ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-03

Differential equations

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Topics to be Covered

- DEFINITION & TYPES
- ORDER & DEGREE OF DE
- SOLUTION OF DE
- FORMATION OF DE
- WRONSKIAN & LD/LI SOLUTIONS
- METHODS OF SOLVING DE
- PARTIAL DIFFERENTIAL EQUATIONS

Methods of Solving D.E. :-

ODE
1) Observation Method
2) D.E. of first order & first degree
   a) Variable separable mtd.
   b) Homogenous D.E. mtd.
   c) Linear D.E. mtd.
   d) Bernoulli D.E. mtd.
3) Exact differential equations (Non exact D.E. $\rightarrow$ Exact D.E.)
4) L.D.E. of n^{th} order with
   - constant coefficients
   - variable coefficients

   (C.F. + P.I.)
5) Methods for solving non-linear D.E.

PDE
6) Methods for solving P.D.E.

Ex:- $\frac{dy}{dx}+\frac{y}{x}\cdot(\log y)=\frac{y(\log y)^2}{x}$

Divide by $y(\log y)^2$.

$$\frac{1}{y(\log y)^2}\cdot\frac{dy}{dx}+\frac{1}{\log y}\cdot\frac{1}{x}=\frac{1}{x}$$

$$\frac{1}{\log y}=z$$

$$-\frac{1}{(\log y)^2}\cdot\frac{1}{y}\frac{dy}{dx}=\frac{dz}{dx}$$

$$\frac{1}{y(\log y)^2}\cdot\frac{dy}{dx}=-\frac{dz}{dx}$$

$$-\frac{dz}{dx}+\frac{1}{x}\cdot z=\frac{1}{x}$$

$$\frac{dz}{dx}\underbrace{-\frac{1}{x}}_{P}z=\underbrace{-\frac{1}{x}}_{Q}$$

I.F. $=x^{-1}$

$$z\cdot x^{-1}=\int-\frac{1}{x}\cdot\frac{1}{x}\Rightarrow\boxed{\frac{1}{(\log y)x}=+\frac{1}{x}+c}$$

3) Exact differential equations

A D.E. is said to be exact if it can be obtained directly by differentiating its primitive soln. w/o any further operation of elimination or reduction.

$M\,dx + N\,dy = 0$ is exact iff

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$ (Exact D.E.)

Solution

$$\int_{y\ \text{constant}} M\,dx + \int N \underbrace{\left(\text{those terms which}\right)}_{\text{are free from } x} dy = c$$

3) Exact differential equations

Ex:- $\underbrace{(1+e^{y})\cos x}_{M}\,dx + \underbrace{\sin x\, e^{y}}\,dy = 0$

$\frac{\partial M}{\partial y} = \cos x \cdot e^{y}$

$\frac{\partial N}{\partial x} = \cos x \cdot e^{y}$

Exact

$(1+e^{y})\sin x$

$\int (1+e^{y}).\cos x\, dx + \int 0\, dy = 0$

$(1+e^{y})\sin x = c$

$\int \underbrace{x}_{N} dy + \underbrace{y}_{M} dx$

$= xy$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1$

$\therefore$ Exact

$\int y\, dx + \int 0\, dy$

$yx + c$

METHODS OF SOLVING DE

$$d(x^2\sin y) = \underbrace{x^2\cos y}\,dy + \underbrace{2x\sin y}\,dx$$

Equations which are reducible to exact form

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \qquad (\text{Non-exact}) \xrightarrow[\text{I.F.}]{\text{Multiply by}} (\text{Exact})$$

CaseI:- If eqn. is homogenous & non-exact, then

$$\boxed{\text{I.F.} = \frac{1}{Mx+Ny}}$$

$$\frac{1}{Mx+Ny} = \frac{1}{(y^3-3xy^2)x+(2x^2y-xy^2)y} = -\frac{1}{x^2 \cdot y^2}$$

Equations which are reducible to exact form

$$\frac{\partial M}{\partial y} = 3y^2 - 6xy$$
$$\frac{\partial N}{\partial x} = 4xy - y^2$$
Non-exact

Ex:- $\left((y^3 - 3xy^2)dx + (2x^2y - xy^2)dy = 0\right)$ I.F.

Homogenous & non-exact

After multiplying by I.F., eqn. is exact $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$\left(-\frac{y}{x^2} + \frac{3}{x}\right)dx + \left(-\frac{2}{y} + \frac{1}{x}\right)dy = 0$$

Exact D.E.

$$\int\left(-\frac{y}{x^2} + \frac{3}{x}\right)dx + \int -\frac{2}{y}\,dy = \int 0$$

$$\frac{y}{x} + 3\log x - 2\log y = c$$

$$\boxed{\frac{y}{x} + \log\frac{x^3}{y^2} = c}$$

METHODS OF SOLVING DE

CaseII:- If eqn. is of the form $y\,f(x,y)\,dx + x\,f(x,y)\,dy = c$ & non-exact,

then $\boxed{I.F. = \dfrac{1}{Mx - Ny}}$

METHODS OF SOLVING DE

Ex:- $(x^2y^3 + 2y)dx + (2x - 2x^3y^2)dy = 0$

$$y(x^2y^2+2)\,dx + x(2-2x^2y^2)\,dy = 0$$

$$\text{I.F.} = \frac{1}{Mx-Ny} = \frac{1}{(x^2y^3+2y)x - (2x-2x^3y^2)y} = \frac{1}{3x^3y^3}$$

After multiplying;

$$\int\left(\frac{1}{3x} + \frac{2}{3x^3y^2}\right)dx + \int\left(\cancel{\frac{2}{3x^2y^3}} - \frac{2}{3y}\right)dy = 0$$

$$\frac{1}{3}\log x + \frac{2}{3y^2}\left(\frac{x^{-3+1}}{-3+1}\right) - \frac{2}{3}\log y = c$$

METHODS OF SOLVING DE

CaseIII:- If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ alone, then I.F. $= e^{\int f(x)\,dx}$

CaseIV:- If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = f(y)$ alone, then I.F. $= e^{\int f(y)\,dy}$

Ex:- $\underbrace{(x^2+y^2+x)}_{M}\,dx+\underbrace{xy}_{N}\,dy=0$

Check; $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{xy}(2y-y)=\frac{1}{x}$ $f(x)$ alone

Case III ✓ I.F. $= e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} = x$

$\int (x^3+xy^2+x^2)\,dx + \int (x^2 \cancel{y})\,dy$ [Exact D.E.]

$\frac{x^4}{4}+\frac{x^2}{2}\cdot y^2+\frac{x^3}{3}=c$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Ex:- $(x+2y^3)\dfrac{dy}{dx} = y + 2x^3 y^2$

$$\underbrace{(y+2x^3y^2)}_{M}\,dx \underbrace{-(x+2y^3)}_{N}\,dy = 0$$

(3) $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{-(x+2y^3)}\left(1+4x^3y-(-1)\right) \neq f(x)$ alone Case III fails

(4) $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y+2x^3y^2}\left(-1-(1+4x^3y)\right) = \frac{-2\cancel{(1+2x^3y)}}{y\cancel{(1+2x^3y)}} = -\frac{2}{y} = f(y)$ alone

Case IV ✓

$$\text{I.F.} = e^{\int -\frac{2}{y}dy} = y^{-2}$$

$$\left(\frac{1}{y}+2x^3\right)dx - \cancel{\frac{x}{y^2}} - 2y\,dy = 0$$

$$x + \frac{2x^4}{4} - \frac{2y^2}{2} + c = 0$$

Case V :- If eqn. is of the form

$$x^a y^b (py\,dx + qx\,dy) + x^c y^d (ry\,dx + sx\,dy)$$

then I.F. $= x^h y^K$ after multiplying; $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, find h and K.

Ex :-

$$xy^3(y\,dx + 2x\,dy) + (3y\,dx + 5x\,dy) = 0$$

$$(xy^4\,dx + 2x^2y^3\,dy) + (3y\,dx + 5x\,dy) = 0$$

$$\left\{\underbrace{(xy^4 + 3y)}\,dx + \underbrace{(2x^2y^3 + 5x)}\,dy = 0\right\} x^h y^K$$

$$\partial M/\partial y \neq \partial N/\partial x$$

$$\underbrace{\left(x^{h+1} \cdot y^{K+4} + 3x^h \cdot y^{K+1}\right)}_{M} dx + \underbrace{\left(2x^{h+2} \cdot y^{K+3} + 5x^{h+1} \cdot y^K\right)}_{N} dy = 0$$

Exact

Now eqn. is exact; $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$(K+4)\underbrace{y^{K+3}.x^{h+1}} + 3(K+1)\underbrace{y^{K}.x^{h}} = 2(h+2)\underbrace{x^{h+1}y^{K+3}} + 5(h+1)\underbrace{x^{h}.y^{K}}$$

$$\left.\begin{array}{ll} ① & 2(h+2) = K+4 \\ ② & 5(h+1) = 3(K+1) \end{array}\right\} \text{On comparing}$$

On solving $h = 2,\ K = 4$

Exact $\int (x^3 y^8 + 3x^2 y^5)\,dx + \int (\cancel{2x^4 y^7} + \cancel{5x^3 y^4})\,dy = 0$

$$\frac{x^4}{4}.y^8 + 3\frac{x^3}{3}.y^5 = C$$

METHODS OF SOLVING DE

4) L.D.E of nth order with

- Constants coefficients
- Variable coefficients

METHODS OF SOLVING DE

4) Linear DE of Higher order with constant coefficients

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots \dots \dots \; a_n y = Q$$

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots \dots \; a_n) y = Q$$

$$\boxed{f(D)\, y = Q} \rightarrow n^{th} \text{ order L.D.E}$$

$$y = C.F. + P.I.$$

Complete soln. of above eqn.

$\frac{d}{dx} \rightarrow D$

$\frac{d^2}{dx^2} \rightarrow D^2$

$\frac{d^3}{dx^3} \rightarrow D^3$

$\int \rightarrow \frac{1}{D}$

$\iint \rightarrow \frac{1}{D^2}$

C.F. $\rightarrow f(D)$

P.I. $\rightarrow f(D), Q$

If $Q = 0$; P.I. $= 0$; $y =$ C.F.

If $Q \neq 0$; $y =$ C.F. + P.I.

How to find complementary function (C.F) :-

Replace $y \to 1$ & $Q \to 0$

Replace $D \to m$

$\boxed{f(m) = 0}$ → Auxiliary eqn. (It is algaebric eqn. of degree n.)

Solve this eqn. & find roots

Q.

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

$$(D^2 + 5D + 6)y = 0$$

$$(m^2 + 5m + 6)1 = 0 \rightarrow \text{Auxiliary eqn.}$$

$$(m+3)(m+2) = 0$$

$$m = -2, -3 \; \} \text{ Real \& distinct.}$$

$$y = C.F. = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$= C_1 e^{-2x} + C_2 e^{-3x}$$

Real equal

If $m = 5, 5$; $y = (C_1 + C_2 x) e^{5x}$

METHODS OF SOLVING DE

Complementary function corresponding to different nature of roots

1) Roots are real & distinct
$\rightarrow m_1, m_2, m_3$ — $y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$

2) Roots are real & equal
$\rightarrow m, m$ — $y = (C_1 + C_2 x) e^{mx}$
$\rightarrow m, m, m$ — $y = (C_1 + C_2 x + C_3 x^2) e^{mx}$
$\rightarrow m, m, m, m_1, m_2$ — $y = (C_1 + C_2 x + C_3 x^2) e^{mx} + C_4 e^{m_1 x} + C_5 e^{m_2 x}$

3) Roots are imaginary & distinct.
$m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ — $y = e^{\alpha x}[C_1 \cos \beta x + C_2 \sin \beta x]$

4) Roots are imaginary & equal.
$\alpha \pm i\beta \quad \alpha \pm i\beta$ — $y = e^{\alpha x}[(C_1 + C_2 x)\cos \beta x + (C_3 + C_4 x)\sin \beta x]$

5) Roots are irrational & distinct.
$m_1 = \alpha + \sqrt{\beta}$; $m_2 = \alpha - \sqrt{\beta}$ — $y = e^{\alpha x}[C_1 \cos \sqrt{\beta}\, x + C_2 \sin \sqrt{\beta}\, x]$

6) Roots are irrational & equal.
$\alpha \pm \sqrt{\beta} \quad \alpha \pm \sqrt{\beta}$ — $y = e^{\alpha x}[(C_1 + C_2 x)\cos \sqrt{\beta}\, x + (C_3 + C_4 x)\sin \sqrt{\beta}\, x]$

METHODS OF SOLVING DE

Ex:- $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$

$(D^3 - 3D^2 + 3D - 1)y = 0$

$m^3 - 3m^2 + 3m - 1 = 0$

$(m-1)^3 = 0$

$m = 1, 1, 1$ (Roots → Real & equal)

$y = (c_1 + c_2 x + c_3 x^2)e^{1x}$

Ex:-

$m = 2, 2, 3, 5$

$y = (c_1 + c_2 x)e^{2x} + c_3 e^{3x} + c_4 e^{5x}$

Ex:- $(D^4 - n^4)y = 0$

$$m^4 - n^4 = 0$$

$$(m^2 + n^2)(m^2 - n^2) = 0$$

$$(m + ni)(m - ni)(m + n)(m - n) = 0$$

$$m = +n, -n, 0 \pm ni$$

$$y = c_1 e^{nx} + c_2 e^{-nx} + e^{0x}[c_3 \cos nx + c_4 \sin nx]$$

METHODS OF SOLVING DE

Ex:- $(D^4 - 4D^3 - 8D^2 - 8D + 4)y = 0$

$$m^4 - 4m^3 - 8m^2 - 8m + 4 = 0$$

$$(m^2 - 2m + 2)^2 = 0$$

$$m = 1 \pm i, \; 1 \pm i$$ Roots are imaginary & equal

$$y = e^{1x}[(c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x]$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ $ax^2 + bx + c = 0$

$$\frac{2 \pm \sqrt{4 - 4.1.2}}{2.1} = \frac{2 \pm 2i}{2} = 1 \pm i$$

METHODS OF SOLVING DE

Ex:- $(D^2 - 4D + 1)y = 0$

$$m^2 - 4m + 1 = 0$$

$$m = \frac{4 \pm \sqrt{16 - 4}}{2 \times 1}$$

$$= \frac{2 \pm \sqrt{12}}{2}$$

$$= \underset{\alpha}{\underset{\downarrow}{2}} \pm \underset{\sqrt{\beta}}{\underset{\downarrow}{\sqrt{3}}}$$

$$y = e^{2x}\left[c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right]$$

Thank you

GW Soldiers !