x_n,y_n). \tag{59}$$

That is, it is to show that $f(x_i, y_i) - f(x_j, y_j) < f(x_m, y_m) - f(x_n, y_n)$ and this inequality holds because

$$\begin{aligned} f(x_i,y_i)-f(x_j,y_j) &= \underbrace{(\hat{c_1}+\underbrace{\hat{c_3}}_{<0}\underbrace{(x_i+y_i)}_{<(x_m+y_m)}}_{>\hat{c_3}(x_m+y_m)})\underbrace{(\underbrace{(x_i-y_i)^2}_{=(x_m-y_m)^2}-\underbrace{(x_j-y_j)^2}_{=(x_n-y_n)^2})}_{<0} \\ &< (\hat{c_1}+\hat{c_3}(x_m+y_m))((x_m-y_m)^2-(x_n-y_n)^2) \\ &= f(x_m,y_m)-f(x_n,y_n) \end{aligned} \tag{60}$$

Case 3: $\boldsymbol{\hat{c_3} > 0}$ ***and*** $\boldsymbol{x_i + y_i > -\frac{\hat{c_1}}{\hat{c_3}}}$. We have to show that $|f(x_i,y_i)-f(x_j,y_j)| < |f(x_m,y_m)-f(x_n,y_n)|$.

Again, let $|x_i-y_i| < |x_j-y_j|$ and thus $|x_m-y_m| < |x_n-y_n|$ without loss of generality. Then $f(x_i,y_i) < f(x_j,y_j)$ and $f(x_m,y_m) < f(x_n,y_n)$, so the inequality that is to show can be rewritten as

$$\text{To show: } f(x_i,y_i)-f(x_j,y_j) = -|f(x_i,y_i)-f(x_j,y_j)| > -|f(x_m,y_m)-f(x_n,y_n)| = f(x_m,y_m)-f(x_n,y_n). \tag{61}$$

That is, it is to show that $f(x_i,y_i)-f(x_j,y_j) > f(x_m,y_m)-f(x_n,y_n)$ and this inequality holds because

$$\begin{aligned} f(x_i,y_i)-f(x_j,y_j) &= \underbrace{(\hat{c_1}+\underbrace{\hat{c_3}}_{>0}\underbrace{(x_i+y_i)}_{<(x_m+y_m)}}_{<\hat{c_3}(x_m+y_m)})\underbrace{(\underbrace{(x_i-y_i)^2}_{=(x_m-y_m)^2}-\underbrace{(x_j-y_j)^2}_{=(x_n-y_n)^2})}_{<0} \\ &> (\hat{c_1}+\hat{c_3}(x_m+y_m))((x_m-y_m)^2-(x_n-y_n)^2) \\ &= f(x_m,y_m)-f(x_n,y_n) \end{aligned} \tag{62}$$

Case 4: $\boldsymbol{\hat{c_3} < 0}$ ***and*** $\boldsymbol{x_i + y_i > -\frac{\hat{c_1}}{\hat{c_3}}}$. We have to show that $|f(x_i,y_i)-f(x_j,y_j)| < |f(x_m,y_m)-f(x_n,y_n)|$.

Again, let $|x_i-y_i| < |x_j-y_j|$ and thus $|x_m-y_m| < |x_n-y_n|$ without loss of generality. Then $f(x_i,y_i) > f(x_j,y_j)$ and $f(x_m,y_m) > f(x_n,y_n)$, so the inequality that is to show can be rewritten as

$$\text{To show: } f(x_i,y_i)-f(x_j,y_j) = |f(x_i,y_i)-f(x_j,y_j)| < |f(x_m,y_m)-f(x_n,y_n)| = f(x_m,y_m)-f(x_n,y_n). \tag{63}$$

That is, it is to show that $f(x_i,y_i)-f(x_j,y_j) < f(x_m,y_m)-f(x_n,y_n)$ and this inequality holds because

$$
\begin{aligned}
f(x_i, y_i) - f(x_j, y_j) &= \underbrace{\Big(\hat{c_1} + \underbrace{\hat{c_3}}_{<0}\, \underbrace{(x_i + y_i)}_{<(x_m+y_m)}}_{>\hat{c_3}(x_m+y_m)}\Big)\underbrace{\Big(\underbrace{(x_i - y_i)^2}_{=(x_m-y_m)^2} - \underbrace{(x_j - y_j)^2}_{=(x_n-y_n)^2}\Big)}_{<0} \\
&< \big(\hat{c_1} + \hat{c_3}(x_m + y_m)\big)\big((x_m - y_m)^2 - (x_n - y_n)^2\big) \\
&= f(x_m, y_m) - f(x_n, y_n)
\end{aligned}
\tag{64}
$$

$\square$

D OSF Material D: Rising Ridge Versions of the Hypotheses - How to Investigate Simple and Complex Congruence Effects Combined With Main Effects

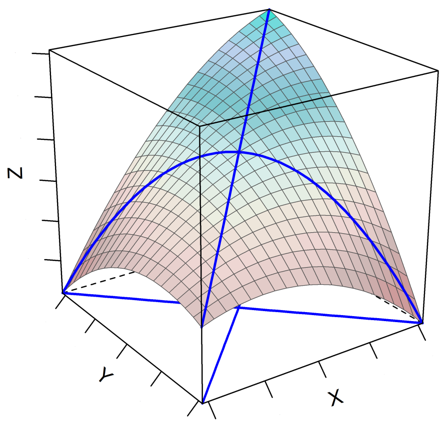
In many psychological domains, not only is the degree of congruence expected to affect the outcome, but the level of the predictor variables is also expected to do so. For example, when considering self-views and reputation concerning intelligence, it might be reasonable to expect that not only is congruence beneficial for psychological adjustment but that, in addition, out of two people with the same discrepancy between self-view and reputation, the person with the higher predictor levels should be happier. A formal definition of this *rising ridge simple congruence hypothesis* can be found in Table 2 in the paper.

Starting from the strict simple congruence model (Equation 1 in Section A.1 above), such a linear level effect can be allowed by including the term $(x + y)$, reflecting the mean predictor level, as a linear predictor:

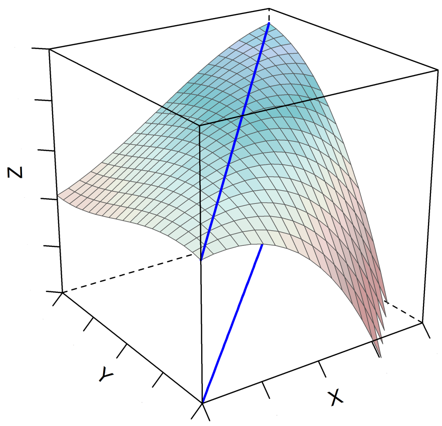
Rising ridge simple congruence model:

$$\begin{aligned} z &= c_0 + c_{level}(x+y) + c_1(x-y)^2 + \varepsilon \\ &= c_0 + c_{level}x + c_{level}y + c_1x^2 - 2c_1xy + c_1y^2 + \varepsilon \end{aligned} \tag{65}$$

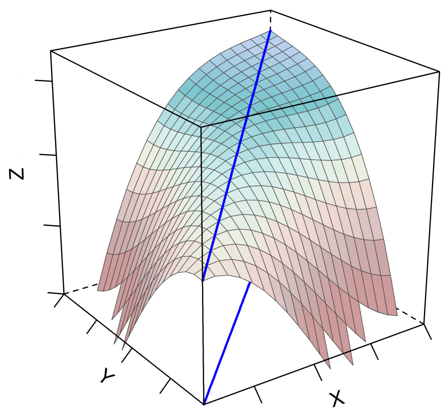
Figure 3a shows an example graph of a rising ridge simple congruence model with a positive linear level effect c_{level}. Owing to this level effect, the surface rises toward the back corner of the coordinate cube.



(a) Rising ridge simple congruence model:
$\hat{z} = \hat{c_0} + \hat{c}_{level}(x + y) + \hat{c_1}(x - y)^2$.



(b) Rising ridge asymmetric congruence model:
$\hat{z} = \hat{c_0} + \hat{c}_{level}(x + y) + \hat{c_1}(x - y)^2 + \hat{c_2}(x - y)^3$.



(c) Rising ridge level-dependent congruence model:
$\hat{z} = \hat{c_0} + \hat{c}_{level}(x+y) + \hat{c_1}(x-y)^2 + \hat{c_3}(x+y)(x-y)^2$.

Figure 3. Prototypical rising ridge congruence models.

The only difference between the rising ridge simple congruence model (Equation 65) and the strict simple congruence model (Equation 1) is that the two linear predictors x and y have equal coefficients that are not necessarily zero (see Section E.4 for details on why this equality constraint cannot be omitted). Accordingly, the model constraints to be tested in Step 1 differ only in that b_1 and b_2 are no longer restricted to be zero, but they are instead restricted to be equal ($b_2 = b_1$). The other constraints are the same as for the strict simple congruence model. The complete set of constraints for the rising ridge simple congruence model is thus:
$b_2 = b_1, b_4 = -2b_3$, and $b_5 = b_3$ (see Table 1 below).

In Step 2 of investigating the rising ridge simple congruence model, analogous to the strict simple congruence model, $\hat{c_1}$ (which equals $\hat{b_3}$ when the model is notated as a constrained full model), indicates the existence and direction of a congruence effect (see Table 1; see Section E.2 for the proof of this claim). In addition, a test of the linear level effect must be conducted to determine whether it is significant and goes in the suggested direction. This effect is reflected in the parameter $\hat{a_1} = \hat{b_1} + \hat{b_2}$. Graphically, $\hat{a_1}$ equals the slope of the surface above the LOC and above each line that is parallel to it. When a positive effect of the predictor level is expected, $\hat{a_1}$ must be significantly positive to be in line with the hypothesis; this is the case in Figure 3a. A negative effect of the level would be reflected in a significantly negative parameter $\hat{a_1}$ (see Section E.3 for the proof).

When the aim is to investigate rising ridge versions of the two complex congruence hypotheses (see Table 2 in the paper for formal definitions), the strict asymmetric congruence model and the strict level-dependent congruence model can be extended in an analogous way as was demonstrated for the strict simple congruence model, providing the following model equations:

Rising ridge asymmetric congruence model:

$$\begin{aligned} z &= c_0 + c_{level}(x+y) + c_1(x-y)^2 + c_2(x-y)^3 + \varepsilon \qquad (66) \\ &= c_0 + c_{level}x + c_{level}y + c_1x^2 - 2c_1xy + c_1y^2 + c_2x^3 - 3c_2x^2y + 3c_2xy^2 - c_2y^3 + \varepsilon \end{aligned}$$

Rising ridge level-dependent congruence model:

$$\begin{aligned} z &= c_0 + c_{level}(x+y) + c_1(x-y)^2 + c_3(x+y)(x-y)^2 + \varepsilon \qquad (67) \\ &= c_0 + c_{level}x + c_{level}y + c_1x^2 - 2c_1xy + c_1y^2 + c_3x^3 - c_3x^2y - c_3xy^2 + c_3y^3 + \varepsilon \end{aligned}$$

Tables 1 to 5 provide overviews of the steps for investigating the rising ridge versions of the simple, asymmetric, and level-dependent congruence hypothesis, respectively. As can be seen in these tables, the steps for investigating the rising ridge versions differ from the steps for the strict versions in three points: First, the model constraints $b_1 = 0$, $b_2 = 0$ in the strict versions must be replaced by the constraint $b_1 = b_2$ in all rising ridge models, which is a consequence of the additional linear level effect. Second, it must be tested whether the estimated model indicates the expected linear level effect. Third, for the rising ridge asymmetric congruence model, the outcome predictions for points behind E_2 must be compared to the outcome prediction of the point on E_2 that has the same level, and not only to an arbitrary point on E_2 as was the case for the strict asymmetric congruence hypothesis.

All other constraints and tests presented here are analogous to the strict models described in the main article.

Table 1
Steps for Investigating a Rising Ridge Simple Congruence Hypothesis

Hypothesis	Step 1:	Step 2: Inspect the Rising Ridge Simple Congruence Model	
	Test the model constraints on the full second-order model	There should be a congruence effect in the expected direction	There should be a linear level effect in the expected direction
Congruence is associated with the highest outcome values and there is a positive effect of the predictor level	$b_2 = b_1$, $b_4 = -2b_3$, $b_5 = b_3$	$\hat{c_1} = \hat{b_3} < 0$	$\hat{a_1} = \hat{b_1} + \hat{b_2} > 0$
Congruence is associated with the lowest outcome values and there is a positive effect of the predictor level	$b_2 = b_1$, $b_4 = -2b_3$, $b_5 = b_3$	$\hat{c_1} = \hat{b_3} > 0$	$\hat{a_1} = \hat{b_1} + \hat{b_2} > 0$
Congruence is associated with the highest outcome values and there is a negative effect of the predictor level	$b_2 = b_1$, $b_4 = -2b_3$, $b_5 = b_3$	$\hat{c_1} = \hat{b_3} < 0$	$\hat{a_1} = \hat{b_1} + \hat{b_2} < 0$
Congruence is associated with the lowest outcome values and there is a negative effect of the predictor level	$b_2 = b_1$, $b_4 = -2b_3$, $b_5 = b_3$	$\hat{c_1} = \hat{b_3} > 0$	$\hat{a_1} = \hat{b_1} + \hat{b_2} < 0$

Table 2

*Steps for Investigating a Rising Ridge Asymmetric Congruence Hypothesis With **Positive** Linear Level Effect*

Hypothesis	Step 1:	Step 2: Inspect the Rising Ridge Asymmetric Congruence Model			
	Test the model constraints on the full third-order model	There should be a congruence effect in the expected direction	There should be an asymmetry effect in the expected direction	There should be a linear level effect in the expected direction	There should be no data point behind E_2 with significantly higher outcome predictions than the point on E_2 which has the same level
Congruence is associated with the highest outcome values, incongruence $x > y$ is associated with lower outcome values than incongruence $x < y$, and there is a positive effect of the predictor level.	$b_2 = b_1$, $b_4 = -2b_3$, $b_5 = b_3$, $b_7 = -3b_6$, $b_8 = 3b_6$, $b_9 = -b_6$	$\hat{c_1} = \hat{b_3} < 0$	$\hat{c_2} = \hat{b_6} < 0$	$\hat{u_1} = \hat{b_1} + \hat{b_2} > 0$	There should be no data point (x_i, y_i) with $y_i > x_i + \frac{2\hat{c_1}}{3\hat{c_2}}$ and $\hat{z_i} \notin (-\infty,\ \hat{z}_{ir} + t_{n-p}(1-\alpha)se(\hat{z}_{ir})]$
Congruence is associated with the highest outcome values, incongruence $x > y$ is associated with higher outcome values than incongruence $x < y$, and there is a positive effect of the predictor level.	$b_2 = b_1$, $b_4 = -2b_3$, $b_5 = b_3$, $b_7 = -3b_6$, $b_8 = 3b_6$, $b_9 = -b_6$	$\hat{c_1} = \hat{b_3} < 0$	$\hat{c_2} = \hat{b_6} > 0$	$\hat{u_1} = \hat{b_1} + \hat{b_2} > 0$	There should be no data point (x_i, y_i) with $y_i < x_i + \frac{2\hat{c_1}}{3\hat{c_2}}$ and $\hat{z_i} \notin (-\infty,\ \hat{z}_{ir} + t_{n-p}(1-\alpha)se(\hat{z}_{ir})]$
Congruence is associated with the lowest outcome values, incongruence $x > y$ is associated with lower outcome values than incongruence $x < y$, and there is a positive effect of the predictor level.	$b_2 = b_1$, $b_4 = -2b_3$, $b_5 = b_3$, $b_7 = -3b_6$, $b_8 = 3b_6$, $b_9 = -b_6$	$\hat{c_1} = \hat{b_3} > 0$	$\hat{c_2} = \hat{b_6} < 0$	$\hat{u_1} = \hat{b_1} + \hat{b_2} > 0$	There should be no data point (x_i, y_i) with $y_i < x_i + \frac{2\hat{c_1}}{3\hat{c_2}}$ and $\hat{z_i} \notin [\hat{z}_{ir} - t_{n-p}(1-\alpha)se(\hat{z}_{ir}),\ \infty)$
Congruence is associated with the lowest outcome values, incongruence $x > y$ is associated with higher outcome values than incongruence $x < y$, and there is a positive effect of the predictor level.	$b_2 = b_1$, $b_4 = -2b_3$, $b_5 = b_3$, $b_7 = -3b_6$, $b_8 = 3b_6$, $b_9 = -b_6$	$\hat{c_1} = \hat{b_3} > 0$	$\hat{c_2} = \hat{b_6} > 0$	$\hat{u_1} = \hat{b_1} + \hat{b_2} > 0$	There should be no data point (x_i, y_i) with $y_i > x_i + \frac{2\hat{c_1}}{3\hat{c_2}}$ and $\hat{z_i} \notin [\hat{z}_{ir} - t_{n-p}(1-\alpha)se(\hat{z}_{ir}),\ \infty)$

Note. E_2 denotes the second extremum line of the surface, consisting of all predictor combinations (x, y) with $y = x + \frac{2\hat{c_1}}{3\hat{c_2}}$. $\hat{z_{ir}}$ denotes the predicted outcome at the point of intersection (x_{ir}, y_{ir}) between E_2 and g_k with $k = (x_i + y_i)/2$ (i.e., k is the level of (x_i, y_i)).

Table 3

*Steps for Investigating a Rising Ridge Asymmetric Congruence Hypothesis With **Negative** Linear Level Effect*

Hypothesis	Step 1:	Step 2: Inspect the Rising Ridge Asymmetric Congruence Model			
	Test the model constraints on the full third-order model	There should be a congruence effect in the expected direction	There should be an asymmetry effect in the expected direction	There should be a linear level effect in the expected direction	There should be no data point behind E_2 with significantly higher outcome predictions than the point on E_2 which has the same level
Congruence is associated with the highest outcome values, incongruence $x > y$ is associated with lower outcome values than incongruence $x < y$, and there is a negative effect of the predictor level.	$b_2 = b_1$, $b_4 = -2b_3$, $b_5 = b_3$, $b_7 = -3b_6$, $b_8 = 3b_6$, $b_9 = -b_6$	$\hat{c_1} = \hat{b_3} < 0$	$\hat{c_2} = \hat{b_6} < 0$	$\hat{u_1} = \hat{b_1} + \hat{b_2} < 0$	There should be no data point (x_i, y_i) with $y_i > x_i + \frac{2\hat{c_1}}{3\hat{c_2}}$ and $\hat{z_i} \notin (-\infty,\ \hat{z}_{ir} + t_{n-p}(1-\alpha)se(\hat{z}_{ir})]$
Congruence is associated with the highest outcome values, incongruence $x > y$ is associated with higher outcome values than incongruence $x < y$, and there is a negative effect of the predictor level.	$b_2 = b_1$, $b_4 = -2b_3$, $b_5 = b_3$, $b_7 = -3b_6$, $b_8 = 3b_6$, $b_9 = -b_6$	$\hat{c_1} = \hat{b_3} < 0$	$\hat{c_2} = \hat{b_6} > 0$	$\hat{u_1} = \hat{b_1} + \hat{b_2} < 0$	There should be no data point (x_i, y_i) with $y_i < x_i + \frac{2\hat{c_1}}{3\hat{c_2}}$ and $\hat{z_i} \notin (-\infty,\ \hat{z}_{ir} + t_{n-p}(1-\alpha)se(\hat{z}_{ir})]$
Congruence is associated with the lowest outcome values, incongruence $x > y$ is associated with lower outcome values than incongruence $x < y$, and there is a negative effect of the predictor level.	$b_2 = b_1$, $b_4 = -2b_3$, $b_5 = b_3$, $b_7 = -3b_6$, $b_8 = 3b_6$, $b_9 = -b_6$	$\hat{c_1} = \hat{b_3} > 0$	$\hat{c_2} = \hat{b_6} < 0$	$\hat{u_1} = \hat{b_1} + \hat{b_2} < 0$	There should be no data point (x_i, y_i) with $y_i < x_i + \frac{2\hat{c_1}}{3\hat{c_2}}$ and $\hat{z_i} \notin [\hat{z}_{ir} - t_{n-p}(1-\alpha)se(\hat{z}_{ir}),\ \infty)$
Congruence is associated with the lowest outcome values, incongruence $x > y$ is associated with higher outcome values than incongruence $x < y$, and there is a negative effect of the predictor level.	$b_2 = b_1$, $b_4 = -2b_3$, $b_5 = b_3$, $b_7 = -3b_6$, $b_8 = 3b_6$, $b_9 = -b_6$	$\hat{c_1} = \hat{b_3} > 0$	$\hat{c_2} = \hat{b_6} > 0$	$\hat{u_1} = \hat{b_1} + \hat{b_2} < 0$	There should be no data point (x_i, y_i) with $y_i > x_i + \frac{2\hat{c_1}}{3\hat{c_2}}$ and $\hat{z_i} \notin [\hat{z}_{ir} - t_{n-p}(1-\alpha)se(\hat{z}_{ir}),\ \infty)$

Note. E_2 denotes the second extremum line of the surface, consisting of all predictor combinations (x, y) with $y = x + \frac{2\hat{c_1}}{3\hat{c_2}}$. $\hat{z_{ir}}$ denotes the predicted outcome at the point of intersection (x_{ir}, y_{ir}) between E_2 and g_k with $k = (x_i + y_i)/2$ (i.e., k is the level of (x_i, y_i)).

Table 4

Steps for Investigating a Rising Ridge Level-Dependent Congruence Hypothesis With ***Positive*** *Linear Level Effect*

Hypothesis	Step 1:	Step 2: Inspect the Rising Ridge Level-Dependent Congruence Model		
	Test the model constraints on the full third-order model	There should be a level-dependency effect in the expected direction	There should be a linear level effect in the expected direction	The intersection areas of the data with the regions of (negative, non-, positive) significance should be in line with the suggested hypothesis
At high predictor levels, congruence is associated with the highest outcome values, and this effect is weaker or nonexistent at lower predictor levels. In addition, there is a positive effect of the predictor level.	$b_2 = b_1$, $b_4 = -2b_3$, $b_5 = b_3$, $b_7 = -b_6$, $b_8 = -b_6$, $b_9 = b_6$	$\hat{c_3} = \hat{b_6} < 0$	$\hat{u_1} = \hat{b_1} + \hat{b_2} > 0$	There should be a region of negative significance that intersects with the data, and if a region of positive significance exists, there should be no data in it.
At low predictor levels, congruence is associated with the highest outcome values, and this effect is weaker or nonexistent at higher predictor levels. In addition, there is a positive effect of the predictor level.	$b_2 = b_1$, $b_4 = -2b_3$, $b_5 = b_3$, $b_7 = -b_6$, $b_8 = -b_6$, $b_9 = b_6$	$\hat{c_3} = \hat{b_6} > 0$	$\hat{u_1} = \hat{b_1} + \hat{b_2} > 0$	There should be a region of negative significance that intersects with the data, and if a region of positive significance exists, there should be no data in it.
At low predictor levels, congruence is associated with the lowest outcome values, and this effect is weaker or nonexistent at higher predictor levels. In addition, there is a positive effect of the predictor level.	$b_2 = b_1$, $b_4 = -2b_3$, $b_5 = b_3$, $b_7 = -b_6$, $b_8 = -b_6$, $b_9 = b_6$	$\hat{c_3} = \hat{b_6} < 0$	$\hat{u_1} = \hat{b_1} + \hat{b_2} > 0$	There should be a region of positive significance that intersects with the data, and if a region of negative significance exists, there should be no data in it.
At high predictor levels, congruence is associated with the lowest outcome values, and this effect is weaker or nonexistent at lower predictor levels. In addition, there is a positive effect of the predictor level.	$b_2 = b_1$, $b_4 = -2b_3$, $b_5 = b_3$, $b_7 = -b_6$, $b_8 = -b_6$, $b_9 = b_6$	$\hat{c_3} = \hat{b_6} > 0$	$\hat{u_1} = \hat{b_1} + \hat{b_2} > 0$	There should be a region of positive significance that intersects with the data, and if a region of negative significance exists, there should be no data in it.

Table 5

Steps for Investigating a Rising Ridge Level-Dependent Congruence Hypothesis With ***Negative*** *Linear Level Effect*

Hypothesis	Step 1: Test the model constraints on the full third-order model	Step 2: Inspect the Rising Ridge Level-Dependent Congruence Model — There should be a level-dependency effect in the expected direction	There should be a linear level effect in the expected direction	The intersection areas of the data with the regions of (negative, non-, positive) significance should be in line with the suggested hypothesis
At high predictor levels, congruence is associated with the highest outcome values, and this effect is weaker or nonexistent at lower predictor levels. In addition, there is a negative effect of the predictor level.	$b_2 = b_1$, $b_4 = -2b_3$, $b_5 = b_3$, $b_7 = -b_6$, $b_8 = -b_6$, $b_9 = b_6$	$\hat{c_3} = \hat{b_6} < 0$	$\hat{u_1} = \hat{b_1} + \hat{b_2} < 0$	There should be a region of negative significance that intersects with the data, and if a region of positive significance exists, there should be no data in it.
At low predictor levels, congruence is associated with the highest outcome values, and this effect is weaker or nonexistent at higher predictor levels. In addition, there is a negative effect of the predictor level.	$b_2 = b_1$, $b_4 = -2b_3$, $b_5 = b_3$, $b_7 = -b_6$, $b_8 = -b_6$, $b_9 = b_6$	$\hat{c_3} = \hat{b_6} > 0$	$\hat{u_1} = \hat{b_1} + \hat{b_2} < 0$	There should be a region of negative significance that intersects with the data, and if a region of positive significance exists, there should be no data in it.
At low predictor levels, congruence is associated with the lowest outcome values, and this effect is weaker or nonexistent at higher predictor levels. In addition, there is a negative effect of the predictor level.	$b_2 = b_1$, $b_4 = -2b_3$, $b_5 = b_3$, $b_7 = -b_6$, $b_8 = -b_6$, $b_9 = b_6$	$\hat{c_3} = \hat{b_6} < 0$	$\hat{u_1} = \hat{b_1} + \hat{b_2} < 0$	There should be a region of positive significance that intersects with the data, and if a region of negative significance exists, there should be no data in it.
At high predictor levels, congruence is associated with the lowest outcome values, and this effect is weaker or nonexistent at lower predictor levels. In addition, there is a negative effect of the predictor level.	$b_2 = b_1$, $b_4 = -2b_3$, $b_5 = b_3$, $b_7 = -b_6$, $b_8 = -b_6$, $b_9 = b_6$	$\hat{c_3} = \hat{b_6} > 0$	$\hat{u_1} = \hat{b_1} + \hat{b_2} < 0$	There should be a region of positive significance that intersects with the data, and if a region of negative significance exists, there should be no data in it.

E OSF Material E: Mathematical Proofs of Claims About the Rising Ridge Simple Congruence Model

We will now show that the rising ridge simple congruence model is in line with all defining properties of the rising ridge simple congruence hypothesis, as listed in Table 2 in the paper.

E.1 Basics and Notation

Having estimated the regression coefficients of the rising ridge simple congruence model (Equation 65), we write the estimated model equation as a function $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto \hat{z}$:

$$\begin{aligned} f(x,y) = \hat{z} &= \hat{c_0} + \hat{c}_{level}(x+y) + \hat{c_1}(x-y)^2 \\ &= \hat{c_0} + \hat{c}_{level}x + \hat{c}_{level}y + \hat{c_1}x^2 - 2\hat{c_1}xy + \hat{c_1}y^2 \end{aligned} \tag{68}$$

E.2 The Rising Ridge Simple Congruence Model is in Line With a *Congruence Effect*, Reflected in $\hat{c_1}$

We will now show that the rising ridge simple congruence model reflects a congruence effect as defined in Table 2 in the paper: "For two people with the same mean predictor level $(x + y)/2$, the person whose x and y values are closer to one another has the higher (or lower) value of z." Furthermore, we will show that the sign of the estimated coefficient $\hat{c_1}$ affects the direction of the congruence effect. More specifically, we make the following claim about the function f that is defined in Equation 68:

Claim. Let $(x_i, y_i), (x_j, y_j) \in \mathbb{R}^2$ satisfy the following two properties:

(1) $x_i + y_i = x_j + y_j$,

(2) $|x_i - y_i| < |x_j - y_j|$.

Then the following inequalities hold:

$$\begin{cases} f(x_i, y_i) > f(x_j, y_j), & \text{if } \hat{c_1} < 0 \\ f(x_i, y_i) < f(x_j, y_j), & \text{if } \hat{c_1} > 0. \end{cases} \tag{69}$$

Proof. Let $(x_i, y_i), (x_j, y_j) \in \mathbb{R}^2$ satisfy Properties (1) and (2). We have $(x_i - y_i)^2 < (x_j - y_j)^2$ (see Equation 3 in Section A.2 above). We now prove the claim separately for the two cases that are distinguished in Equation 69.

Case 1: $\hat{c_1} < 0$**.** We have

$$\begin{aligned} f(x_i, y_i) &= \hat{c_0} + \hat{c}_{level} \underbrace{(x_i + y_i)}_{=x_j+y_j} + \underbrace{\hat{c_1}}_{<0} \underbrace{(x_i - y_i)^2}_{<(x_j-y_j)^2} \\ &> \hat{c_0} + \hat{c}_{level}(x_j + y_j) + \hat{c_1}(x_j - y_j)^2 \\ &= f(x_j, y_j). \end{aligned} \tag{70}$$

Case 2: $\hat{c_1} > 0$**.** We have

$$\begin{aligned} f(x_i, y_i) &= \hat{c_0} + \hat{c}_{level} \underbrace{(x_i + y_i)}_{=x_j+y_j} + \underbrace{\hat{c_1}}_{>0} \underbrace{(x_i - y_i)^2}_{<(x_j-y_j)^2} \\ &< \hat{c_0} + \hat{c}_{level}(x_j + y_j) + \hat{c_1}(x_j - y_j)^2 \\ &= f(x_j, y_j). \end{aligned} \tag{71}$$

□

E.3 The Rising Ridge Simple Congruence Model is in Line With a *Linear Level Effect*, Reflected in $\hat{c}_{level}$

We will now show that the rising ridge simple congruence model reflects a linear level effect as defined in Table 2 in the paper: "For two people with the same value on the discrepancy $x - y$, the person who has the higher mean predictor level $(x + y)/2$ has the higher (or lower) value of z." Furthermore, we will show that the sign of the estimated coefficient $\hat{c}_{level}$ affects the direction of the linear level effect. More specifically, we make the following claim about the function f that is defined in Equation 68:

Claim. Let $(x_i, y_i), (x_j, y_j) \in \mathbb{R}^2$ satisfy the following two properties:

(1) $x_i - y_i = x_j - y_j$,

(2) $x_i + y_i > x_j + y_j$.

Then the following inequalities hold:

$$\begin{cases} f(x_i, y_i) > f(x_j, y_j), & \text{if } \hat{c}_{level} > 0 \\ f(x_i, y_i) < f(x_j, y_j), & \text{if } \hat{c}_{level} < 0. \end{cases} \tag{72}$$

Proof. Let $(x_i, y_i), (x_j, y_j) \in \mathbb{R}^2$ satisfy Properties (1) and (2). We now prove the claim separately for the two cases that are distinguished in Equation 72.

***Case 1:* $\hat{\mathbf{c}}_{level} > \mathbf{0}$.** We have

$$
\begin{aligned}
f(x_i, y_i) &= \hat{c_0} + \underbrace{\hat{c}_{level}}_{>0} \underbrace{(x_i + y_i)}_{>x_j+y_j} + \hat{c_1} \underbrace{(x_i - y_i)^2}_{=(x_j-y_j)^2} \\
&> \hat{c_0} + \hat{c}_{level}(x_j + y_j) + \hat{c_1}(x_j - y_j)^2 \\
&= f(x_j, y_j).
\end{aligned}
\tag{73}
$$

***Case 2:* $\hat{\mathbf{c}}_{level} < \mathbf{0}$.** We have

$$
\begin{aligned}
f(x_i, y_i) &= \hat{c_0} + \underbrace{\hat{c}_{level}}_{<0} \underbrace{(x_i + y_i)}_{>x_j+y_j} + \hat{c_1} \underbrace{(x_i - y_i)^2}_{=(x_j-y_j)^2} \\
&< \hat{c_0} + \hat{c}_{level}(x_j + y_j) + \hat{c_1}(x_j - y_j)^2 \\
&= f(x_j, y_j).
\end{aligned}
\tag{74}
$$

□

E.4 The Equality Constraint of the Linear Terms Cannot Be Omitted in the Estimation of the Rising Ridge Simple Congruence Model

We now explain why in the rising ridge simple congruence model (Equation 68), we cannot loosen the equality constraint that the coefficients of x and y be equal ($b_1 = b_2$ in the notation of the full polynomial model), because the model predictions would systematically contradict a congruence effect if we did. This objective is very similar to the objective in Section B.6, where we showed why the strict asymmetric congruence model must not contain the term $(x - y)$ as a linear predictor. Accordingly, the reasoning that we will now present follows the same logic as Section B.6.

Specifically, let $\hat{c_0}, \hat{b_1}, \hat{b_2}$, and $\hat{c_1} \in \mathbb{R}$ and consider the function that loosens the equality constraint of the linear terms in the rising ridge simple congruence model:

$$
F(x, y) = \hat{c_0} + \hat{b_1}x + \hat{b_2}y + \hat{c_1}(x - y)^2 \tag{75}
$$

As in Section E.2, let $(x_i, y_i), (x_j, y_j) \in \mathbb{R}$ satisfy the following two properties:

(1) $x_i + y_i = x_j + y_j$,

(2) $|x_i - y_i| < |x_j - y_j|$.

Without loss of generality, consider a situation where $\hat{c_1} < 0$. From the proof in

Section E.2, we know that when $\hat{b_1} = \hat{b_2}$ (i.e., when the function in Equation 75 reduces to the function in Equation 68), then the inequality $F(x_i, y_i) > F(x_j, y_j)$ holds, in line with the congruence effect.

We now show that in case that $\hat{b_1} \neq \hat{b_2}$, the inequality $F(x_i, y_i) > F(x_j, y_j)$ is no longer true for all predictor combinations that satisfy Properties (1) and (2), so the congruence effect would be violated in this case. This observation justifies the equality constraint of the linear terms in the rising ridge simple congruence model. The same reasoning could also be applied to understand the constraint $b_1 = b_2$ in the rising ridge asymmetric congruence model and in the rising ridge level-dependent congruence model. More specifically, we make the following claim:

Claim. Let $\hat{c_1} < 0$ and $\hat{b_1} \neq \hat{b_2}$. Let the function $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined as in Equation 75. Then there exist $(x_i, y_i), (x_j, y_j) \in \mathbb{R}^2$ which satisfy Properties (1) and (2), such that $F(x_i, y_i) < F(x_j, y_j)$.

Proof. To prove the claim, we need to find two elements $(x_i, y_i), (x_j, y_j)$ that satisfy Property (1), Property (2), and $F(x_i, y_i) < F(x_j, y_j)$. We restrict our search for these points to predictor combinations that lie on the line g_0, that is, to elements (x, y) with $y = -x$.

The outcome values that the function in Equation 75 predicts for predictor combinations that lie on g_0 are given by:

$$\begin{aligned} F_{|g_0}(x) &= F(x, -x) = \hat{c_0} + \hat{b_1}x + \hat{b_2}(-x) + \hat{c_1}(x - (-x))^2 \\ &= \hat{c_0} + (\hat{b_1} - \hat{b_2})x + 4\hat{c_1}x^2 \end{aligned} \tag{76}$$

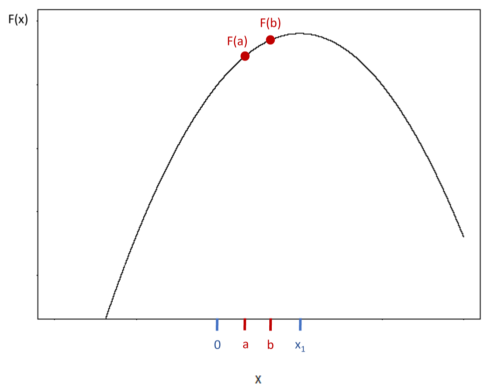


Figure 4. Prototypical graph of $F_{|g_0}$ (Equation 76), when $\hat{c_1} < 0$ and $\hat{b_1} \neq \hat{b_2}$.

Figure 4 shows a prototypical graph of $F_{|g_0}$. The first derivative of $F_{|g_0}$ is

$\frac{\partial F_{|g_0}(x)}{\partial x} = \hat{b_1} - \hat{b_2} + 8\hat{c_1}x$, so the extremum of $F_{\big|g_0}$ lies at $x_1 = \frac{\hat{b_2}-\hat{b_1}}{8\hat{c_1}}$. Because $\hat{b_1} \neq \hat{b_2}$, we have $x_1 \neq 0$. Without loss of generality, let $\hat{b_2} > \hat{b_1}$, so that $x_1 > 0$. Because $\hat{c_1} < 0$, we know that $F_{\big|g_0}$ has a maximum at x_1 and is thus strictly monotonically increasing on $(-\infty, x_1]$.

Now choose two arbitrary values $a, b \in [0, x_1]$ such that $a < b$ (see Figure 4). Define $(x_i, y_i) := (a, -a)$ and $(x_j, y_j) := (b, -b)$. Then (x_i, y_i) and (x_j, y_j) satisfy Property (1) because $x_i + y_i = a - a = 0 = b - b = x_j + y_j$. They also satisfy Property (2) because

$$|x_i - y_i| = |2a| < |2b| = |x_j - y_j| \tag{77}$$

Moreover, we have

$$F(x_i, y_i) = F(a, -a) = F_{|g_0}(a) < F_{|g_0}(b) = F(b, -b) = F(x_j, y_j), \tag{78}$$

where the inequality in the middle of Equation 78 follows because $F_{|g_0}$ is strictly monotonically increasing on $[0, x_1] \subset (-\infty, x_1]$. □

F OSF Material F: Mathematical Proofs of Claims About the Rising Ridge Asymmetric Congruence Model

We will now show that the rising ridge asymmetric congruence model is in line with all defining properties of the rising ridge asymmetric congruence hypothesis, as listed in Table 2 in the paper.

F.1 Basics and Notation

Having estimated the regression coefficients of the rising ridge asymmetric congruence model (Equation 66), we write the estimated model equation as a function $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto \hat{z}$:

$$\begin{aligned} f(x,y) = \hat{z} &= \hat{c_0} + \hat{c}_{level}(x+y) + \hat{c_1}(x-y)^2 + \hat{c_2}(x-y)^3 \\ &= \hat{c_0} + \hat{c}_{level}x + \hat{c}_{level}y + \hat{c_1}x^2 - 2\hat{c_1}xy + \hat{c_1}y^2 + \hat{c_2}x^3 - 3\hat{c_2}x^2y + 3\hat{c_2}xy^2 - \hat{c_2}y^3 \end{aligned} \tag{79}$$

F.2 The Rising Ridge Asymmetric Congruence Model is in Line With a *Congruence Effect*, Reflected in $\hat{c_1}$

Consider the rising ridge asymmetric congruence model as defined in Equation 79, and let $\hat{c_1} \neq 0$ and $\hat{c_2} \neq 0$. Let $D(\hat{c_1}, \hat{c_2}) \subset \mathbb{R}^2$ be defined as in Equation 17 in Section B.4.

We will now show that the predictions of f (Equation 79) for predictor combinations $(x, y) \in D(\hat{c_1}, \hat{c_2})$ are in line with a congruence effect as defined in Table 2 in the paper: "For two people who either both have $x > y$ or both have $x < y$ and who have the same mean predictor level $(x + y)/2$, the person whose x and y values are closer to one another has the higher (or lower) value of z." Furthermore, we will show that the sign of the estimated coefficient $\hat{c_1}$ affects the direction of the congruence effect. Specifically, we claim the following:

Claim. Let $(x_i, y_i), (x_j, y_j) \in D(\hat{c_1}, \hat{c_2})$ satisfy the following properties:

(1) $(x_i > y_i$ and $x_j > y_j)$ or $(x_i < y_i$ and $x_j < y_j)$,

(2) $x_i + y_i = x_j + y_j$,

(3) $|x_i - y_i| < |x_j - y_j|$.

Then the following inequalities hold:

$$\begin{cases} f(x_i, y_i) > f(x_j, y_j), & \text{if } \hat{c_1} < 0 \\ f(x_i, y_i) < f(x_j, y_j), & \text{if } \hat{c_1} > 0. \end{cases} \tag{80}$$

Proof. Let $(x_i, y_i), (x_j, y_j) \in D(\hat{c_1}, \hat{c_2})$ satisfy Properties (1) to (3).

Define $k := \frac{x_i+y_i}{2} = \frac{x_j+y_j}{2}$ (where the equality follows from Property (2)).

Consider the auxiliary function

$$h : \mathbb{R} \to \mathbb{R}, d \mapsto \hat{c_0} + 2\hat{c}_{level}k + \hat{c_1}d^2 + \hat{c_2}d^3. \tag{81}$$

Then we have

$$\begin{aligned} h(x_i - y_i) &= \hat{c_0} + 2\hat{c}_{level}k + \hat{c_1}(x_i - y_i)^2 + \hat{c_2}(x_i - y_i)^3 \\ &= \hat{c_0} + 2\hat{c}_{level}\tfrac{x_i+y_i}{2} + \hat{c_1}(x_i - y_i)^2 + \hat{c_2}(x_i - y_i)^3 \\ &= \hat{c_0} + \hat{c}_{level}(x_i + y_i) + \hat{c_1}(x_i - y_i)^2 + \hat{c_2}(x_i - y_i)^3 \\ &= f(x_i, y_i) \end{aligned} \tag{82}$$

and, analogously, $f(x_j, y_j) = h(x_j - y_j)$.

The first derivative of h is $h'(d) = 2\hat{c_1}d + 3\hat{c_2}d^2 = (2\hat{c_1} + 3\hat{c_2}d)d$, so the extrema of h are located at $d_1 = 0$ and $d_2 = -\frac{2\hat{c_1}}{3\hat{c_2}}$.

The remainder of the proof is exactly the same as for the strict asymmetric congruence model (see Section B.4). □

F.3 The Rising Ridge Asymmetric Congruence Model is in Line With a *Linear Level Effect*, Reflected in $\hat{c}_{level}$

The rising ridge asymmetric congruence model as defined in Equation 79 reflects a linear level effect as defined in Table 2 in the paper: "For two people with the same value on the discrepancy $x - y$, the person who has the higher mean predictor level $(x + y)/2$ has the higher (or lower) value of z." Furthermore, the sign of the estimated coefficient $\hat{c}_{level}$ affects the direction of the linear level effect.

The corresponding technical claim and the proof are exactly the same as those for the rising ridge simple congruence model, so we refer readers to Section E.3 at this point.

F.4 The Rising Ridge Asymmetric Congruence Model is in Line With an *Asymmetry Effect*, Reflected in $\hat{c_2}$

We will now show that the predictions of the rising ridge asymmetric congruence model (Equation 79) are in line with an asymmetry effect as defined in Table 2 in the paper: "For two people with the same mean predictor level $(x+y)/2$ and whose discrepancy values $x-y$ are equal in magnitude but opposite in sign, the person with $x<y$ has the higher (or lower) value of z." Furthermore, we will show that the sign of the estimated coefficient $\hat{c_2}$ affects the direction of the asymmetry effect. Specifically, we consider the following claim:

Claim. Let $(x_i, y_i), (x_j, y_j) \in \mathbb{R}^2$ satisfy the following two properties:

(1) $x_i + y_i = x_j + y_j$,

(2) $x_i - y_i = -(x_j - y_j)$.

Without loss of generality, let $x_i < y_i$, so that $x_i - y_i < 0$ and $x_j - y_j > 0$. Then the following inequalities hold:

$$\begin{cases} f(x_i, y_i) > f(x_j, y_j), & \text{if } \hat{c_2} < 0 \\ f(x_i, y_i) < f(x_j, y_j), & \text{if } \hat{c_2} > 0. \end{cases} \tag{83}$$

Proof. Let $(x_i, y_i), (x_j, y_j) \in \mathbb{R}^2$ satisfy Properties (1) and (2).

Case 1: $\hat{c_2} < 0$. We have

$$\begin{aligned} f(x_i, y_i) &= \hat{c_0} + \hat{c}_{level}(x_i + y_i) + \hat{c_1}(x_i - y_i)^2 + \hat{c_2}(x_i - y_i)^3 \\ &= \hat{c_0} + \hat{c}_{level}(x_j + y_j) + \hat{c_1}(x_j - y_j)^2 - \underbrace{\hat{c_2}}_{<0}\underbrace{(x_j - y_j)^3}_{>0} \\ &> \hat{c_0} + \hat{c}_{level}(x_j + y_j) + \hat{c_1}(x_j - y_j)^2 + \hat{c_2}(x_j - y_j)^3 \\ &= f(x_j, y_j) \end{aligned} \tag{84}$$

***Case 2:* $\hat{c_2} > 0$.** We have

$$\begin{aligned}
f(x_i, y_i) &= \hat{c}_0 + \hat{c}_{level}(x_i + y_i) + \hat{c}_1(x_i - y_i)^2 + \hat{c}_2(x_i - y_i)^3 \\
&= \hat{c}_0 + \hat{c}_{level}(x_j + y_j) + \hat{c}_1(x_j - y_j)^2 - \underbrace{\hat{c}_2}_{>0}\underbrace{(x_j - y_j)^3}_{>0} \\
&< \hat{c}_0 + \hat{c}_{level}(x_j + y_j) + \hat{c}_1(x_j - y_j)^2 + \hat{c}_2(x_j - y_j)^3 \\
&= f(x_j, y_j)
\end{aligned} \tag{85}$$

□

G OSF Material G: Mathematical Proofs of Claims About the Rising Ridge Level-Dependent Congruence Model

We will now show that the rising ridge level-dependent congruence model is in line with all defining properties of the rising ridge level-dependent congruence hypothesis, as listed in Table 2 in the paper.

G.1 Basics and Notation

Having estimated the regression coefficients of the rising ridge level-dependent congruence model (Equation 67), we write the estimated model equation as a function $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto \hat{z}$:

$$\begin{aligned} f(x,y) = \hat{z} &= \hat{c_0} + \hat{c}_{level}(x+y) + \hat{c_1}(x-y)^2 + \hat{c_3}(x+y)(x-y)^2 \\ &= \hat{c_0} + \hat{c}_{level}x + \hat{c}_{level}y + \hat{c_1}x^2 - 2\hat{c_1}xy + \hat{c_1}y^2 + \hat{c_3}x^3 - \hat{c_3}x^2y - \hat{c_3}xy^2 + \hat{c_3}y^3 \end{aligned} \tag{86}$$

G.2 The Rising Ridge Level-Dependent Congruence Model is in Line With a *Congruence Effect*

The rising ridge level-dependent congruence model as defined in Equation 86 reflects a congruence effect as defined in Table 2 in the paper: “For two people with the same mean predictor level $(x + y)/2$, the person whose x and y values are closer to one another has the higher (or lower) value of z.”

The corresponding technical claim and the proof are exactly the same as those for the strict level-dependent congruence model (see Section C.3). We only need to note that when (x_i, y_i) and (x_j, y_j) are chosen as in the claim in Section C.3, then $\hat{c}_{level}(x_i + y_i) = \hat{c}_{level}(x_j + y_j)$.

G.3 The Rising Ridge Level-Dependent Congruence Model is in Line With a *Linear Level Effect*, Reflected in $\hat{c}_{level}$

The rising ridge level-dependent congruence model as defined in Equation 86 reflects a linear level effect as defined in Table 2 in the paper: “For two people with the same value on the discrepancy $x - y$, the person who has the higher mean predictor level $(x + y)/2$ has the higher (or lower) value of z.” Furthermore, the sign of the estimated coefficient $\hat{c}_{level}$ affects the direction of the linear level effect.

The corresponding technical claim and the proof are exactly the same as those for the rising ridge simple congruence model, so we refer readers to Section E.3 at this point.

G.4 The Rising Ridge Level-Dependent Congruence Model is in Line With a *Level-Dependency Effect*

The rising ridge level-dependent congruence model as defined in Equation 86 reflects a level-dependency effect as defined in Table 2 in the paper: “For four people A, B, C, and D of which A and B, as well as C and D, respectively, have equal mean predictor levels $(x+y)/2$, where A and B have a lower mean predictor level than C and D, and of which A and C, as well as B and D, respectively, have equal discrepancy values $x-y$, the absolute difference between A’s and B’s z values is larger (or smaller) than the absolute difference between C’s and D’s z values.”

The corresponding technical claim and the proof are exactly the same as those for the strict level-dependent congruence model (see Section C.4). We only need to note that when $(x_i, y_i), (x_j, y_j), (x_m, y_m)$, and (x_n, y_n) are chosen as in the claim in Section C.4, then $\hat{c}_{level}(x_i + y_i) - \hat{c}_{level}(x_j + y_j) = 0$ and $\hat{c}_{level}(x_m + y_m) - \hat{c}_{level}(x_n + y_n) = 0$.

H OSF Material H: R-Code Walkthrough

We implemented the four cubic models (asymmetric and level-dependent, strict and rising ridge versions, respectively) into the R package *RSA* (Schönbrodt & Humberg, 2020). We also implemented additional functions for investigating whether a hypothesis is supported in the area of realistic predictor combinations. In the OSF project, we provide simulated data and an example script that, for each of the four hypotheses, goes through the whole process of investigating the respective hypothesis (see the file "illustration.R"). In the following, we guide readers through this code.

H.1 R Code to Investigate an Asymmetric Congruence Hypothesis

As a realistic psychological example, imagine that we are interested in the adaptiveness of self-other agreement about intellectual ability, and that we posit a strict asymmetric congruence effect: The harmony of people's social interactions z should be higher the closer a person's self-viewed ability x is to his or her friend-viewed ability y, and maintaining a self-view that is higher than the friend-view ($x > y$) should be even worse than a self-view that is lower than the friend-view ($x < y$). Having assessed and prepared the variables x, y, and z in a suitable way (see the paper for more details), we need to conduct two steps to investigate the hypothesis.

In the first step, we must test the constraints posed by the strict asymmetric congruence model by comparing the fit of this model with the fit of the full third-order model (see Table 4 in the paper). We implemented both models in the *RSA* package, with abbreviations `CA` ("cubic asymmetry") for the strict asymmetric congruence model, and `cubic` for the full third-order model, respectively. To estimate the two models, we use the `RSA()` function and label the resulting *RSA* object as `ca_myrsa`:

```
> ca_myrsa <- RSA(Z ~ X*Y, data=ca_data, model=c("CA","cubic"),
                  missing="fiml", out.rm=TRUE)
```

The model formula `Z ∼ X*Y` specifies that harmony z is the outcome variable and that self-view x and friend-view y are the predictor variables. The `model` option specifies that the models `CA` and `cubic` should be estimated. The `RSA()` function estimates the two models in *lavaan*. Missing values are treated with FIML. When estimating complex regression models, the data should be screened for influential cases to rule out the possibility that single data points are strongly influencing the results. By default, the *RSA*() function identifies and removes influential cases according to a strategy that Edwards (2002) recommended for RSA.[1] This strategy can be modified by using the `out.rm` option.

To test the model constraints, the `compare2()` function from the *RSA* package provides the χ^2 likelihood ratio test statistic:

```
> compare2(ca_myrsa, "CA", "cubic")
```

For the example data, the output of this command indicates that the strict asymmetric congruence model did not fit the data significantly worse than the full model ($\chi^2(7) = 7.06, p = .42$; see Table 6). That is, the data support the constraints imposed by the strict asymmetric congruence model. In Table 6, we also report adjusted R^2 values (also extracted from the `compare2()` output) as complementary information to the relative model test.

Table 6
Cubic RSA Results for Simulated Example Data

Model	$\hat{b_0}$	$\hat{b_1}$	$\hat{b_2}$	$\hat{b_3}$	$\hat{b_4}$	$\hat{b_5}$	$\hat{b_6}$	$\hat{b_7}$	$\hat{b_8}$	$\hat{b_9}$	$\Delta\chi^2$	R^2
CA model	3.51	0	0	-0.065	0.13	-0.06	-0.02	0.06	-0.06	0.02	7.06	0.035
p-values	<.001			<.001	<.001	<.001	.001	.001	.001	.001	.422	
full model	3.53	-0.13	0.06	-0.06	0.13	-0.11	0	0.06	-0.02	0		0.043
p-values	<.001	.061	.434	.032	.003	<.001	.815	.035	.579	.842		

Note. CA model = strict asymmetric congruence model. full model = full third-order polynomial model.
The coefficient estimates $\hat{b_0}$ to $\hat{b_9}$ refer to the full third-order polynomial model
$z = b_0 + b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2 + b_6x^3 + b_7x^2y + b_8xy^2 + b_9y^3$.
$\Delta\chi^2$ = difference of the χ^2 values of the two models.

In the second step, we must inspect the estimated model coefficients of the strict asymmetric congruence model (see Table 4 in the paper). The coefficient estimates can be extracted with the `getPar()` function:

```
> getPar(ca_myrsa, model="CA")
```

[1] That is, a data point is categorized as an influential case if the following three conditions hold for $dfFit$, Cook's distance D, and the hat value (Edwards, 2002): (a) $|dfFit| > 3\sqrt{(k/(n-k))}$, (b) $D >$ 50th percentile of $F(k, n-k)$, and (c) $hat > 3k/n$, where k is the number of estimated parameters in the respective model, and n denotes the sample size.

The estimates are presented in Table 6. First, we must test whether $\hat{c_1} = \hat{b_3}$ is significantly negative to determine whether the model indicates a congruence effect in the suggested direction. We use an α level of $\alpha = .01$ for the coefficient tests. The coefficient $\hat{b_3}$ was significantly negative in the present example ($\hat{b_3} = -0.06, p < .001$). Second, to test the suggested asymmetry effect, we must test whether $\hat{c_2} = \hat{b_6}$ is significantly negative, which was also the case ($\hat{b_6} = -0.02, p = .001$).

Finally, we must inspect whether the effect is present for the whole range of realistic predictor combinations. To this aim, we implemented the function `caRange()`, which can be applied to the `RSA()` output object `ca_myrsa`:

```
> caRange(ca_myrsa, model="CA", alpha = 0.01, alphacorrection="Bonferroni")
```

In the options of this function, we specify that the test of the universal null hypothesis stating that no data points contradict the hypothesis should be conducted at an α level of $\alpha = .01$, and with a Bonferroni correction. For the present example data, the output of `caRange()` indicates that 2% of the predictor combinations in the data are positioned behind the second extremum line (E_2) but that none of the outcome predictions at these points lie outside of the confidence interval $(-\infty, 3.78]$ of $\hat{z_r}$. Overall, the example data support the strict asymmetric congruence hypothesis.

To investigate the rising ridge instead of the strict version of the asymmetric congruence hypothesis, the model name `CA` needs to be replaced by `RRCA` ("rising ridge cubic asymmetry") in all commands (see the illustrative R code). In addition, we would need to inspect the linear level effect $\hat{u_1}$, which is provided in the `getPar()` output of this model.

H.2 R Code to Investigate a Level-Dependent Congruence Hypothesis

Turning to another example, imagine that we posit a strict level-dependent congruence hypothesis: Congruence between targets' self-reported security importance (x) and their partners' estimate of the targets' security importance (y) is expected to be associated with the lowest feelings of emotional distance in the targets (z), and this effect is expected to be weaker or nonexistent for lower levels of self- and partner-reported value importance than for higher levels. Again, we would assess and prepare the variables x, y, and z in a suitable way and conduct the steps for investigating the hypothesis by use of the *RSA* package.

In the first step, we estimate the strict level-dependent congruence model (`CL` = "cubic level") and the full model (`cubic`) and test them against each other:

```
> cl_myrsa <- RSA(Z ~ X*Y, data=cl_data, model=c("CL","cubic"),
                  missing="fiml", out.rm=TRUE)
> compare2(cl_myrsa, "CL", "cubic")
```

For the present example data, the constraints of the strict level-dependent congruence model are supported ($\chi^2(7) = 3.02, p = .88$; see Table 7).

Table 7
Cubic RSA Results for Simulated Example Data

Model	$\hat{b_0}$	$\hat{b_1}$	$\hat{b_2}$	$\hat{b_3}$	$\hat{b_4}$	$\hat{b_5}$	$\hat{b_6}$	$\hat{b_7}$	$\hat{b_8}$	$\hat{b_9}$	$\Delta\chi^2$	R^2
CL model	4.02	0	0	0.09	-0.19	0.09	0.06	-0.06	-0.06	0.06	3.02	0.043
p-values	<.001			.005	.005	.005	<.001	<.001	<.001	<.001	.883	
full model	4.01	-0.07	0.14	0.1	-0.16	0.1	0.07	-0.08	-0.04	0.03		0.049
p-values	<.001	.478	.182	.025	.024	.037	.005	.19	.571	.428		

Note. CL model = strict level-dependent congruence model. full model = full third-order polynomial model.
The coefficient estimates $\hat{b_0}$ to $\hat{b_9}$ refer to the full third-order polynomial model $z = b_0 + b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2 + b_6x^3 + b_7x^2y + b_8xy^2 + b_9y^3$.
$\Delta\chi^2$ = difference of the χ^2 values of the two models.

In inspecting the coefficients of the of the strict level-dependent congruence model with `getPar(cl_myrsa, model="CL")` in the second step, we find that $\hat{c_3} = \hat{b_6}$ is significantly positive ($\hat{b_6} = 0.06, p < .001$), in line with the expected level-dependency effect (see Table 7; see Table 5 in the paper for the conditions).

To consider the regions of significance, we apply the function `clRange()` that we implemented into the RSA package:

```
> clRange(cl_myrsa, model="CL", alpha = 0.01)
```

The output of this function for the present example is presented in Table 8. Here, the curvature of the surface is significantly negative above all g_k with $k \in (-\infty, -2.33]$, nonsignificant for $k \in [-2.33, -0.08]$, and significantly positive for $k \in [-0.08, \infty)$. As can be seen in the last column of Table 8, 0% of the data points' mean predictor levels k lie in the region of negative significance, 35% lie in the region of nonsignificance, and 65% lie in the region of positive significance. These findings are in line with the hypothesis (see Table 5 in the paper). Overall, the example data support the suggested strict level-dependent congruence hypothesis.

Table 8

Output of the clRange() Function for Simulated Example Data

region	lower_bound	upper_bound	data_points	percent_data
neg. sign.	-Inf	-2.33	0	0.00
nonsign.	-2.33	-0.08	140	35.00
pos. sign.	-0.08	Inf	260	65.00

Note. neg. sign. = region of negative significance. nonsign. = region of nonsignificance. pos. sign. = region of positive significance. lower_bound / upper_bound = k value of the lower/upper boundary of the respective interval. data_points = number of data points in the respective interval. percent_data = percentage of data points in the respective interval. Inf = infinity.

The rising ridge version of the level-dependent congruence hypothesis can be investigated by specifying `RRCL` ("rising ridge cubic level") as the model of interest in all commands and by extracting the linear level effect $\hat{u}_1$ with the `getPar()` function (see the illustrative R code).

I OSF Material I: Examples for Hypotheses that can be Investigated with (Quadratic or Cubic) RSA (Examples Corresponding to the Formal Definitions in Table 2 of the Article)

Table 9

Formal Definitions and Examples of the Hypotheses that can be Investigated with (Quadratic or Cubic) RSA

Hypotheses definitions	Examples*
Strict simple congruence hypothesis (see also Figure 1 in the paper)	
For two people, the person whose x and y values are closer to one another has the higher (or lower) value of z. [*congruence effect*]	People whose values (x) are more congruent with the organizational values (y) of the work place are more satisfied with their job (z) than people with less congruent values.
Strict asymmetric congruence hypothesis (see also Figure 4 in the paper)	
For two people who either both have $x > y$ or both have $x < y$, the person whose x and y values are closer to one another has the higher (or lower) value of z, [*congruence effect*]	Women experience lower feelings of shame (z) when their actual weight (x) matches the weight they evaluate as ideal (y),
and for two people whose discrepancy values $x - y$ are equal in magnitude but opposite in sign, the person with $x < y$ has the higher (or lower) value of z. [*asymmetry effect*]	and women feel more ashamed when they weigh “more than ideal” in comparison with weighing “less than ideal”.
Strict level-dependent congruence hypothesis (see also Figure 6 in the paper)	
For two people with the same mean predictor level $(x + y)/2$, the person whose x and y values are closer to one another has the higher (or lower) value of z, [*congruence effect*]	Children whose neuroticism levels (x) resemble their mothers' neuroticism levels (y) experience more externalizing problems (z) in adulthood,
and for four people A, B, C, and D of which A and B, as well as C and D, respectively, have equal mean predictor levels $(x + y)/2$, where A and B have a lower mean predictor level than C and D, and of which A and C, as well as B and D, respectively, have equal discrepancy values $x - y$, the absolute difference between A's and B's z values is larger (or smaller) than the absolute difference between C's and D's z values. [*level-dependency effect*]	and this similarity effect is stronger for child-mother combinations with low average neuroticism levels and weaker for child-mother combinations with higher average neuroticism levels.

Note. *see Table 2 in the main article for references to papers where these example hypotheses were discussed.

Table 10

Formal Definitions and Examples of the Hypotheses that can be Investigated with (Quadratic or Cubic) RSA [Continued]

Hypotheses definitions	Examples*
Rising ridge simple congruence hypothesis (see also Figure 8a in the paper)	
For two people with the same mean predictor level $(x+y)/2$, the person whose x and y values are closer to one another has the higher (or lower) value of z, [*congruence effect*]	People are better adjusted (z) when their true extraversion level (x) is similar to their reputation concerning extraversion (y),
and for two people with the same value on the discrepancy $x-y$, the person who has the higher mean predictor level $(x+y)/2$ has the higher (or lower) value of z. [*linear level effect*]	and people with higher average levels of true and other-perceived extraversion are better adjusted than people with lower levels.
Rising ridge asymmetric congruence hypothesis (see also Figure 8b in the paper)	
For two people who either both have $x > y$ or both have $x < y$ and who have the same mean predictor level $(x+y)/2$, the person whose x and y values are closer to one another has the higher (or lower) value of z, [*congruence effect*]	Women's relationship satisfaction (z) is higher on days when the support that they provide for their respective partner (x) is similar to the support that the respective partner provides for them (y)
and for two people with the same value on the discrepancy $x-y$, the person who has the higher mean predictor level $(x+y)/2$ has the higher (or lower) value of z, [*linear level effect*]	and women are more satisfied on days when the average support level in the relationship is higher than on days when their average support level is lower,
and for two people with the same mean predictor level $(x+y)/2$ and whose discrepancy values $x-y$ are equal in magnitude but opposite in sign, the person with $x < y$ has the higher (or lower) value of z. [*asymmetry effect*]	and providing more support than the partner comes at a greater cost for women's relationship satisfaction than providing less support than the partner.
Rising ridge level-dependent congruence hypothesis (see also Figure 8c in the paper)	
For two people with the same mean predictor level $(x+y)/2$, the person whose x and y values are closer to one another has the higher (or lower) value of z, [*congruence effect*]	Children whose narcissism level (x) is similar to their friend's narcissism level (y) like their friend more (z) than children whose narcissism level is more discrepant from their friend's narcissism level,
and for two people with the same value on the discrepancy $x-y$, the person who has the higher mean predictor level $(x+y)/2$ has the higher (or lower) value of z, [*linear level effect*]	child-friend combinations with higher average narcissism levels like each other less than child-friend combinations with lower average narcissism levels,
and for four people A, B, C, and D of which A and B, as well as C and D, respectively, have equal mean predictor levels $(x+y)/2$, where A and B have a lower mean predictor level than C and D, and of which A and C, as well as B and D, respectively, have equal discrepancy values $x-y$, the absolute difference between A's and B's z values is larger (or smaller) than the absolute difference between C's and D's z values. [*level-dependency effect*]	and the effect of narcissism similarity is stronger for children with high narcissism levels than for children with lower levels.

Note. *see Table 2 in the main article for references to papers where these example hypotheses were discussed.

References

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