ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-7

Linear Algebra

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Topics to be Covered

- System of Linear Equations
- Augmented Matrix
- Non-Homogenous Linear Equations
- Solution of Non-Homogenous Linear Equations
- Homogenous Linear Equations
- Solution of Homogenous Linear Equations

VECTOR SPACE

Ordered pair :- (a, b) is an ordered pair in which position of a and b are fixed.

Ordered triplet :- (a, b, c) is ordered triplet in which positions of a, b and c are fixed.

Ordered n-tuple :- Set of ordered n numbers in which their positions are fixed.

$1\hat{i} + 2\hat{j}$

$\begin{bmatrix} 1 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\hat{i} + 5\hat{j} - 6\hat{k}$

$\begin{bmatrix} 1 & 5 & -6 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 5 \\ -6 \end{bmatrix}$

$x_1, x_2, x_3 \ldots\ldots x_n$

$\begin{bmatrix} 1 & 2 & -5 & \ldots\ldots & 3 \end{bmatrix}$

LINEAR DEPENDENCE AND INDEPENDENCE OF VECTORS

→ If $\vec{X_1}, \vec{X_2}, \vec{X_3} \dots \vec{X_n}$ are n-vectors such that they all can be expressed as a linear combination where $K_1, K_2, K_3 \dots$ are all not 0 simultaneously.

$$K_1 X_1 + K_2 X_2 + K_3 X_3 + \dots K_n X_n = 0$$

(Relation exist between them)

These set of vectors $X_1\ X_2\ X_3 \dots$ are L.D.

→ If $\vec{X_1}, \vec{X_2} \dots \vec{X_n}$ are n-vectors such that they can be expressed as $\overset{0}{\cancel{K_1}} X_1 + \overset{0}{\cancel{K_2}} X_2 + \overset{0}{\cancel{K_3}} X_3 + \dots \overset{0}{\cancel{K_n}} X_n = 0$ where $K_1, K_2, K_3 \dots K_n$ are all 0 simultaneously.

(No relation exist between them)

These set of vectors $X_1, X_2, X_3 \dots$ are L.I.

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 5 & 10 & 15 \end{bmatrix} \quad \begin{bmatrix} 1 & 5 & 0 \\ 2 & 10 & -1 \\ 4 & 20 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & -1 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 3 & -3 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 & 5 \\ -7 & 1 & -6 \\ 1 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 4 & 2 & 3 \\ 1 & 7 & 5 \\ 5 & 9 & 8 \end{bmatrix}$$

$R_2 = 3R_1$, $R_3 = 5R_1$ | $5C_1 = C_2$ | $R_1 = R_2$ | $C_1 = C_3$ | $C_1 + C_2 = C_3$ | $R_1 + R_2 = R_3$

No. of relations	2	1	1	1	1	1
Nullity	2	1	1	1	1	1
Rank	1	2	2	2	2	2

Order of square matrix = Rank + Nullity

Rank → LI rows/columns

Nullity → LD rows/columns

2- Dimension Space:-

Ex:- $\vec{X_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad \vec{X_2} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

$\vec{X_2} = K\vec{X_1}$ $\quad \begin{bmatrix} 5 \\ 10 \end{bmatrix} = K\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\therefore K = 5 \Rightarrow \vec{X_2} = 5\vec{X_1}$

$\{5X_1 - X_2 = 0\}$

Ex:- $\vec{X_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad \vec{X_2} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$

$\vec{X_2} = K\vec{X_1}$ $\quad \begin{bmatrix} -3 \\ 7 \end{bmatrix} = K\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\therefore K = -3$ and 3.5

$\therefore \vec{X_1}$ and $\vec{X_2}$ have no relation $\because$ K does not have unique value.

NOTE:-

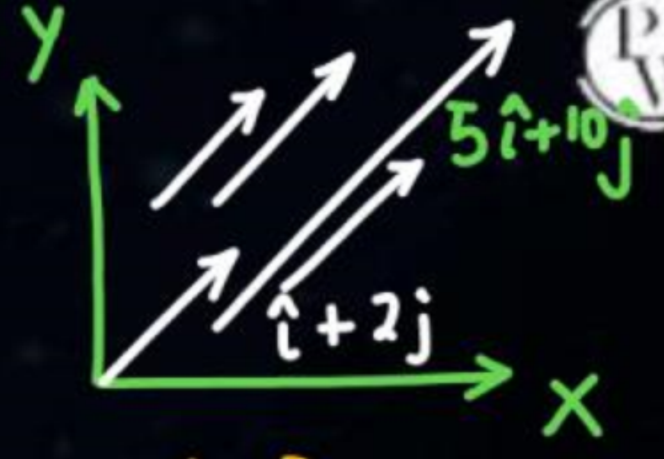
- If $\vec{X_2} = K\vec{X_1}$

$\Rightarrow \vec{X_1}$ and $\vec{X_2}$ are co-linear

$\Rightarrow \vec{X_1}$ and $\vec{X_2}$ are L. D.

$\Rightarrow A = [X_1 \; X_2] = \begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix}$

$\rho(A) = 1 <$ No. of vectors (2)



1-D

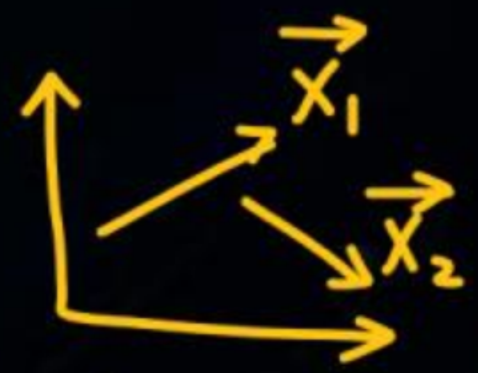
- If $\vec{X_2} \neq K\vec{X_1}$

$\Rightarrow \vec{X_1}$ and $\vec{X_2}$ are non-colinear

$\Rightarrow \vec{X_1}$ and $\vec{X_2}$ are L. I.

$\Rightarrow A = [X_1 \; X_2] = \begin{bmatrix} 1 & -3 \\ 2 & 7 \end{bmatrix}$

$\rho(A) = 2 =$ No. of vectors



2-D

NOTE:- ① If $\rho(A) <$ No. of vectors

$\Rightarrow$ Then the set of vectors are L.D.

$\Rightarrow |A| = 0$

$A \rightarrow$ Set of vectors

$[X_1 \; X_2 \; X_3 \ldots]$

② If $\rho(A) =$ No. of vectors

$\Rightarrow$ Then the set of vectors are L.I.

$\Rightarrow |A| \neq 0$

NOTE:- Dimension of each vector $<$ No. of vectors

$\Rightarrow$ Then the vectors are L.D.

Dimension of each vector $\geq$ No. of vectors

$\Rightarrow$ Then the vectors may or may not be L.I.

$\vec{X_1} \rightarrow$ 1-D

$\vec{X_1}, \vec{X_2}$ — $\rightarrow \vec{X_1} = K\vec{X_2}$ (1-D) L.D.

$\rightarrow \vec{X_1} \neq K\vec{X_2}$ (2-D) L.I.

X_1, X_2, X_3 — $\rightarrow \rho(A) = 1$ (1-D) (any 2, 3 are L.D.)

$\rightarrow \rho(A) = 2$ (2-D) (all 3 are L.D.)

$\rightarrow \rho(A) = 3$ (3-D) all 3 are L.I.

3-D space :-

Ex:- Check set of vectors are L D or L I :-

i) $\vec{X_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\vec{X_2} = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}$ $\vec{X_3} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$A = [X_1 \ X_2 \ X_3]$

$A = \begin{bmatrix} 1 & 5 & 1 \\ -1 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

$|A| = 24 \neq 0 \ \therefore \rho(A) = 3$

$\therefore \rho(A) = \text{No. of vectors} = 3 \ \{\because \text{L.I.}\}$

3-D span

$K_1 X_1 + K_2 X_2 + K_3 X_3 = 0$

Ex:- ii) $\vec{X_1} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ $\vec{X_2} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ $\vec{X_3} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ $\vec{X_4} = \begin{bmatrix} -3 \\ 7 \\ 2 \end{bmatrix}$

$$A = [X_1 \ X_2 \ X_3 \ X_4] = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\therefore \rho(A) < \text{No. of vectors}$

$3 < 4$

$\Rightarrow$ Set of vectors are L.D.

$$K_1 X_1 + K_2 X_2 + K_3 X_3 + K_4 X_4 = 0$$

$$K_1\begin{bmatrix}1\\2\\4\end{bmatrix} + K_2\begin{bmatrix}2\\-1\\3\end{bmatrix} + K_3\begin{bmatrix}0\\1\\2\end{bmatrix} + K_4\begin{bmatrix}-3\\7\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

3-D span

Undetermined system

$$K_1 + 2K_2 - 3K_4 = 0$$
$$2K_1 - K_2 + K_3 + 7K_4 = 0$$
$$4K_1 + 3K_2 + 2K_3 + 2K_4 = 0$$

$\neq 0$

$$K_1 X_1 + K_2 X_2 + K_3 X_3 + K_4 X_4 = 0$$

X_1, X_2, X_3, X_4 are L.D.

$$\Downarrow$$

$$K_1 + 2K_2 - 3K_4 = 0$$
$$-5K_2 + K_3 + 13K_4 = 0$$
$$K_3 + K_4 = 0$$

Let $K_4 = t$

$\Rightarrow \boxed{K_3 = -t}$

$-5K_2 - t + 13t = 0 \Rightarrow \boxed{K_2 = \frac{12}{5}t}$

$\Rightarrow K_1 + 2\left(\frac{12}{5}t\right) - 3t = 0 \Rightarrow \boxed{K_1 = -\frac{9}{5}t}$

$K_1 = -9/5\, t$
$K_2 = 12/5\, t$
$K_3 = -t$
$K_4 = t$

$-\frac{9}{5}X_1 + \frac{12}{5}X_2 - X_3 + X_4 = 0 \Rightarrow \boxed{9X_1 - 12X_2 + 5X_3 - 5X_4 = 0}$

v) $\vec{X_1} = \begin{bmatrix} 1 \\ 5 \\ -7 \end{bmatrix}$ $\vec{X_2} = \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}$ $\vec{X_3} = \begin{bmatrix} 5 \\ 17 \\ -1 \end{bmatrix}$

2-D span

$A = \begin{bmatrix} 1 & 2 & 5 \\ 5 & 6 & 17 \\ -7 & 3 & -1 \end{bmatrix}$ $|A|_{3\times3} = 0 \therefore \rho(A) < 3$

$|A|_{2\times2} \neq 0 \therefore \rho(A) = 2$

$\rho(A) = 2 <$ No. of vectors (3)

$\therefore X_1, X_2$ and X_3 are L.D.

$$\vec{X_1} + 2\vec{X_2} - \vec{X_3} = 0$$

vi) $A = [a_1, a_2, a_3 \ldots\ldots a_m]$ m vectors

$a \rightarrow$ ordered n-tuple $(n < m)$

$\Rightarrow$ Then the set of vectors are L.D.

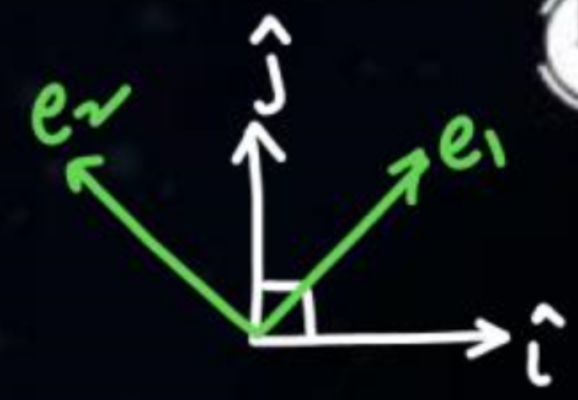
Max. $\rho(A) \leq n$

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_m \\ b_1 & b_2 & & \vdots \\ \vdots & \vdots & & \vdots \\ n & n & & n \end{bmatrix}_{n\times m}$$

$\rightarrow |A|_{n\times n} \neq 0$; $\rho(A) = n$ LI vectors

$\rightarrow |A|_{n\times n} = 0$; $\rho(A) < n$

[BASIS]



Vector Space :- The set of vectors span a region

Ex :- R^1, R^2, R^3

OR Space spanned by set of vectors.

Basis :- The set of vectors form a basis (B) if

i) The set of vectors are LI.

ii) These set of vectors spans vector space V.

(any vector V can be expressed as a linear combination of vectors in B)

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\rho(A) = 3$

$\rho(A) = n = 3$

i) These set of vectors are LI.

$\vec{X} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$

$$\vec{X} = K_1 e_1 + K_2 e_2 + K_3 e_3$$

$$\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = K_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + K_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + K_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$5 = K_1$$
$$-1 = K_2$$
$$2 = K_3$$

$$\vec{X} = 5e_1 - e_2 + 2e_3$$

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Will e_1, e_2 form basis :-

i) e_1 & e_2 are L.I.

ii) ✓

Ex:- $e_1 = (1\ 0\ 2)$ $e_2 = (0\ 1\ 0)$ & $e_3 = (-2\ 0\ 1)$ forms an orthogonal basis of R^3 then $u = (4, 3, -3)$ in terms of e_1, e_2 and e_3

$$B = [e_1\ e_2\ e_3] = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$|A|_{3\times3} \neq 0$

$\therefore \rho(A) = n = 3$

$\therefore e_1, e_2$ and e_3 are L.I.

$$\vec{u} = K_1 e_1 + K_2 e_2 + K_3 e_3$$

$$\begin{bmatrix} 4 \\ 3 \\ -3 \end{bmatrix} = K_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + K_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + K_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$4 = K_1 - 2K_3$$

$$3 = K_2$$

$$-3 = 2K_1 + K_3$$

On solving $K_1 = -\frac{2}{5}, K_2 = 3, K_3 = -\frac{11}{5}$

$$\vec{u} = -\frac{2}{5} e_1 + 3 e_2 - \frac{11}{5} e_3$$

$e_1 \perp e_2 \perp e_3$

$\Rightarrow e_1, e_2, e_3$ LI and forms a basis.

$\Rightarrow e_1, e_2, e_3$ spans 3-D.

SYSTEM OF LINEAR EQUATIONS

Homogenous system

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0$$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = O$$

Non-Homogenous system

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$AX = B$$

$A \rightarrow$ Coefficient matrix

$X \rightarrow$ Matrix of unknowns

$B \rightarrow$ Constant matrix

Homogenous Linear Equations

① $\rho(A) =$ No. of unknowns

$\Rightarrow$ Consistent unique solution

$\Rightarrow$ Trivial soln. (Zero soln.)

$\Rightarrow$ In case ①, no LI soln

($\because n - r = 0$)

② $\rho(A) <$ No. of unknowns

$\Rightarrow$ Consistent infinte solution

$\Rightarrow$ Non-trivial (non-zero soln.)

$\Rightarrow$ In case ②, $(n-r)$ LI soln

$\because n - r \neq 0$

$\rightarrow$ 1 (K) LI soln

$\rightarrow$ 2 (K_1, K_2) "

$\rightarrow$ 3 (K_1, K_2, K_3) "

NOTE :-

- Homogenous system is always consistent.
- Nullity (No. of LI solns.) = Order (No. of unknowns) − Rank
- Trivial soln $\rightarrow$ Zero soln ($x = y = z = 0$)
- Non-trivial soln $\rightarrow$ Non-zero soln
- No. of equations < No. of unknowns $\Rightarrow$ Infinite solution (Under-determined system)

2 variables homogenous system :-

① $$\begin{aligned} x + y &= 0 \\ x - 2y &= 0 \end{aligned}$$

Soln :- $x = 0, y = 0$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

$\rho(A) = 2 =$ No. of unknowns

$\Rightarrow$ Consistent unique solution

$\Rightarrow$ Trivial solution (Zero solution)

② $$\begin{aligned} x + 2y &= 0 \\ 5x + 10y &= 0 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix} \quad |A|_{2\times 2} = 0$$

$\rho(A) = 1 <$ No. of unknowns (2)

$\Rightarrow$ Consistent infinite solution.

$n - r = 2 - 1 = 1$ (one arbitrary variable will be assigned).

Let $y = K$ then $x = -2K$ where K can take any value.

$\Rightarrow$ Non trivial soln (Non-zero soln)

3 variable system :-

① $$\begin{aligned} x+2y-2z&=0\\ -x+3y\quad &=0\\ -2y+z&=0 \end{aligned}$$

3 planes cut at same point.

$$A=\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix} \qquad |A|_{3\times3}\neq 0$$

$\therefore\ \rho(A)=3=$ No. of unknowns (3)

$\Rightarrow$ Consistent unique solution

$\Rightarrow$ Trivial soln. (Zero solution)

$x=0,\ y=0,\ z=0$

② $$\begin{aligned} x+y+z&=0\\ x-5y+6z&=0\\ 3x-15y+18z&=0 \end{aligned}$$

$$A=\begin{bmatrix}1 & 1 & 1\\ 1 & -5 & 6\\ 3 & -15 & 18\end{bmatrix} \qquad |A|_{3\times3}=0,\quad |A|_{2\times2}\neq 0$$

$\therefore\ \rho(A)=2<$ No. of unknowns

$\Rightarrow$ Consistent infinite solution

$\Rightarrow$ Non-trivial soln (Non-zero soln.)

$n-r=3-2=1$ (one arbitrary variable will be assigned)

$x=-\frac{11}{6}K \quad y=\frac{5}{6}K \quad z=K$

Evaluation of solution :-

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -5 & 6 \\ 3 & -15 & 18 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -6 & 5 \\ 0 & -18 & 15 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -6 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x + y + z &= 0 \\ x - 5y + 6z &= 0 \\ 3x - 15y + 18z &= 0 \end{aligned} \quad \xRightarrow[\text{elimination}]{\text{Gauss}} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & -6 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x + y + z &= 0 \\ -6y + 5z &= 0 \end{aligned}$$

Let $z = K$ by back substitution,

$y = \frac{5}{6} z = \frac{5}{6} K$

$x = -y - z = -\frac{5}{6} K - K = -\frac{11}{6} K$

Infinite solution

$x = -\frac{11}{6} K ; y = \frac{5}{6} K ; z = K$

where $K \in R$

1 LI soln

4 variable system :-

①

$$x+y+z+w=0$$
$$x+3y-2z+w=0$$
$$2x+\;\;-3z+2w=0$$
$$x+y+w=0$$

$$A=\begin{bmatrix}1&1&1&1\\1&3&-2&1\\2&0&-3&2\\1&1&0&1\end{bmatrix}\xrightarrow{\text{Row E. Trans.}}\begin{bmatrix}1&1&1&1\\0&2&-3&0\\0&0&-8&0\\0&0&0&0\end{bmatrix}$$

which is Echelon form

$\rho(A)=3<$ No. of unknowns $=4$

$\Rightarrow$ Consistent infinite solution (Non-trivial soln.)

$\Rightarrow$ $n-r=4-3=1$ (Only assign one variable)

Evaluation of solution

$$\begin{bmatrix}1&1&1&1\\0&2&-3&0\\0&0&-8&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$x+y+z+w=0$$
$$2y-3z=0$$
$$-8z=0$$

Let $x=K$

then $y=0,z=0,w=-K$

(x,y,z,w)

$(K,0,0,-K)$

$K\in R$

1 LI soln.

(2) $3x + 4y - z - 6w = 0$
$2x + 3y + 2z - 3w = 0$
$2x + y - 14z - 9w = 0$
$x + 3y + 13z + 3w = 0$

$$A = \begin{bmatrix} 3 & 4 & -1 & -6 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 1 & 3 & 13 & 3 \end{bmatrix} \xrightarrow[\text{E. Trans.}]{\text{Row}} \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is in Echelon form

$\rho(A) = 2 <$ No. of unknowns (4)

$\Rightarrow$ Consistent infinite soln. (Non-trivial soln.)

$\Rightarrow$ $n - r = 4 - 2 = 2$ (two arbitrary variables will be assigned)

Evaluation of Solution

$$\begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + 3y + 13z + 3w = 0$
$y + 8z + 3w = 0$

Let $z = K_1$, $w = K_2$
then $y = -8K_1 - 3K_2$ & $x = 11K_1 + 6K_2$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 11K_1 + 6K_2 \\ -8K_1 - 3K_2 \\ K_1 \\ K_2 \end{pmatrix} = \overbrace{K_1 \begin{bmatrix} 11 \\ -8 \\ 1 \\ 0 \end{bmatrix} + K_2 \begin{bmatrix} 6 \\ -3 \\ 0 \\ 1 \end{bmatrix}}^{\text{2 LI soln.}}$$

$K_1, K_2 \in R$

Non-Homogenous Linear Equations

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$A \quad X = B$$

Augmented matrix $= (A : B) = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$

Augmented Matrix

Non - homogenous system

$\rho(A) = \rho(A|B) = n$

$\Rightarrow$ Consistent unique solution

$\rho(A) = \rho(A|B) < n$

$\Rightarrow$ Consistent infinite solution

$\rho(A) \neq \rho(A|B)$

$\Rightarrow$ Inconsistent system (No solution)

NOTE:-

i) Try to convert augmented matrix $\rightarrow$ Echelon form using elementary transformations

ii) Check condition & evaluate soln by back substitution method

2-variable non-homogenous system:-

$$x + y = 10$$
$$x - y = 4$$

$$(A|B) = \left[\begin{array}{cc|c} 1 & 1 & 10 \\ 1 & -1 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 10 \\ 0 & -2 & -6 \end{array}\right]$$

Echelon form

$$\rho(A) = 2$$
$$\rho(A|B) = 2$$
$$\rho(A) = \rho(A|B) = n = 2$$

$\rightarrow$ Consistent unique soln.

$$x = 7,\ y = 3$$

$$x + 2y = 5$$
$$2x + 4y = 10$$

$$(A|B) = \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & 4 & 10 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 0 & 0 \end{array}\right]$$

Echelon form

$x + 2y = 5$

Let $y = K$

$\Rightarrow x = 5 - 2K$

$$\rho(A) = 1$$
$$\rho(A|B) = 1$$
$$\rho(A) = \rho(A|B) = 1 < n(2)$$

$\rightarrow$ Consistent infinite soln.

$$(x, y) \rightarrow (5 - 2K, K) \quad K \in R$$

$$x + 3y = 5$$
$$2x + 6y = 12$$

$$(A|B) = \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & 6 & 12 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 0 & 2 \end{array}\right]$$

Echelon form

$$\rho(A) = 1$$
$$\rho(A|B) = 2$$
$$\rho(A) \neq \rho(A|B)$$

$\rightarrow$ Inconsistent system (No solution exists).

3-variable non-homogenous system:-

(1)

$$x + y + z = 9$$
$$2x + 5y + 7z = 52$$
$$2x + y - z = 0$$

$$(A|B) = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & 5 & 7 & 52 \\ 2 & 1 & -1 & 0 \end{array}\right] \xrightarrow[\text{transf.}]{\text{Row E.}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -1 & -3 & -18 \\ 0 & 0 & -4 & -20 \end{array}\right]$$

$$\rho(A) = \rho(A|B) = n = 3$$

Consistent unique soln.

Evaluation of Solution

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -18 \\ -20 \end{bmatrix}$$

$$x + y + z = 9$$
$$-y - 3z = -18$$
$$-4z = -20$$

On solving by back substitution, $z = 5$, $y = 3$, $x = 1$

2

$$x - y + 2z = 1$$
$$2x + 2z = 2$$
$$x - 3y + 4z = 2$$

$$(A|B) = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 0 & 2 & 2 \\ 1 & -3 & 4 & 2 \end{array}\right] \underset{\text{transformations}}{\overset{\text{E. row}}{\sim}} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\rho(A) = 2 \quad ; \rho(A|B) = 3$$

$\rho(A) \neq \rho(A|B) \Rightarrow$ Inconsistent system
(No solution)

3

$$4x - 2y + 6z = 8$$
$$x + y + 3z = -1$$
$$15x - 3y + 9z = 21$$

$$(A|B) = \left[\begin{array}{ccc|c} 4 & -2 & 6 & 8 \\ 1 & 1 & 3 & -1 \\ 15 & -3 & 9 & 21 \end{array}\right] \underset{\text{transf.}}{\overset{\text{E. row}}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & 3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = \rho(A|B) = 2 < n(3)$

$\Rightarrow$ Consistent infinite solution.

$\Rightarrow$ $n - r = 3 - 2 = 1$ (one variable will be assigned)

Evaluation of Solution

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & -6 & 18 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 12 \\ 0 \end{bmatrix}$$

$x + y + 3z = -1$

$-6y + 18z = 12$

Let $z = K$

then $y = 3K - 2$

then $x = -6K + 1$

$(x, y, z) \rightarrow (-6K+1, 3K-2, K)$

$K \in R$

Thank you

GW Soldiers !