ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-5

Linear Algebra

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Topics to be Covered

- Vector Space
- Basis
- LU Decomposition
- Computation of Inverse by E- Transformation
- Linear dependence and Independence of Vectors

PROPERTIES OF RANK

1. $A_{m\times n}$ Max. Rank $= \min(m,n)$

   Let $m < n \Rightarrow$ Possible ranks of $A \rightarrow m, m-1, \ldots\ldots$

2. Rank of null matrix is always 0 (not defined)

3. $A_{n\times n}$ and if A is non-singular $[|A| \neq 0] \Rightarrow \rho(A) = n$

* 4. $\rho(A) = \rho(A^T) = \rho(A^*) = \rho(A^{-1}) = \rho(AA^T) = \rho(A^TA) = \rho(AA^*) = \rho(A^*A)$

5. If $A \rightarrow$ Non-zero column matrix $\Rightarrow \rho(A) = 1$

6. If $A \rightarrow$ Non-zero row matrix $\Rightarrow \rho(A) = 1$

7. If all the elements of $A_{n\times n}$ are same $\Rightarrow \rho(A) = 1$

   Ex:- $\begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$

PROPERTIES OF RANK

8. If all rows/columns are proportional then $\rho(A)$ is 1.

$A_{n\times n} \Rightarrow \rho(A) = 1$ $\qquad$ $A_{m\times n}\ (m<n) \Rightarrow \rho(A) = 1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 5 & 10 & 15 \end{bmatrix}_{3\times 3} \qquad \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 3 & 6 & 9 & 3 \end{bmatrix}_{3\times 4}$$

9. Rank of diagonal matrix

$\hookrightarrow$ No. of non-zero diagonal elements

Ex:-

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow 3 \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix} \rightarrow 2$$

10. If $A \rightarrow$ column matrix
$B \rightarrow$ Row matrix
then $\rho(AB) = 1$

$$A_{n\times 1}\ B_{1\times m} = AB_{n\times m}$$

PROPERTIES OF RANK

11. $\text{Rank}(AB) \le \rho(A)$

$\rho(AB) \le \rho(B)$

12. $\rho(A+B) \le \rho(A) + \rho(B)$

$A_{3\times3} \rightarrow 3$ $\quad B_{3\times3} \rightarrow 2$ $\quad (A+B)_{3\times3} \rightarrow 3$

13. $\rho(A-B) \ge$ Difference in ranks of A and B.

$|\rho(A) - \rho(B)|$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

14. If $A_{n\times n}$

$\rightarrow \rho(A) = n \quad ; \rho(\text{Adj} A) = n$

$\rightarrow \rho(A) = n-1 \quad ; \rho(\text{Adj} A) = 1$

$\rightarrow \rho(A) \le n-2 \quad ; \rho(\text{Adj} A) = 0$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 0 \end{bmatrix}_{3\times3}$$

$\rho(A) = 2$

$$\text{Adj} A = \begin{bmatrix} -6 & 0 & 2 \\ +3 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}^T$$

$$\rho(\text{Adj} A) = \begin{bmatrix} -6 & 3 & 0 \\ 0 & 0 & 0 \\ 2 & -1 & 0 \end{bmatrix}$$

$\rho(\text{Adj} A) \rightarrow 1$

15. $A_{n\times n}$ $\quad \rho(AB) \ge \rho(A) + \rho(B) - n$

PROPERTIES OF RANK

16. Rank = Order of largest non-zero minor.
17. = No. of non-zero rows in Row echelon form [Row Rank]
18. = No. of non-zero columns in Column echelon form [Column Rank]
19. = Order of identity matrix in normal (Canonical) form.

20. Rank
    - No of linearly independent rows [LI rows]
    - No. of linearly independent columns [LI columns]

21. Nullity
    - No. of linearly dependent rows [LD rows]
    - No. of linearly dependent columns [LD columns]

22. Elementary operations do not change the rank
23. Rank of scalar and identity matrix of $n \times n$ order is n.

LINEAR DEPENDENCE AND INDEPENDENCE OF VECTORS

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 5 & 10 & 15 \end{bmatrix}$$

$R_2 = 2R_1$
$R_3 = 5R_1$
(Proportional)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & -1 & 3 \end{bmatrix}$$

$R_2 = 2R_1$
(Proportional)

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & -1 & 5 \end{bmatrix}$$

$R_2 = R_1$
(Identical)

$$\begin{bmatrix} 1 & -17 & 1 \\ 5 & 9 & 5 \\ 6 & 8 & 6 \end{bmatrix}$$

$C_1 = C_3$
(Identical)

$$\begin{bmatrix} 1 & 4 & 9 \\ 2 & 8 & 18 \\ 3 & 1 & 5 \end{bmatrix}$$

$C_1 + 2C_2 = C_3$
(Relationship)

$$\begin{bmatrix} 5 & 7 & 2 \\ 1 & 3 & 3 \\ -6 & -10 & -5 \end{bmatrix}$$

$R_1 + R_2 = -R_3$
(Relationship)

No. of relations	2	1	1	1	1	1
Nullity	2	1	1	1	1	1
Rank	1	2	2	2	2	2

ELEMENTARY MATRICES

A matrix obtained from the unit matrix by applying to it any of the three elementary transformations is called an elementary matrix.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{C_1 \leftrightarrow C_3} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Elementary matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to 2R_1} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

ELEMENTARY MATRICES

Row Equivalent Matrix

A matrix B is said to be row equivalent to A, if it is obtained by performing elementary row operations on A.

$$A \xrightarrow[\text{transformations}]{\text{Row}} B$$

Row equivalent matrix

ELEMENTARY MATRICES

Column Equivalent Matrix

A matrix B is said to be column equivalent to A, if it is obtained by performing elementary column operations on A.

$$A \xrightarrow[\text{transformations}]{\text{Column}} B$$

Column equivalent matrix

COMPUTATION OF INVERSE BY E-TRANSFORMATION

Find A^{-1} when $A = \begin{bmatrix} 1 & 2 & 6 \\ 4 & 0 & 3 \\ 0 & 5 & 0 \end{bmatrix}$

$$AA^{-1} = I$$

$$A = IA$$

$$\begin{bmatrix} 1 & 2 & 6 \\ 4 & 0 & 3 \\ 0 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 6 \\ \boxed{0} & -8 & -21 \\ \boxed{0} & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$C_2 \to C_2 - 2C_1$

$C_3 \to C_3 - 6C_1$

$$\begin{bmatrix} 1 & \boxed{0} & \boxed{0} \\ \boxed{0} & -8 & -21 \\ \boxed{0} & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -6 \\ -4 & 9 & 24 \\ 0 & 0 & 1 \end{bmatrix} A$$

$C_2 \to \dfrac{C_2}{-8}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -21 \\ 0 & -5/8 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1/4 & -6 \\ -4 & -9/8 & 24 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -21 \\ 0 & 0 & \frac{105}{8} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{4} & -6 \\ -4 & -\frac{9}{8} & 24 \\ -\frac{5}{2} & -\frac{45}{64} & 16 \end{bmatrix} A \qquad R_3 \rightarrow R_3 + \frac{5}{8} R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{105}{8} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{4} & -\frac{3}{4} \\ -4 & -\frac{9}{8} & \frac{3}{8} \\ -\frac{5}{2} & -\frac{45}{64} & \frac{79}{64} \end{bmatrix} A \qquad C_3 \rightarrow C_3 + 21 C_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & \frac{1}{4} & -\frac{3}{4} \\ -4 & -\frac{9}{8} & \frac{3}{8} \\ -\frac{4}{21} & -\frac{3}{56} & \frac{79}{840} \end{bmatrix}} A \qquad R_3 \rightarrow \frac{R_3}{105/8}$$

$$I = A^{-1} A$$

This method is used mainly for 4x4 matrices.

[ROW RANK AND COLUMN RANK OF A MATRICES]

If $A = [a_{ij}]_{m\times n}$ be any $m \times n$ matrix, then the number of maximum linearly independent rows (or row vectors) of matrix A is called its row rank. And the number of maximum linearly independent columns (or column vectors) of matrix A is called its columns rank.

Row E-operations $\Rightarrow$ Row Rank

Column E- operations $\Rightarrow$ Column Rank

$$\boxed{\text{Row rank} = \text{Column Rank} = \text{Rank}}$$

[ROW RANK AND COLUMN RANK OF A MATRICES]

NOTE: Row rank, column rank and rank of a matrix are same.

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 3 & -2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

No. of non-zero rows = Row Rank = 2

→ If we want to find column rank, find row rank of A^T.

Row Rank of (A^T) = Column rank of (A) = 2

$$A^T = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 3 & 4 \\ -1 & -2 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 2 \\ -1 & -2 & -3 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 4 \\ 0 & -6 & -6 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + \frac{R_2}{6}} \begin{bmatrix} 1 & 3 & 4 \\ 0 & -6 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

ROW RANK AND COLUMN RANK OF A MATRICES

H.W.

Find the row rank, column rank and rank of a matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$$

Chetan Sir (CE) off
AI PYQ ✓
Rank of M PYQ ✓

Thank you

GW Soldiers !