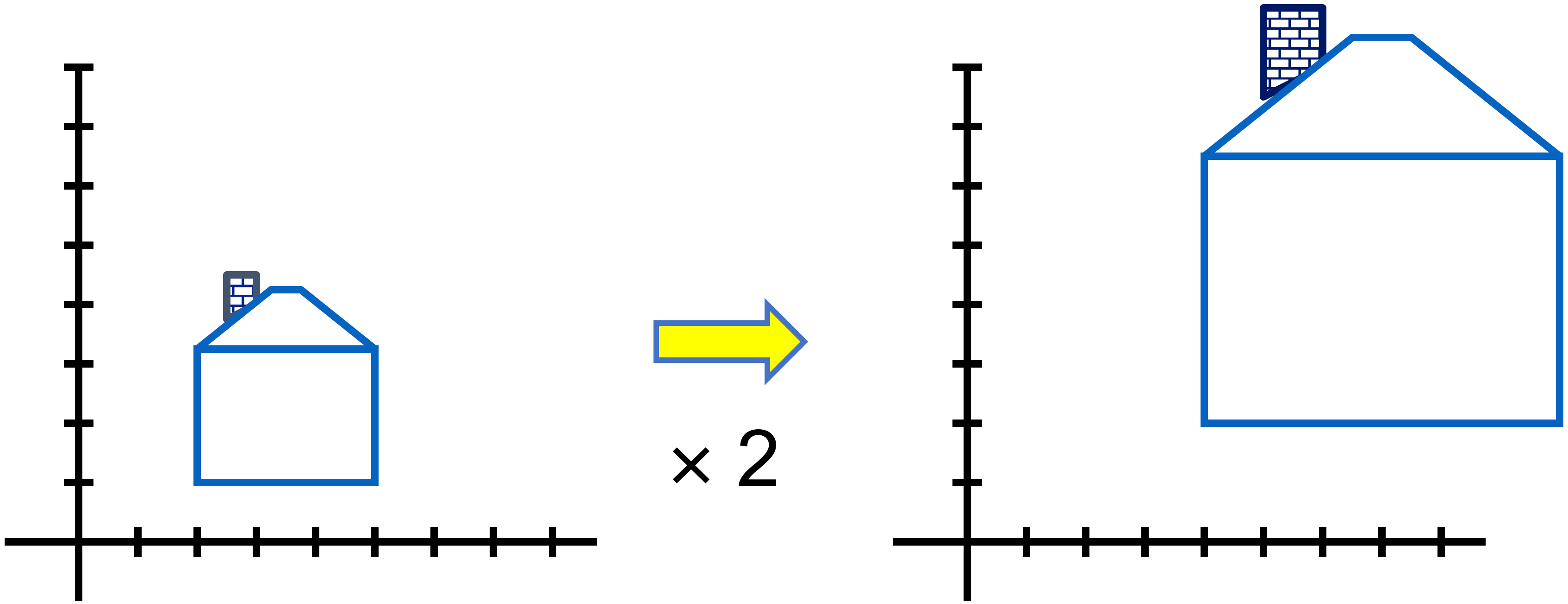
Eigenvalues and Eigenvectors

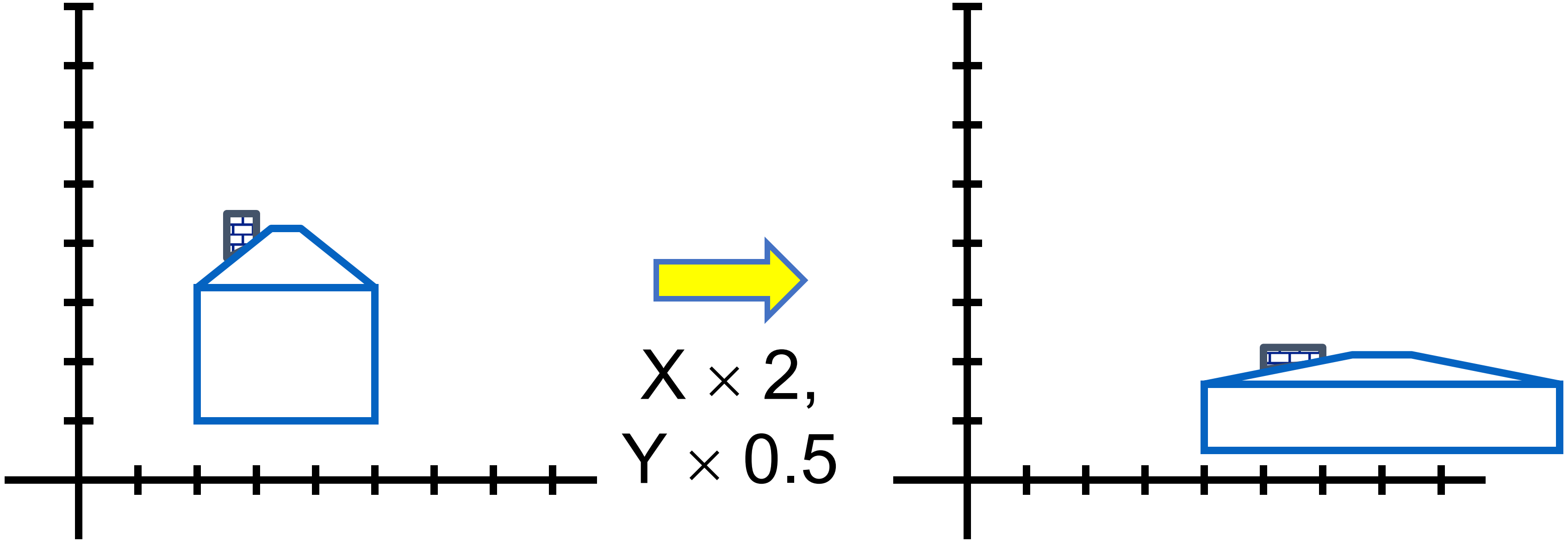
Scaling

- *Scaling* a coordinate means multiplying each of its components by a scalar
- *Uniform scaling* means this scalar is the same for all components:



Scaling

- *Non-uniform scaling*: different scalars per component:



Scaling

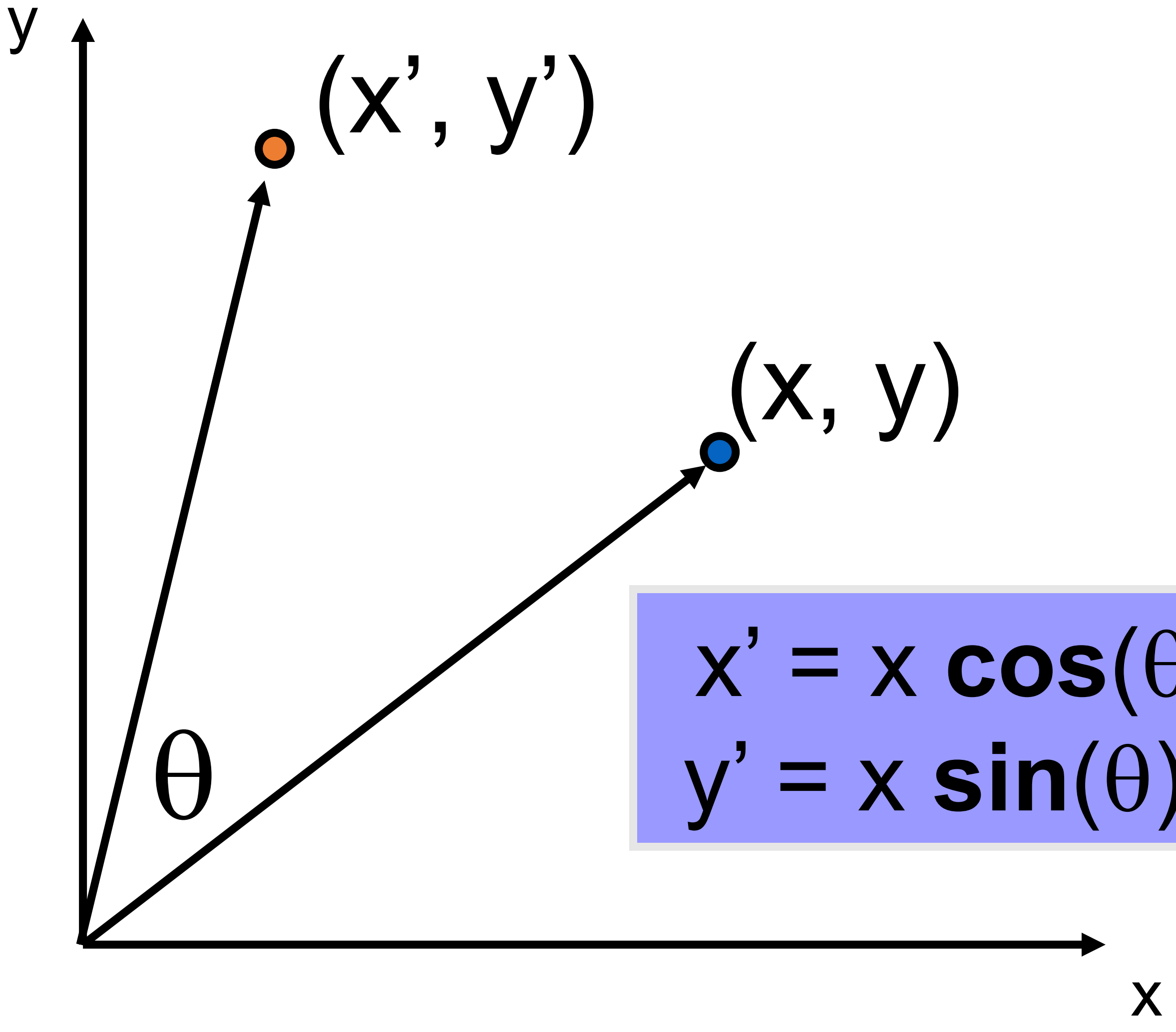
Scaling operation:

$$x' = s_x x$$
$$y' = s_y y$$

Or, in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}}_{\textit{scaling matrix } S} \begin{bmatrix} x \\ y \end{bmatrix}$$

2-D Rotation



$$x' = x\,\mathbf{cos}(\theta) - y\,\mathbf{sin}(\theta)$$
$$y' = x\,\mathbf{sin}(\theta) + y\,\mathbf{cos}(\theta)$$

2-D Rotation

In matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} x \\ y \end{bmatrix}$$

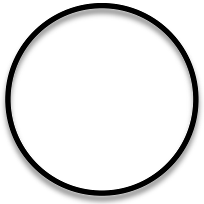
Even though sin(θ) and cos(θ) are nonlinear functions of θ,

- ***x' is a linear combination of x and y***
- ***y' is a linear combination of x and y***

What is the inverse transformation?

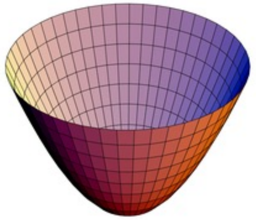
- Rotation by –θ
- For rotation matrices

$$\mathbf{R}^{-1} = \mathbf{R}^{T}$$



Equation of a circle

$$1 = x^2 + y^2$$



Equation of a 'bowl' (paraboloid)

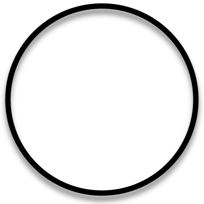
$$f(x, y) = x^2 + y^2$$

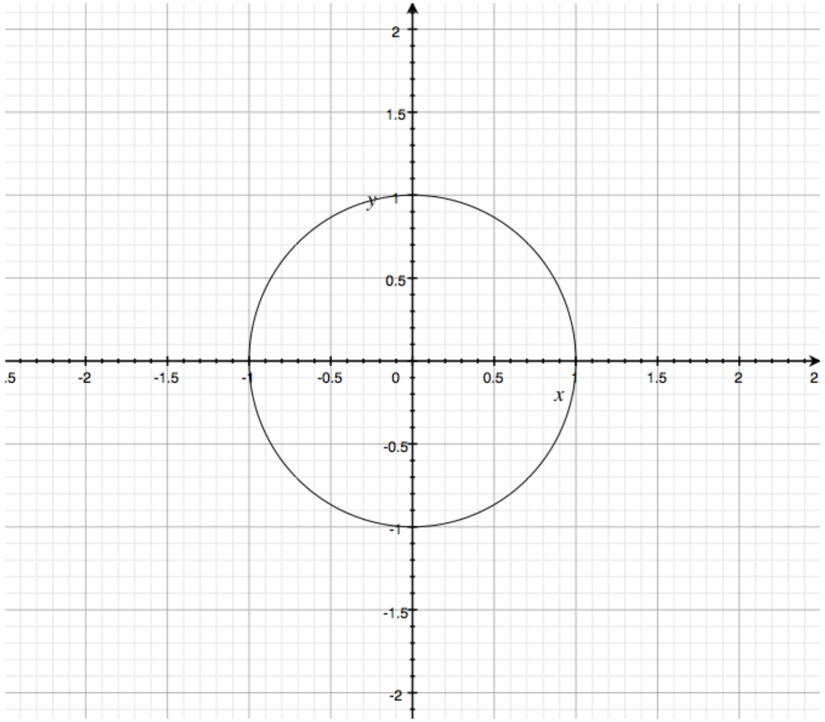
$$f(x, y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

If you slice the bowl at

$$f(x, y) = 1$$

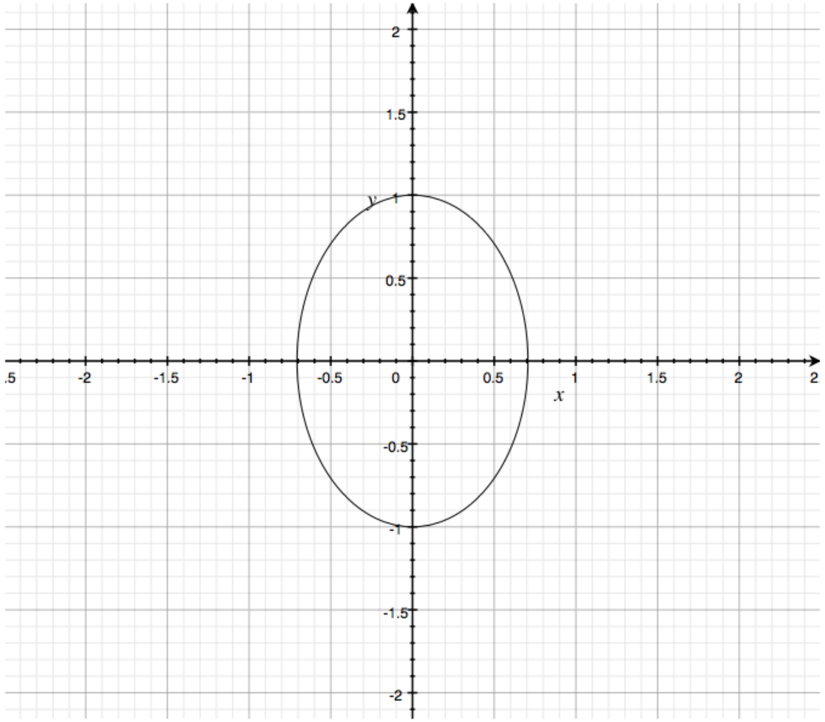
what do you get?





$$f(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

'sliced at 1'

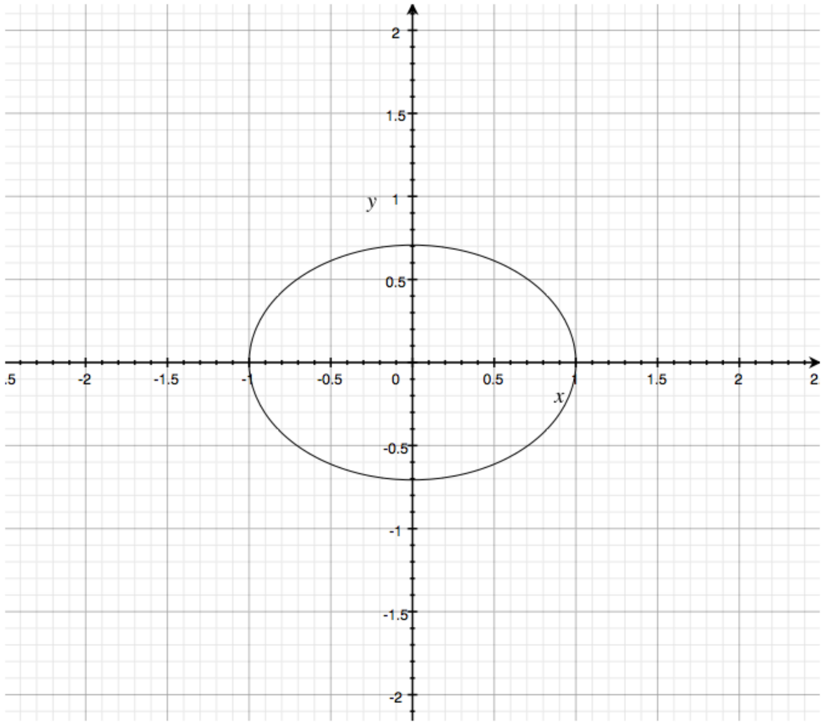


*What happens if you **increase** coefficient on **x**?*

$$f(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and slice at 1

decrease width in **x!**



*What happens if you **increase** coefficient on **y**?*

$$f(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and slice at 1

decrease width in **y**

$$f(x,y) = x^2 + y^2$$

can be written in matrix form like this…

$$f(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Result of Singular Value Decomposition (SVD)

eigenvectors

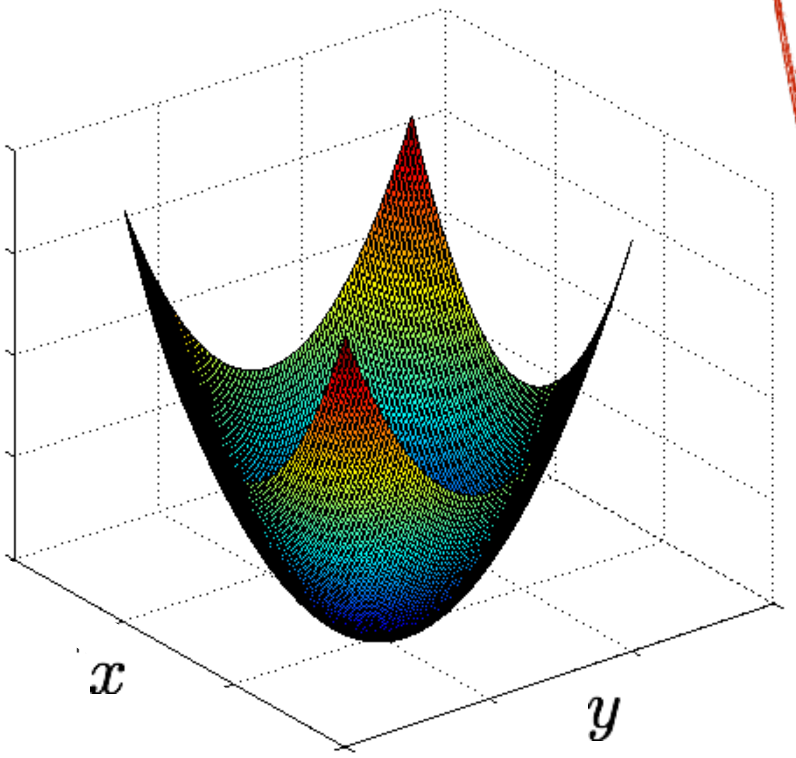
eigenvalues along diagonal

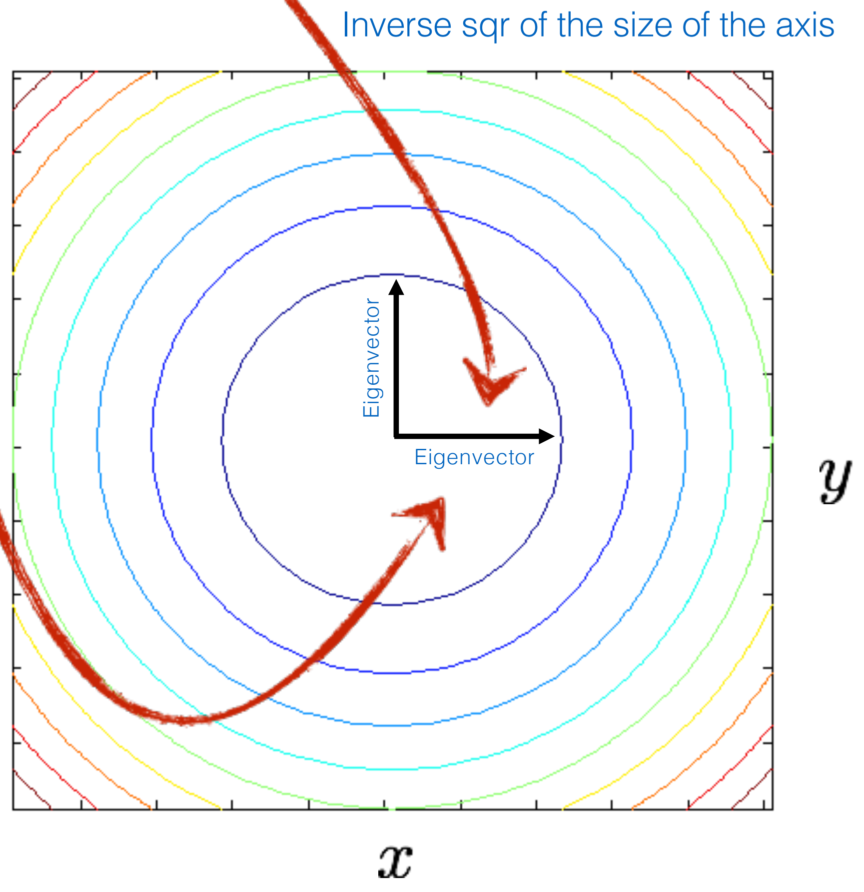
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{\top}$$

axis of the 'ellipse slice'

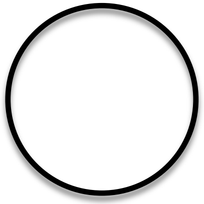
Inverse sqr of length of the quadratic along the axis

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \overset{\text{Eigenvectors}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \overset{\text{Eigenvalues}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T \ \text{Eigenvectors}$$



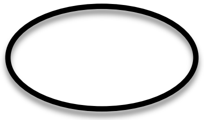


Recall:



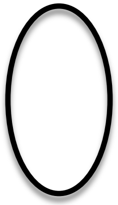
$$f(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

you can smash this bowl in the **y** direction



$$f(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

you can smash this bowl in the **x** direction

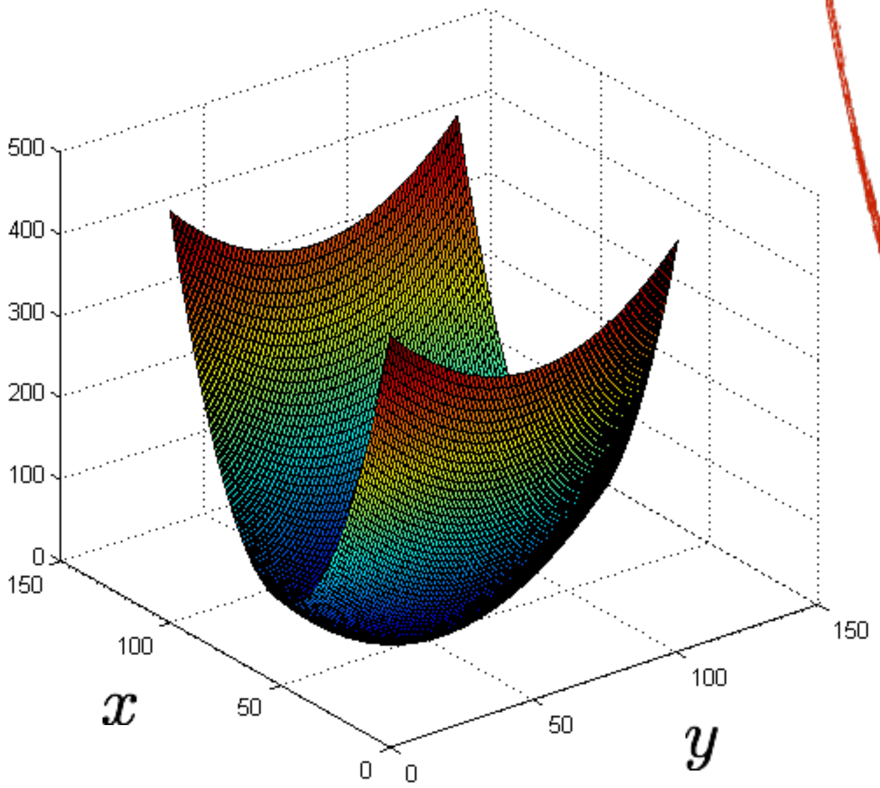


$$f(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

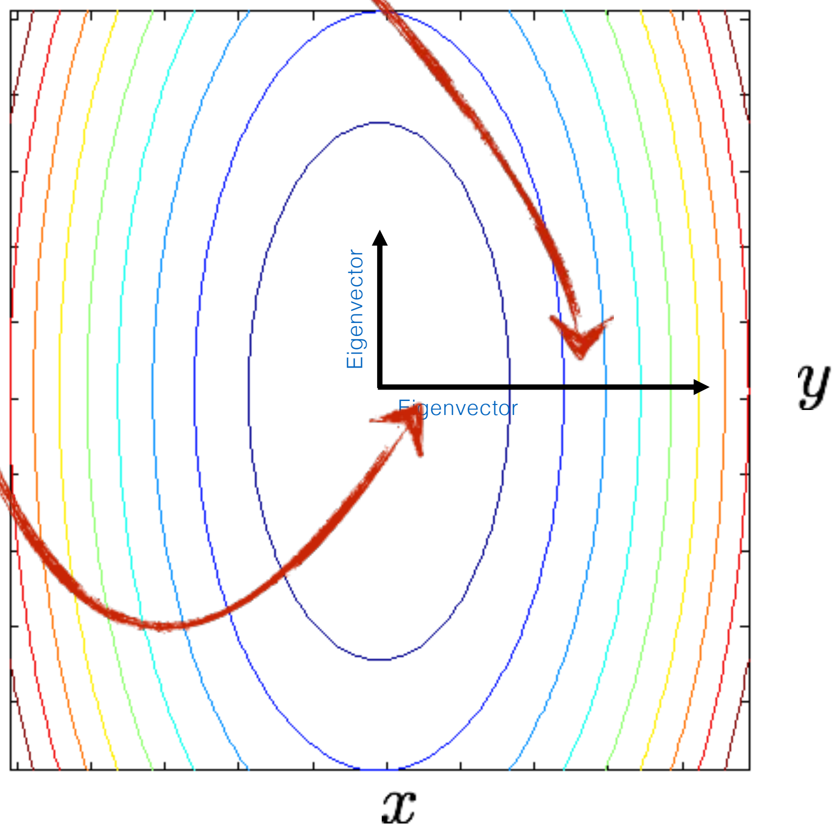
Eigenvalues

$$\mathbf{A} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T$$

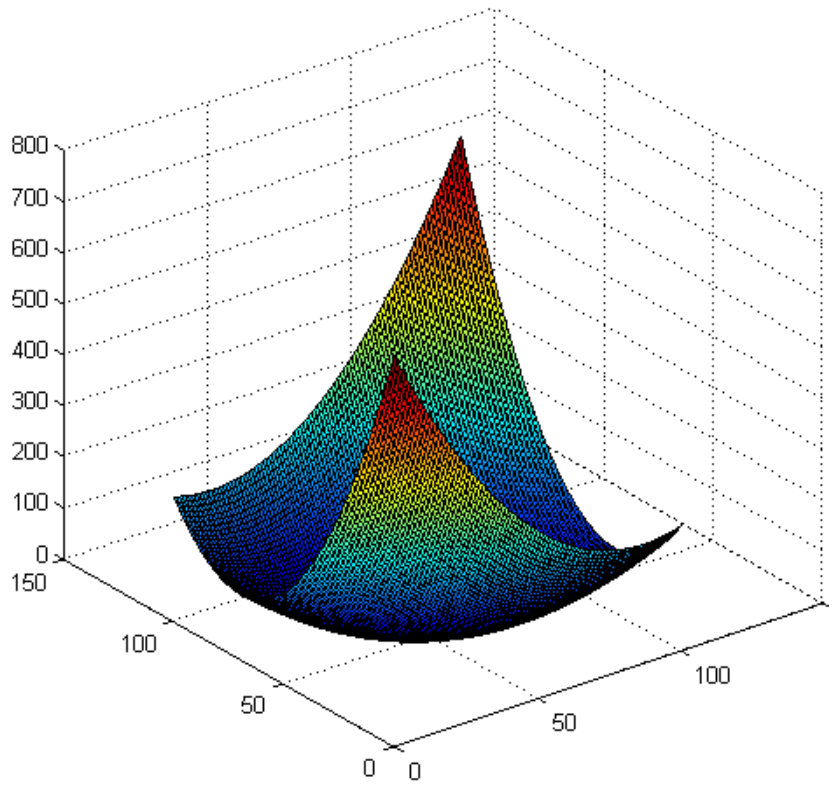
Eigenvectors Eigenvectors

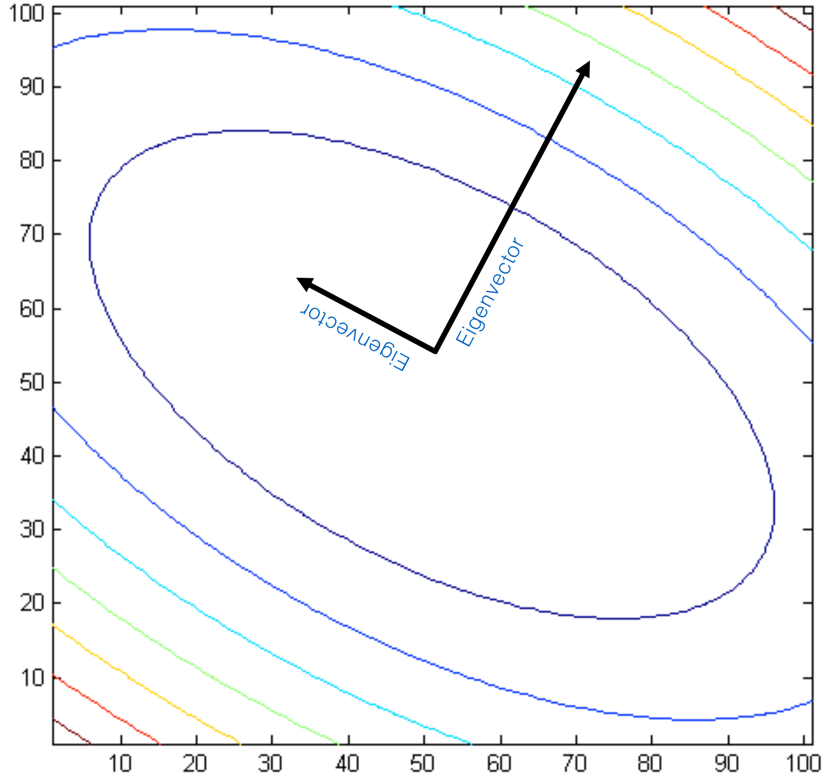


Inverse sqr of length of axis

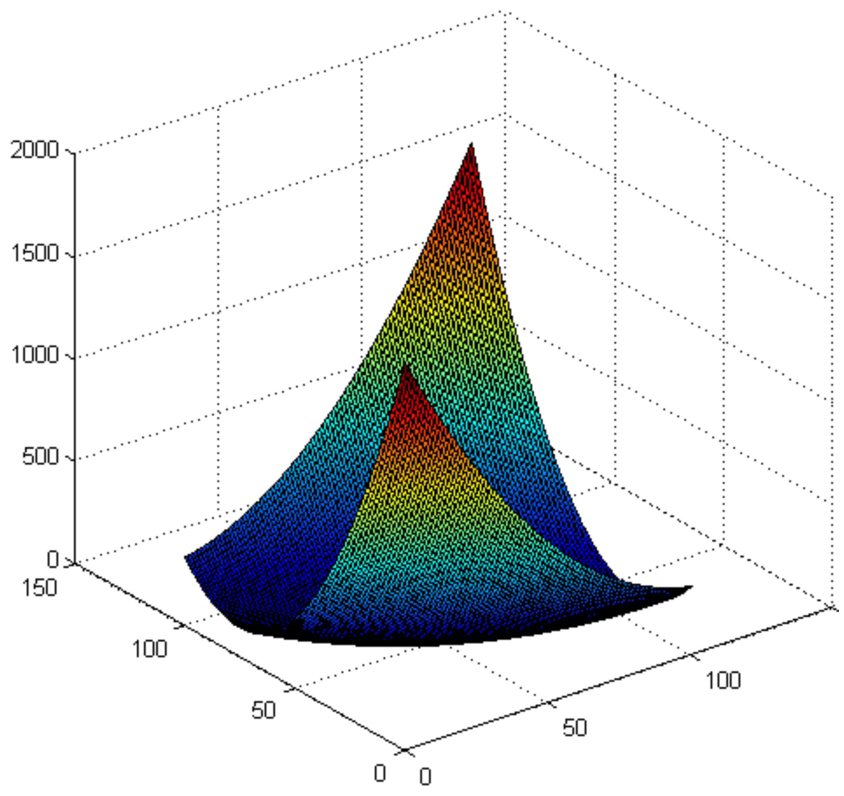


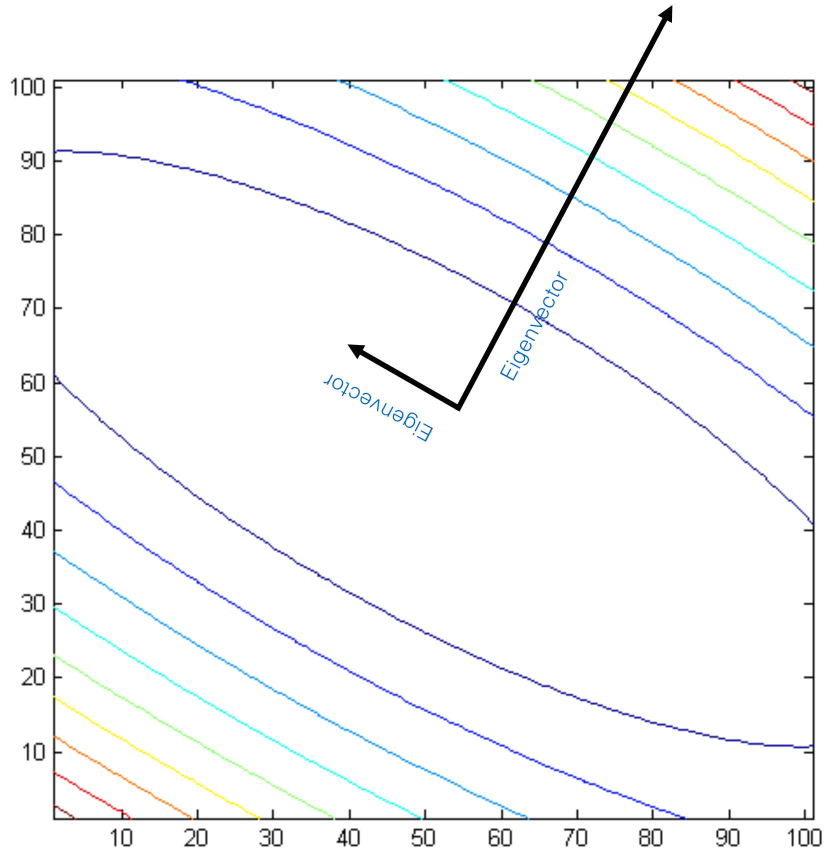
$$\mathbf{A} = \begin{bmatrix} 3.25 & 1.30 \\ 1.30 & 1.75 \end{bmatrix} = \underbrace{\begin{bmatrix} 0.50 & -0.87 \\ -0.87 & -0.50 \end{bmatrix}}_{\text{Eigenvectors}} \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}}^{\text{Eigenvalues}} \underbrace{\begin{bmatrix} 0.50 & -0.87 \\ -0.87 & -0.50 \end{bmatrix}^{T}}_{\text{Eigenvectors}}$$





$$\mathbf{A} = \begin{bmatrix} 7.75 & 3.90 \\ 3.90 & 3.25 \end{bmatrix} = \underbrace{\begin{bmatrix} 0.50 & -0.87 \\ -0.87 & -0.50 \end{bmatrix}}_{\text{Eigenvectors}} \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}}^{\text{Eigenvalues}} \underbrace{\begin{bmatrix} 0.50 & -0.87 \\ -0.87 & -0.50 \end{bmatrix}^T}_{\text{Eigenvectors}}$$





Linear Algebra: Recap

$$M = \sum_{x,y} w(x,y) \begin{bmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{bmatrix}$$

M is symmetric around the diagonal.
Symmetric matrices have orthogonal eigenvectors (i.e., a basis).

M is square. Square matrices are diagonalizable if some matrix R exists s.t. $M = R^{-1}AR$ where A has only diagonal entries and R represents a change of basis (in 2D, a rotation).

$$M = R^{-1} \underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{A} R$$

Interpreting the Second Moment Matrix

Consider a horizontal "slice" of $E(u, v)$:

This defines an ellipse.

$$[u \;\; v] \; M \begin{bmatrix} u \\ v \end{bmatrix} = \text{const}$$

Diagonalization of M:

$$M = R^{-1} \overbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}^{A} R$$

Form of standard ellipse:

Centered at origin and oriented along the axes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad [x \quad y] \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1$$

$$\lambda_{max} = \tfrac{1}{a^2} \implies a = (\lambda_{max})^{-1/2}$$

$$\lambda_{min} = \tfrac{1}{b^2} \implies b = (\lambda_{min})^{-1/2}$$

James Hays

Interpreting the Second Moment Matrix

Consider a horizontal "slice" of $E(u, v)$: $\quad [u \;\; v] \; M \begin{bmatrix} u \\ v \end{bmatrix} = \text{const}$

This defines an ellipse.

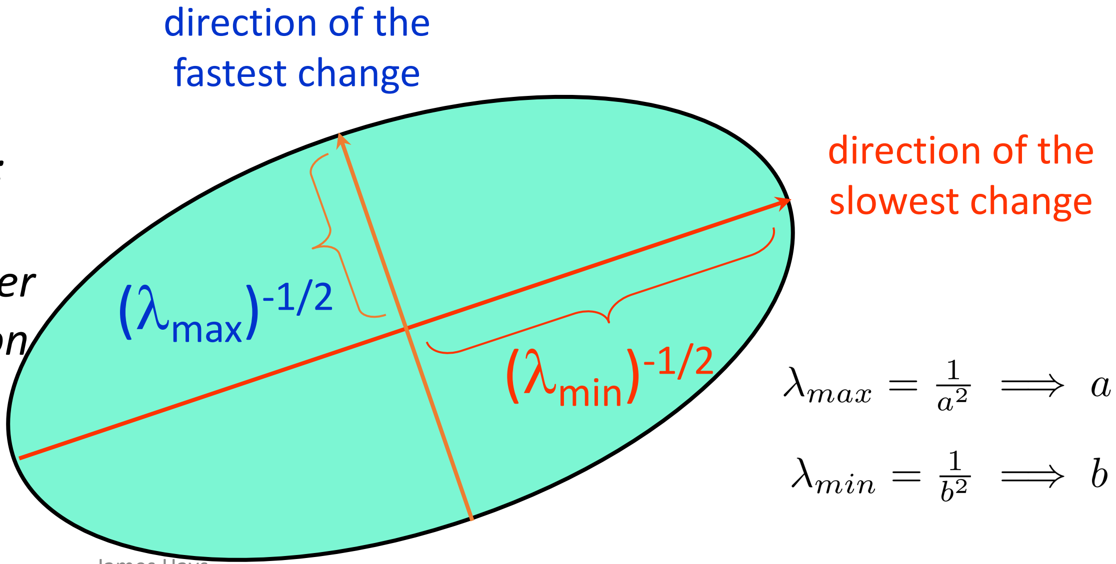
Diagonalization of M: $\quad M = R^{-1} \overset{A}{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}} R$

The axis lengths of the ellipse are determined by the eigenvalues,

and the orientation is determined by a rotation matrix R.

Note inverse relationship:

larger eigenvalue = smaller ellipse radii in visualization



$$\lambda_{max} = \frac{1}{a^2} \implies a = (\lambda_{max})^{-1/2}$$

$$\lambda_{min} = \frac{1}{b^2} \implies b = (\lambda_{min})^{-1/2}$$