OSF-Material B

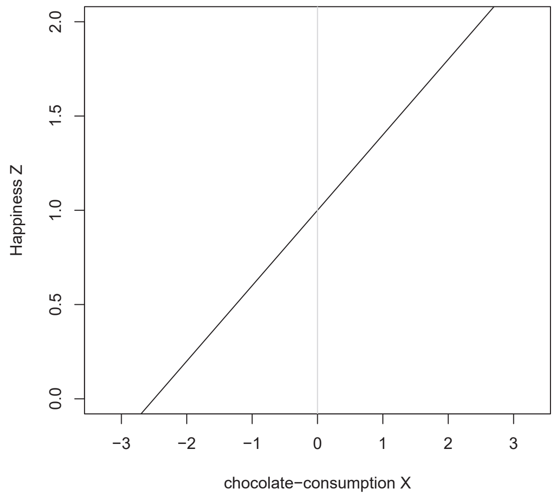
The Fundamentals of RSA Interpretation: Quadratic Equations

RSA interpretation includes several elements that require basic knowledge about quadratic equations. We therefore briefly recall these basics in this supplementary document. As a hypothetical example, imagine that we want to investigate the relation between people's chocolate-consumption and happiness. If we assume that chocolate-consumption X affects happiness Z in a quadratic way, we would assess the respective data and fit a quadratic regression model to the data:

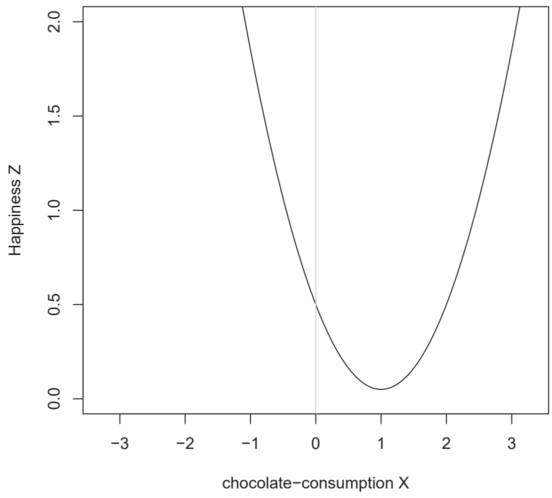
$$Z = q_0 + q_1 X + q_2 X^2, \tag{1}$$

where q_0, q_1, and q_2 are the estimated coefficients, and Z denotes the predicted happiness for a person with chocolate-consumption X. To see what the (estimated) coefficients of such a *quadratic equation* reveal about the association of X and Z, one should first consider the coefficient q_2 of the quadratic term. When q_2 is zero, Equation 1 simplifies to $Z = q_0 + q_1 X$, that is, the association of X and Z is linear. The corresponding graph is a straight line (see Figure 1a) with a slope of q_1 and an intercept of q_0.

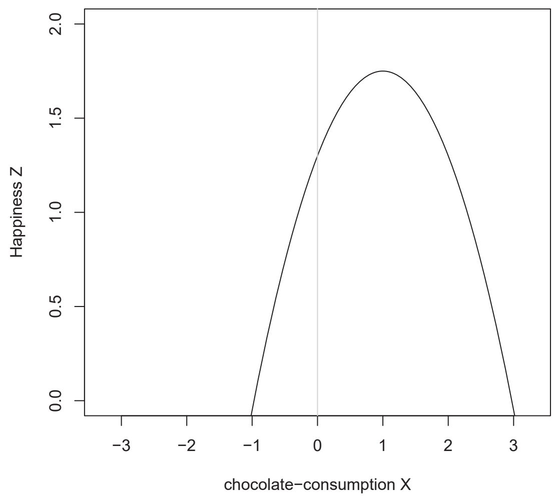
When q_2 is non-zero, the graph of Equation 1 is a *parabola*. When q_2 is positive, the graph has a U-shape (see Figure 1b), and when q_2 is negative, the graph has an inverted U-shape (see Figure 1c). The magnitude of q_2 reflects how "strong" the curvilinear effect of X on Z is: The closer q_2 is to zero, the wider the parabola is (see Figure 1d), and the larger q_2 is in magnitude, the steeper the parabola is (Figure 1c).



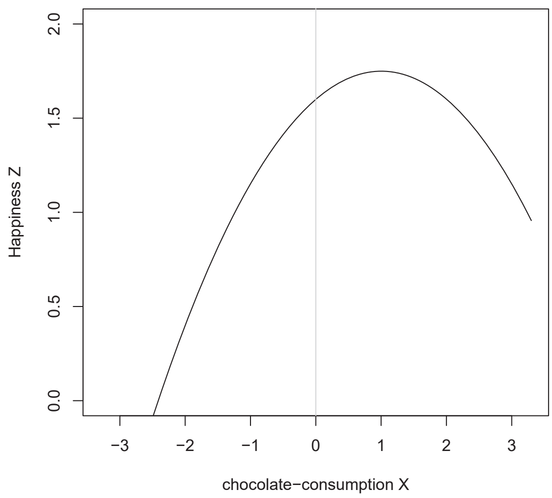
(a) Graph of the equation $Z = 1 + 0.4X + 0X^2$.



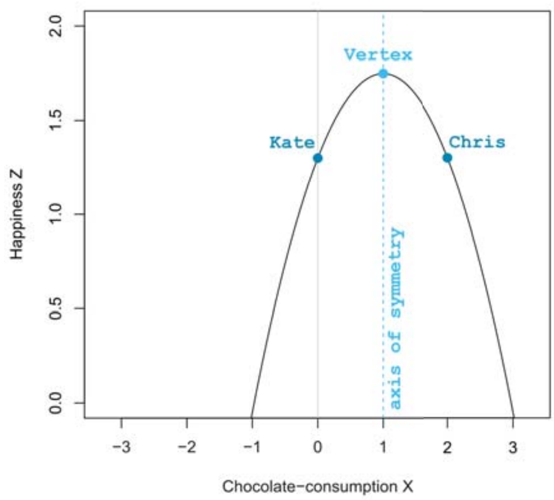
(b) Graph of the equation $Z = 0.5 - 0.9X + 0.45X^2$.



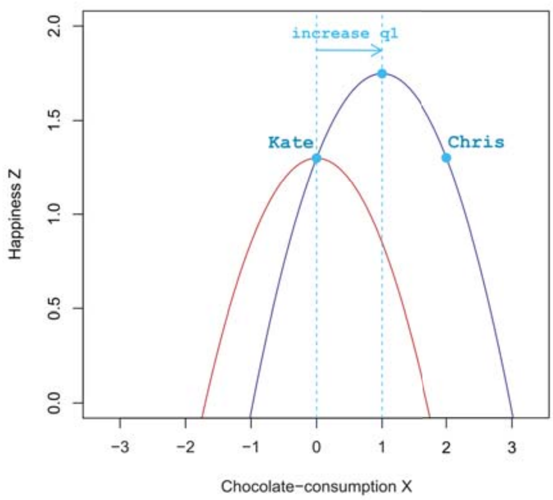
(c) Graph of the equation $Z = 1.3 + 0.9X - 0.45X^2$.



(d) Graph of the equation $Z = 1.6 + 0.3X - 0.15X^2$.



(e) Graph of the equation $Z = 1.3 + 0.9X - 0.45X^2$.



(f) Graph of the two equations $Z = 1.3 + 0X - 0.45X^2$ (left parabola) and $Z = 1.3 + 0.9X - 0.45X^2$ (right parabola).

Figure 1. Graphs of some quadratic functions $Z = q_0 + q_1X + q_2X^2$.

For the chocolate example, a negative value of q_2 would mean that there is some "optimal value" of chocolate-consumption: In Figure 1c, it looks as if people with a value of $X = 1$ are predicted to be the happiest, whereas happiness is lower the more chocolate-consumption deviates from 1. In technical terms, the parabola has a *vertex*, which is defined as its highest[1] point (see Figure 1e). To determine the exact amount of chocolate-consumption that leads to the highest happiness, we can determine the X value of the vertex via the formula (see OSF-Material E for the derivation)

$$X_{vertex} = \frac{-q_1}{2q_2} \tag{2}$$

In Figure 1c, the vertex is located at $X_{vertex} = \frac{-0.9}{2*(-0.45)} = 1$.

An important property of every parabola is its symmetry: When we draw a vertical line (i.e., a line that is perpendicular to the x-axis) through the vertex of a parabola, the parabola is symmetric around this *axis of symmetry* (see Figure 1e; see OSF-Material F for the mathematical proof). That is, the right-hand side of the axis of symmetry is an exact reflection of the left-hand side. For the chocolate example, this means that a person whose chocolate-consumption is higher than the optimal level to some extent (e.g., at "optimal level + 1" as for Chris) is predicted to be as happy as a person whose consumption is lower than the optimal level by the same amount (e.g., at "optimal level - 1" as for Kate).

When interpreting the results of quadratic equations, it is also crucial to understand the role of q_1 because this coefficient can substantially alter the interpretation in terms of how X is related to Z. Imagine that q_2 is negative, so we know that there is some optimal value of chocolate-consumption X, and we aim at finding out whether the optimal X value is zero. To this aim, we must consider the coefficient q_1. As a mathematical fact, q_1 equals the slope of the parabola at $X = 0$ (e.g., see Edwards, 2007). The vertex of a parabola is positioned at $X = 0$ only if q_1 is zero; in this case (assuming $q_2 < 0$), the parabola rises for negative X values, reverses at $X = 0$, and falls for positive X values (as in the left-hand side parabola in Figure 1f), predicting the highest happiness for people with a chocolate-consumption value of zero (e.g., Kate).

When we change q_1 from zero to a positive value (as in the right parabola in Figure 1f), the

[1] When $q_2 > 0$, the vertex is the *lowest* point of the parabola. Its position can also be computed via Equation 2.

parabola is shifted to the right,[2] such that the slope above $X = 0$ is now positive. In shifting the parabola, its shape and vertical orientation stay perfectly fixed: The axis of symmetry is shifted sideways along with the parabola and thereby remains perpendicular to the x-axis (see Figure 1f). When q_1 is non-zero, the parabola is no longer maximized at $X = 0$, such that Kate's chocolate-consumption value of zero is no longer optimal. Instead, when q_1 is positive, happiness rises when we approach $X = 0$ from negative consumption values, and it continues to rise at a value of zero consumption. To sum, the linear term coefficient q_1 in a quadratic equation (with $q_2 \neq 0$) equals the slope of the parabola at $X = 0$, and the vertex of the parabola is positioned at $X = 0$ only if q_1 is zero.

To conclude, a good strategy for interpreting the (estimated) coefficients of a quadratic equation consists of two steps (see Figure 2). First, consider the coefficient q_2 of the quadratic term, which indicates whether the graph is linear (if $q_2 = 0$), U-shaped (if $q_2 > 0$), or an inverted U-shape (if $q_2 < 0$). Second, consider the coefficient q_1. Its interpretation depends on what was found in the first step: When $q_2 = 0$, q_1 reflects the direction and strength of the linear association of X and Z. When $q_2 \neq 0$, q_1 indicates whether the vertex is positioned at $X = 0$. If this is not the case, the position of the vertex can be computed as $X_{vertex} = \frac{-q_1}{2q_2}$.

[2]and also up so that the intersection point with the y-axis (i.e., the intercept q_0) remains the same.

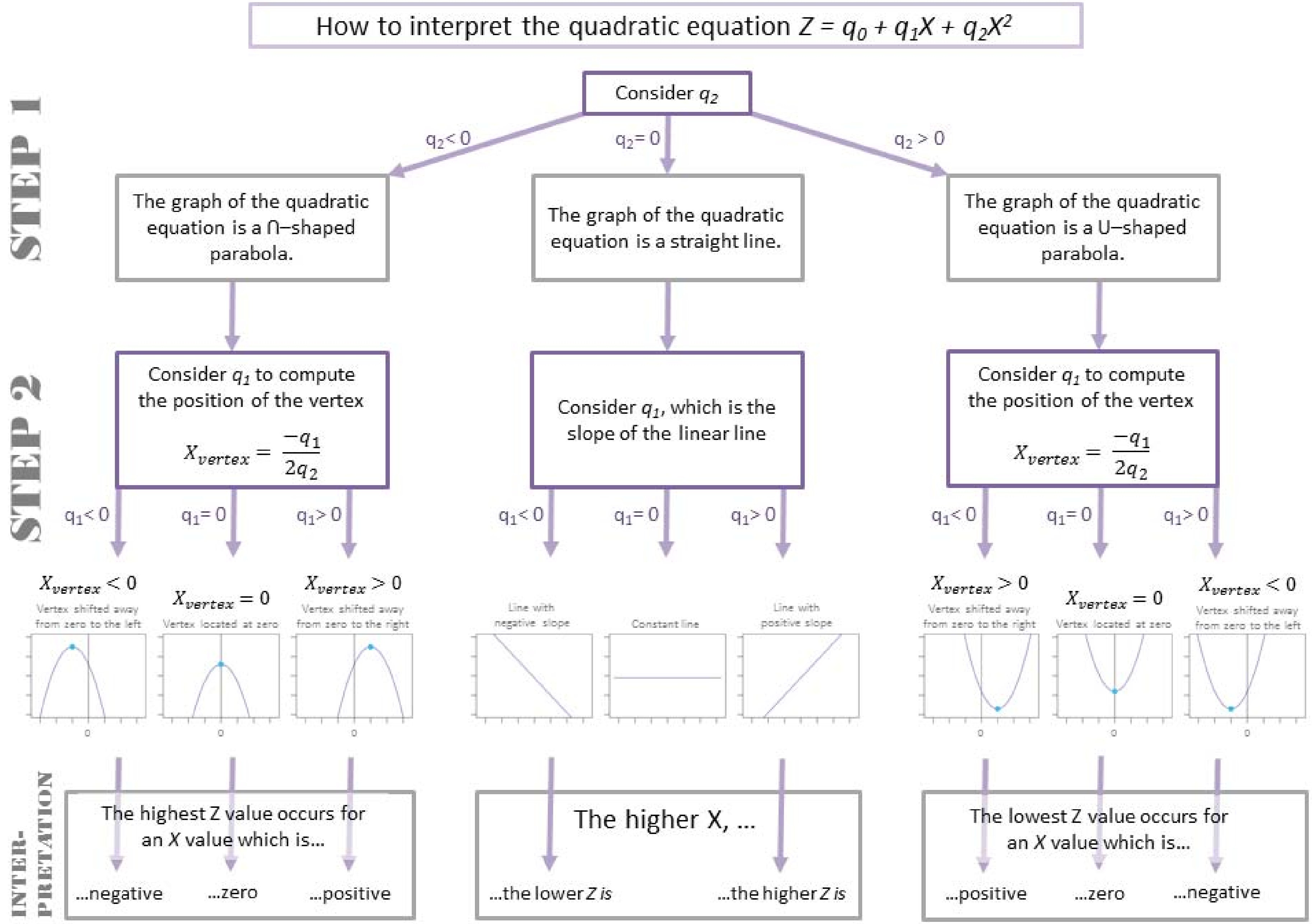


Figure 2. A recommended strategy for interpreting a quadratic equation.