ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-12

Calculus

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Topics to be Covered

- APPLICATION OF INTEGRATIONS
- LENGTH OR PERIMETER OF CURVE
- SURFACE AREA OF REVOLUTION
- VOLUME OF SOLID OF REVOLUTION
- MULTIPLE INTEGRALS

Ex:- $I = \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta \quad = \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)$

$I = \log\left(1 + \frac{1-\tan\theta}{1+\tan\theta}\right) d\theta = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta$

$I = \int \log 2 - \int \log(1+\tan\theta)\, d\theta$

$2I = \log 2\, [\theta]_0^{\pi/4} = \frac{\pi}{4} \log 2$

$\boxed{I = \frac{\pi}{8} \log 2}$

$\tan(A-B)$

$= \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$

$= \frac{\tan \pi/4 - \tan\theta}{1 + \tan\frac{\pi}{4} \cdot \tan\theta}$

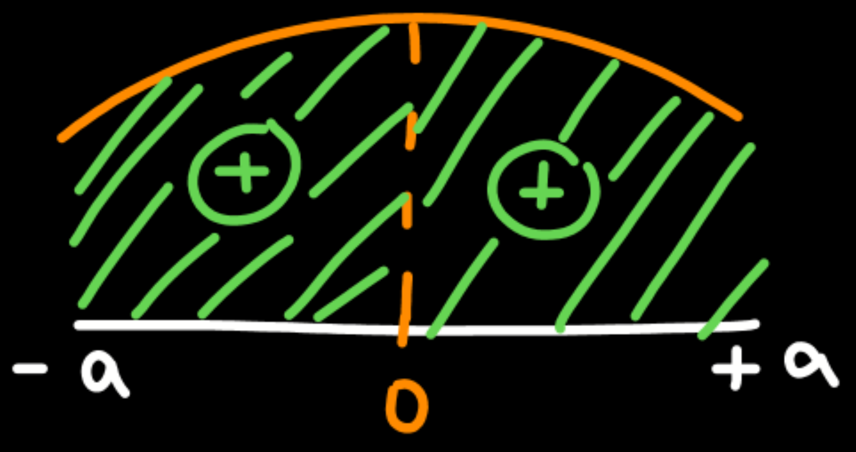
Ex:- $\int \frac{3}{9 + \sin^2\theta}\, d\theta$

Ex:- $\int_{0.25}^{1.25} (x - [x])\, dx$

5) $\displaystyle\int_{-a}^{+a} f(x)\,dx = \begin{cases} 2\displaystyle\int_0^a f(x)\,dx & ; \text{ if } f(x) = f(-x) \\ 0 & ; \text{ if } f(x) = -f(-x) \end{cases}$

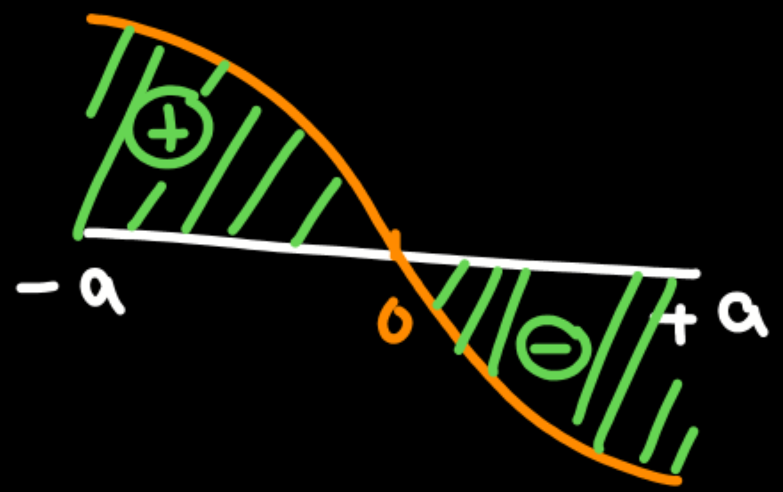
Even function; $f(x) = f(-x)$

Ex:- $x^2, x^4, \cos x$



Odd function; $f(x) = -f(-x)$

Ex:- $x, x^3, x^5, \sin x, \tan x$



$$\int_{-\pi/2}^{\pi/2} \sin dx = \underline{0}$$

$$f(x) = -f(-x)$$

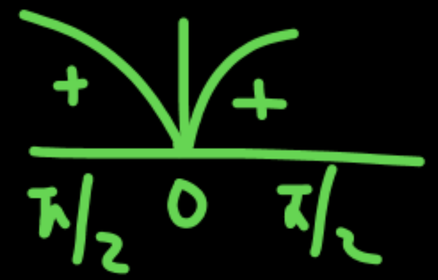
$$\int_{-\pi/2}^{+\pi/2} \overset{\text{Even}}{x^2} \overset{\text{Odd}}{\sin x}\, dx = \int_{-\pi/2}^{+\pi/2} \text{Odd fn.} = \underline{0}$$

$$\int_{-a}^{+a} \overset{\text{Odd}}{\sin x} \overset{\text{Odd}}{\tan x}\, dx =$$

$$\int_{-\pi/2}^{+\pi/2} \overset{\text{Odd}}{\sin x} \overset{\text{Even}}{\cos x}\, dx = \int_{-\pi/2}^{\pi/2} \text{Odd fn.} = 0$$

$$\int_{-\pi/2}^{\pi/2} \overset{\text{Even}}{\sin^2 x}\, dx = 2\int_{0}^{\pi/2} \sin^2 x = \pi/2$$

$$E \times E = E$$
$$O \times O = E$$
$$E \times O = O$$
$$O \times E = O$$

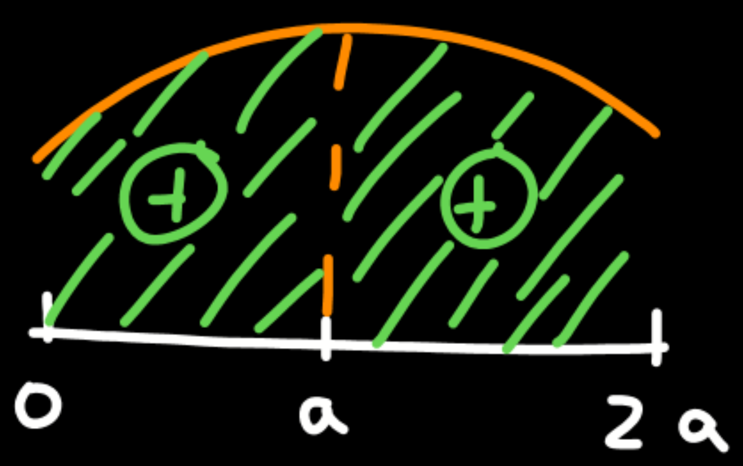


$$\sin^2(-x) = (-\sin x)^2 = \sin^2 x$$

6) $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & ; \quad f(x) = f(2a-x) \\ 0 & ; \quad f(x) = -f(2a-x) \end{cases}$

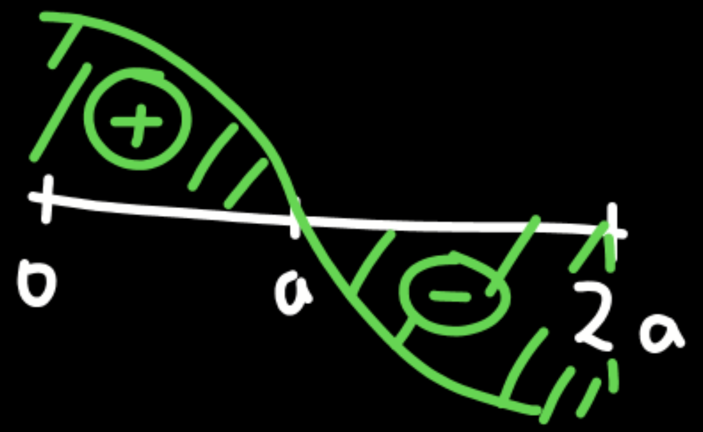
Ex:- $\int_0^{\pi} \sin x\,dx = 2\int_0^{\pi/2} \sin x\,dx$ ($2a \to \pi$)

$f(x) = \sin x \;;\; f(\pi - x) = \sin(\pi - x)$
$= \sin x$



Ex:- $\int_0^{\pi} \cos x\,dx = \underline{0}$

$f(x) = \cos x \;;\; f(\pi - x) = \cos(\pi - x)$
$= -\cos x$



$$\boxed{\int_0^{2a} x\, f(x)\, dx = 2a\int_0^{a} f(x)\, dx \; ; f(x) = f(2a-x)}$$

$$= (2a-x) f(2a-x)\, dx$$

$$= 2a\, f(2a-x) - x\, f(2a-x)$$

$$\int x\, f(x) dx = 2\int a\, f(x) - \int x\, f(x)$$

$$2I = 2a\int_0^{2a} f(x) = 2a\left[2\int_0^{a} f(x)\, dx\right]$$

$$I = 2a\int_0^{a} f(x)\, dx$$

$$\int_0^{2a} f(x)\, dx = \int_0^{2a} f(2a-x)\, dx$$

if $f(x) = f(2a-x)$

Ex:-

$$2a \leftarrow \pi$$

$$\int_0^{\pi} x \underbrace{\sin^6 x \cos^4 x}_{f(x)}\, dx = \pi\int_0^{\pi/2} \sin^6 x \cos^4 x\, dx$$

$$f(x) = \sin^6 x \cos^4 x$$

$$f(\pi-x) = \sin^6(\pi-x)\cos^4(\pi-x)$$

$$= \sin^6 x \cos^6 x$$

$$\int_0^{\pi}(\pi-x)\sin^6(\pi-x)\cos^6(\pi-x)\,dx$$

$$= \pi \sin^6 x \cos^4 x \,\cancel{dx} - x \sin^6 x \cos^4 x \,dx$$

$$2I = \pi\int_0^{\pi}\sin^6 x \cos^4 x\,dx$$

$$\cancel{2}I = \pi\,\cancel{2}\int_0^{\pi/2}\sin^6 x\cos^4 x\,dx$$

Ex:-

$$\int_0^{\pi} x\,\underbrace{\cos^3 x}_{f(x)}\,dx = 0$$

$$f(x) = -f(\pi-x)$$

GAMMA & BETA FUNCTION:-

Mathematician Euler

Ist Eulerian Integral:-

Beta function $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}$

$m,n > 0$ but not necessarily integer

IInd Eulerian Integral:-

Gamma function $\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1} dx$

$\Gamma 1 = \int_0^\infty e^{-x} \cdot x^{1-1} dx = -\left[e^{-x}\right]_0^\infty = -(0-1) = 1$

$$\boxed{\Gamma(n+1) = n\,\Gamma(n) = n!}$$

$\Gamma(1) = 1$

$\Gamma(0) = \infty$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$

Ex:-

- $\Gamma(5) = 4\,\Gamma(4) = 4.3\,\Gamma(3) = 4.3.2\,\Gamma(2) = 4.3.2.1\,\cancelto{1}{\Gamma(1)} = 4!$

$$\int_0^\infty e^{-x} \cdot x^{5-1}\,dx$$

Ex:-

- $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\,\Gamma\left(\frac{3}{2}\right) = \frac{3}{2} \cdot \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}$

- $\Gamma(7) = 6!$

Ex:- $\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx$

$\sqrt{x} = t$

$\frac{1}{2\sqrt{x}} dx = dt$

$\int_0^\infty e^{-t} \cdot t^{1/2} \cdot 2t\, dt$

$2\int_0^\infty e^{-t}\, t^{3/2} = \Gamma\frac{5}{2} = 2 \cdot \frac{3}{4} \cdot \sqrt{\pi}$

$\frac{5}{2} - 1$

$\int_0^\infty e^{-x} \cdot x^{3} = \Gamma 4 = 3!$ $\quad 4-1$

$\int_0^\infty e^{-x} \cdot x^{5} = \Gamma 6 = 5!$ $\quad 6-1$

$\int_0^\infty e^{-x} \cdot x^{6} = \Gamma 7 = 6!$ $\quad 7-1$

Ex:- $\int_0^\infty \sqrt{x} \cdot e^{-\sqrt[3]{x}}\, dx$

$\sqrt[3]{x} = t$

$x = t^3$

$dx = 3t^2 dt$

$e^{-t}\, t^{3/2} \cdot 3t^2 dt$

$3\int_0^\infty e^{-t}\, t^{7/2} dt = 3 \cdot \Gamma\frac{9}{2} = 3 \cdot \frac{7}{2}\Gamma\frac{7}{2} = 3 \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\Gamma\frac{1}{2} = \frac{315}{16}\sqrt{\pi}$

Ex :-

$$\int_0^\infty \frac{x^a}{a^x}\,dx$$

$$a^x = e^t$$

$$x \log a = t \log e$$

$$x \log a = t$$

$$dx = \frac{dt}{\log a}$$

$$\int_0^\infty e^{-t}\left(\frac{t}{\log a}\right)^a\left(\frac{dt}{\log a}\right)$$

$$\frac{1}{(\log a)^{a+1}}\int_0^\infty e^{-t}\,t^{a}\,dt = \frac{\Gamma(a+1)}{(\log a)^{a+1}} = \frac{a!}{(\log a)^{a+1}}$$

$$t^{a} = t^{\overbrace{a+1}-1}$$

if a is integer.

Relation b/w Gamma & Beta function :-

$$\boxed{\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}}$$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$\beta(5,6) = \frac{\Gamma 5 \, \Gamma 6}{\Gamma 11} = \frac{4! \, 5!}{10!}$$

Properties of Beta & Gamma Function :-

1. $\beta(m,n) = \beta(n,m)$

2. $\displaystyle\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right) = \frac{1}{2}\frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{\Gamma\frac{m+n+2}{2}}$ $\quad m,n > -1$

3. $\beta(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx$

$m = 5,\ n = 10$

Ex:- $\int_0^1 \frac{x^4(1+x^5)}{(1+x)^{15}}\,dx = \int_0^1 \frac{x^{\overset{5-1}{4}} + x^{\overset{10-1}{9}}}{(1+x)^{\underline{15}}}\,dx = \beta(5,10) = \frac{\Gamma 5\,\Gamma 10}{\Gamma 15}$

4. $\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

Ex:- $\int_0^\infty \frac{x^{\overset{8-1}{7}}}{(1+x)^{\underset{8+2}{10}}}\,dx = \beta(8,2) = \frac{\Gamma 2\,\Gamma 8}{\Gamma 10}$

5. Transformation of Gamma Function

$$\int_0^\infty e^{-\lambda x} \cdot x^{n-1} \cdot dx = \frac{\Gamma n}{\lambda^n} = \frac{(n-1)!}{\lambda^n}$$

6. Reflection or complement formula

$$\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

Ex:- $\int_0^\infty e^{-5x} \cdot x^{2 \to 3-1}\, dt = \frac{\Gamma 3}{5^3}$

Ex:- $\int_0^1 (x \ln x)^4\, dx$

$\ln x = -t$

$x = e^{-t}$

$dx = -e^{-t}\, dt$

$\int \left(e^{-t}(-t)\right)^4 \cdot -e^{-t}\, dt$

$-\int_\infty^0 e^{-4t-t} \cdot t^4\, dt = \int_0^\infty e^{-5t}\, t^{4 \to 5-1}\, dt = \frac{\Gamma 5}{5^5}$

$x = 0 \rightarrow 1$

$t = \infty \quad\;\; 0$

Ex:- $\underset{\Gamma n}{\Gamma \tfrac{1}{3}}\ \underset{\Gamma 1-n}{\Gamma \tfrac{2}{3}} = \frac{\pi}{\sin \frac{\pi}{3}} = \frac{2\pi}{\sqrt{3}}$

$\Gamma \tfrac{1}{4}\ \Gamma \tfrac{3}{4} = \frac{\pi}{\sin \frac{\pi}{4}} = \sqrt{2}\,\pi$

Ex:-

$$\int_0^\infty e^{-x^2}\,dx$$

$$x^2 = t$$
$$2x\,dx = dt$$
$$dx = \frac{dt}{2x}$$
$$dx = \frac{dt}{2\sqrt{t}}$$

$$\int_0^\infty e^{-t}\,\frac{dt}{2\sqrt{t}}$$

$$\frac{1}{2}\int_0^\infty e^{-t}\,t^{-1/2}\,dt = \frac{1}{2}\sqrt{\frac{1}{2}} = \boxed{\frac{\sqrt{\pi}}{2}}$$

$$\frac{1}{2}-1$$

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \boxed{\int_0^\infty e^{-x^2}\,dx}\,\boxed{\int_0^\infty e^{-y^2}\,dy} = \frac{\sqrt{\pi}}{2}\cdot\frac{\sqrt{\pi}}{2} = \frac{\pi}{4}$$

$$\int_0^\infty\int_0^\infty\int_0^\infty e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz = \boxed{\int_0^\infty e^{-x^2}\,dx}\,\boxed{\int_0^\infty e^{-y^2}\,dy}\,\boxed{\int_0^\infty e^{-z^2}\,dz} = \left(\frac{\sqrt{\pi}}{2}\right)^3 = \frac{\pi\sqrt{\pi}}{8}$$

TRICK:-

$$\int_0^{\pi/2} \sin^m x\, dx = \int_0^{\pi/2} \cos^m x\, dx$$

$$= \frac{(m-1).(m-3)\ldots 2/1}{m.(m-2)(m-4)\ldots 2/1} . K$$

$$K \begin{cases} \pi/2 \; ; m \to \text{Even} \\ 1 \; ; m \to \text{Odd} \end{cases}$$

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx$$

$$= \frac{(m-1)(m-3)\ldots \quad (n-1).(n-3)}{(m+n)(m+n-2)\ldots} . K$$

$$K \begin{cases} \pi/2, \; m \to \text{Even}, \; n \to \text{Even} \\ 1 \; ; \text{else} \end{cases}$$

APPLICATION OF INTEGRATIONS

$$\int_0^{\pi/2} \sin^6 x \, dx = \frac{5.3.1}{6.4.2} \cdot \frac{\pi}{2}$$

$$\int_0^{\pi/2} \cos^7 x \, dx = \frac{6.4.2}{7.5.3.1} \cdot 1$$

$$\int_0^{\pi/2} \sin^6 x \cos^8 x \, dx = \frac{5.3.1. \quad 7.5.3.1}{14.10.8.6.4.2} \cdot \frac{\pi}{2}$$

$$\int_0^{\pi/2} \sin^6 x \cos^3 x \, dx = \frac{5.3.1. \quad 2}{9.7.5.3.1} \cdot 1$$

Thank you

GW Soldiers !