OSF-Material C

Mathematical Proof: The Six Conditions Stated in Figure 3 are Sufficient for a Strict Congruence Effect

We can understand the second-degree polynomial model that builds the basis of RSA as a function g that maps X and Y to the respective Z value:

$$Z = g(X, Y) = b_0 + b_1 X + b_2 Y + b_3 X^2 + b_4 XY + b_5 Y^2. \tag{3}$$

Now, let all six necessary conditions for a strict congruence effect hold, that is, let

1. $p_{10} = 0$
2. $p_{11} = 1$,
3. $a_2 = 0$,
4. $a_1 = 0$,
5. $a_4 < 0$,
6. $a_3 = 0$.

We will now proof that when all six conditions are satisfied, the model reflects a strict congruence effect in the sense that it predicts higher Z values the smaller the distance between X and Y. In mathematical terms, such an effect means that $g(X, Y) > g(X', Y')$ is equivalent to $|X - Y| < |X' - Y'|$. To proof that this equivalence holds when the six conditions are satisfied, we proceed in six steps.

Step 1: Show that $\mathbf{b_1 = b_2 = 0}$.

From the condition $a_1 = b_1 + b_2 = 0$, it follows that $b_1 = -b_2$. Moreover, because $a_3 = b_1 - b_2 = 0$, we know that $b_1 = b_2 = -b_1$. Consequently, $b_1 = 0$ and $b_2 = 0$.

Step 2: Show that $\mathbf{b_4 \neq 0}$.

We proof this claim by contradiction: Suppose $b_4 = 0$. Then, from $a_2 = b_3 + b_4 + b_5 = 0$, it follows that $b_3 + b_5 = 0$. Also, from $a_4 = b_3 - b_4 + b_5 < 0$, it follows that $b_3 + b_5 < 0$, which is a contradiction. Consequently, b_4 must differ from zero.

Step 3: Show that $\mathbf{b_5 = b_3}$.

From $p_{11} = \frac{b_5 - b_3 + \sqrt{(b_3-b_5)^2+b_4^2}}{b_4} = 1$ (see Edwards, 2007, for the formula of p_{11}), it follows that

$$\begin{aligned}
& b_5 - b_3 + \sqrt{(b_3-b_5)^2+b_4^2} &=& \; b_4 \\
\Leftrightarrow \quad & (b_3-b_5)^2+b_4^2 &=& \; ((b_3-b_5)+b_4)^2 \\
\Leftrightarrow \quad & (b_3-b_5)^2+b_4^2 &=& \; (b_3-b_5)^2+2b_4(b_3-b_5)+b_4^2 \\
\Leftrightarrow \quad & 0 &=& \; 2b_4(b_3-b_5) \\
\Leftrightarrow \quad & 0 &=& \; b_3-b_5 \\
\Leftrightarrow \quad & b_5 &=& \; b_3,
\end{aligned} \tag{4}$$

where the second-last step is valid because $b_4 \neq 0$.

Step 4: Show that $\mathbf{b_4 = -2b_3}$.

From $a_2 = 0$ and $b_5 = b_3$, it follows that

$$\begin{aligned}
& a_2 = b_3 + b_4 + b_5 &=& \; 0 \\
\Leftrightarrow \quad & b_3 + b_4 + b_3 &=& \; 0 \\
\Leftrightarrow \quad & b_4 &=& \; -2b_3
\end{aligned} \tag{5}$$

Step 5: Show that g simplifies to $\mathbf{g(X,Y) = b_0 + b_3(X-Y)^2}$ with $\mathbf{b_3 < 0}$.

Taken together, the function g (Equation 3) is given by

$$\begin{aligned}
g(X,Y) &= b_0 + b_1X + b_2Y + b_3X^2 + b_4XY + b_5Y^2 \\
&= b_0 + 0X + 0Y + b_3X^2 - 2b_3XY + b_3Y^2 \\
&= b_0 + b_3(X-Y)^2.
\end{aligned} \tag{6}$$

From $a_4 < 0$, $b_5 = b_3$, and $b_4 = -2b_3$, it follows that the coefficient b_3 is negative:

$$
\begin{aligned}
& a_4 &<& \quad 0 \\
\Leftrightarrow\quad & b_3 - b_4 + b_5 &<& \quad 0 \\
\Leftrightarrow\quad & b_3 + 2b_3 + b_3 &<& \quad 0 \\
\Leftrightarrow\quad & 4b_3 &<& \quad 0 \\
\Leftrightarrow\quad & b_3 &<& \quad 0.
\end{aligned} \tag{7}
$$

Step 6: Show that $\mathbf{g(X, Y) > g(X', Y')}$ is equivalent to $\mathbf{|X - Y| < |X' - Y'|}$.

We can now use the simplified formula for g to show that there is a strict congruence effect:

$$
\begin{aligned}
& g(X,Y) &>& \quad g(X',Y') \\
\Leftrightarrow\quad & b_0 + b_3(X-Y)^2 &>& \quad b_0 + b_3(X'-Y')^2 \\
\Leftrightarrow\quad & b_3(X-Y)^2 &>& \quad b_3(X'-Y')^2 \\
\Leftrightarrow\quad & (X-Y)^2 &<& \quad (X'-Y')^2 \\
\Leftrightarrow\quad & |X-Y| &<& \quad |X'-Y'|.
\end{aligned} \tag{8}
$$

where the third step is valid because $b_3 < 0$.

To sum up, we showed that the six conditions are sufficient for the existence of a strict congruence effect.