Supplementary Materials

Defining FDR in the same way as we did in Equation 1,

$$FDR = \frac{P(H_0) * \alpha}{P(H_0) * \alpha + (1 - P(H_0)) * \rho},$$

and substituting power of a two-sided z-test, (1 – β), from the Equation 2,

$$\rho = 1 - \Phi_{\mu}(\Phi^{-1}(1 - \frac{\alpha}{2})) + \Phi_{\mu}(\Phi^{-1}(\frac{\alpha}{2})),$$

we arrive to the FDR expressed only in terms of α and $P(H_0)$,

$$FDR = \frac{P(H_0) * \alpha}{P(H_0) * \alpha + (1 - P(H_0)) * (1 - \Phi_{\mu}\left(\Phi^{-1}\left(1 - \frac{\alpha}{2}\right)\right) + \Phi_{\mu}\left(\Phi^{-1}\left(\frac{\alpha}{2}\right)\right))}.$$

Furthermore, we decompose the cumulative density function of normal distribution with mean μ, Φ_{μ}, into

$$\Phi_{\mu}(\mathrm{x}) = \frac{1}{2} * (\mathrm{erf}\left(\frac{x - \mu}{\sqrt{2}}\right)),$$

and the quantile function of standard normal distribution, Φ^{-1}, into

$$\Phi^{-1}(p) = \sqrt{2} * erf^{-1}(2p - 1).$$

Followingly, we re-express the FDR function as,

$$FDR = \frac{P(H_0) * \alpha}{P(H_0) * \alpha + (1 - P(H_0)) * \left(\frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{\sqrt{2} * erf^{-1}(1 - \alpha) - \mu}{\sqrt{2}}\right) + \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{\sqrt{2} * erf^{-1}(\alpha - 1) - \mu}{\sqrt{2}}\right)\right)},$$

where *erf* stands for error function,

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} * \int_0^x e^{-t^2} dt.$$

Then, the gradient of FDR in respect to α can be written as,

$$\frac{\partial FDR}{\partial \alpha} = \frac{\partial (P(H_0) * \alpha)}{\partial \alpha} * \left(P(H_0) * \alpha + (1 - P(H_0)) * \left(\frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{\sqrt{2} * erf^{-1}(1 - \alpha) - \mu}{\sqrt{2}}\right) + \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{\sqrt{2} * erf^{-1}(\alpha - 1) - \mu}{\sqrt{2}}\right)\right)\right)^{-1} + (P(H_0) * \alpha) * \frac{\partial \left(P(H_0) * \alpha + (1 - P(H_0)) * \left(\frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{\sqrt{2} * erf^{-1}(1 - \alpha) - \mu}{\sqrt{2}}\right) + \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{\sqrt{2} * erf^{-1}(\alpha - 1) - \mu}{\sqrt{2}}\right)\right)\right)^{-1}}{\partial \alpha}$$

$$= \frac{P(H_0)}{P(H_0)*\alpha + (1-P(H_0))*\left(\frac{1}{2}-\frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{2}*erf^{-1}(1-\alpha)-\mu}{\sqrt{2}}\right)+\frac{1}{2}+\frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{2}*erf^{-1}(\alpha-1)-\mu}{\sqrt{2}}\right)\right)} + (P(H_0)*\ \alpha)*$$

$$-\left(P(H_0)*\ \alpha\ +\ (1\ -\ P(H_0))*\left(\frac{1}{2}-\frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{2}*erf^{-1}(1-\alpha)-\mu}{\sqrt{2}}\right)+\frac{1}{2}+\right.\right.$$

$$\left.\left.\frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{2}*erf^{-1}(\alpha-1)-\mu}{\sqrt{2}}\right)\right)\right)^{-2} *$$

$$\frac{\partial\left(P(H_0)*\alpha + (1-P(H_0))*\left(\frac{1}{2}-\frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{2}*erf^{-1}(1-\alpha)-\mu}{\sqrt{2}}\right)+\frac{1}{2}+\frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{2}*erf^{-1}(\alpha-1)-\mu}{\sqrt{2}}\right)\right)\right)}{\partial\alpha}$$

$$= \frac{P(H_0)}{P(H_0)*\alpha + (1-P(H_0))*\left(\frac{1}{2}-\frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{2}*erf^{-1}(1-\alpha)-\mu}{\sqrt{2}}\right)+\frac{1}{2}+\frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{2}*erf^{-1}(\alpha-1)-\mu}{\sqrt{2}}\right)\right)} + (P(H_0)*\ \alpha)*$$

$$-\left(P(H_0)*\ \alpha\ +\ (1\ -\ P(H_0))*\left(\frac{1}{2}-\frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{2}*erf^{-1}(1-\alpha)-\mu}{\sqrt{2}}\right)+\frac{1}{2}+\right.\right.$$

$$\left.\left.\frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{2}*erf^{-1}(\alpha-1)-\mu}{\sqrt{2}}\right)\right)\right)^{-2} * \left(P(H_0) - \frac{1}{2}*(1\ -\ P(H_0))*erf'\left(\frac{\sqrt{2}*erf^{-1}(1-\alpha)-\mu}{\sqrt{2}}\right)*\right.$$

$$\left.\frac{\partial\left(\frac{\sqrt{2}*erf^{-1}(1-\alpha)-\mu}{\sqrt{2}}\right)}{\partial\alpha} + \frac{1}{2}*(1\ -\ P(H_0))*erf'\left(\frac{\sqrt{2}*erf^{-1}(\alpha-1)-\mu}{\sqrt{2}}\right)*\ \frac{\partial\left(\frac{\sqrt{2}*erf^{-1}(\alpha-1)-\mu}{\sqrt{2}}\right)}{\partial\alpha}\right)$$

$$= \frac{P(H_0)}{P(H_0)*\alpha + (1-P(H_0))*\left(\frac{1}{2}-\frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{2}*erf^{-1}(1-\alpha)-\mu}{\sqrt{2}}\right)+\frac{1}{2}+\frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{2}*erf^{-1}(\alpha-1)-\mu}{\sqrt{2}}\right)\right)} + (P(H_0)*\ \alpha)*$$

$$-\left(P(H_0)*\ \alpha\ +\ (1\ -\ P(H_0))*\left(\frac{1}{2}-\frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{2}*erf^{-1}(1-\alpha)-\mu}{\sqrt{2}}\right)+\frac{1}{2}+\right.\right.$$

$$\left.\left.\frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{2}*erf^{-1}(\alpha-1)-\mu}{\sqrt{2}}\right)\right)\right)^{-2} * \left(P(H_0) - \frac{1}{2}*(1\ -\ P(H_0))*erf'\left(\frac{\sqrt{2}*erf^{-1}(1-\alpha)-\mu}{\sqrt{2}}\right)*\right.$$

$$\left.erf^{-1'}(1-\alpha)*-1 + \frac{1}{2}*(1\ -\ P(H_0))*erf'\left(\frac{\sqrt{2}*erf^{-1}(\alpha-1)-\mu}{\sqrt{2}}\right)*\ \ erf^{-1}(\alpha-1)\right).$$

Where erf' stands for the derivative of error function,

$$erf'(x) = \frac{2}{\sqrt{\pi}} * e^{-t^2},$$

and stands for the derivative of inverse error function

$$erf^{-1'}(x) = \frac{1}{2} * \sqrt{\pi} * e^{(erf^{-1}(x))^2}.$$

The gradient of FDR with respect to α for a one-sided z-test can be obtained in a similar manner, but by substituting a power of one-sided z-test into the Equation 1,

$$\rho\ =\ 1\ -\ \Phi_{\mu}(\ \Phi^{-1}(1-\alpha)\ ).$$

Then, the gradient of FDR of one-sided z-test in respect to α can be written as,

$$\frac{\partial FDR}{\partial \alpha} = \frac{\partial(P(H_0) * \alpha)}{\partial \alpha} * \left(P(H_0) * \alpha + \left(1 - P(H_0)\right) * \left(\frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{2} * erf^{-1}(1-2\alpha) - \mu}{\sqrt{2}}\right)\right)\right)^{-1} + (P(H_0) * \alpha) * \frac{\partial\left(P(H_0) * \alpha + \left(1 - P(H_0)\right) * \left(\frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{2} * erf^{-1}(1-2\alpha) - \mu}{\sqrt{2}}\right)\right)\right)^{-1}}{\partial \alpha}$$

$$= \cdots = \frac{P(H_0)}{P(H_0) * \alpha + \left(1 - P(H_0)\right) * \left(\frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{2} * erf^{-1}(1-2\alpha) - \mu}{\sqrt{2}}\right)\right)} + (P(H_0) * \alpha) * -\left(P(H_0) * \alpha + \left(1 - P(H_0)\right) * \left(\frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{2} * erf^{-1}(1-2\alpha) - \mu}{\sqrt{2}}\right)\right)\right)^{-2} * \left(P(H_0) - \frac{1}{2} * \left(1 - P(H_0)\right) * erf'\left(\frac{\sqrt{2} * erf^{-1}(1-2\alpha) - \mu}{\sqrt{2}}\right) * \ erf^{-1'}(1-2\alpha) * -2\right).$$

R implementation of the gradient computation and its visualization can be found in the Supplementary materials. We want to note that especially for low μ and α, the FDR approaches 0 with decreasing α in such a rate, that the gradient of FDR becomes to steep to visualize.