ENGINEERING MATHEMATICS

ALL BRANCHES

Calculus
Maxima and Minima
DPP-06 Solution

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Question - 01

The minimum value of $\left(x^2+\frac{250}{x}\right)$ is

A 75

B 50

C 25

D 0

$f(x) = x^2 + \frac{250}{x}$

$f'(x) = 2x - \frac{250}{x^2} = 0$

$\Rightarrow 2x^3 = 250$

$x = 5$ (Turning point)

$f''(x) = 2 + \frac{500}{x^3}$

$f''(5) = 2 + \frac{500}{5^3} = 6 > 0$ ($\therefore$ Minima)

$$f(5) = x^2 + \frac{250}{x}$$

$$= 5^2 + \frac{250}{5}$$

$$= 25 + 50 = 75$$

Question - 02

The maximum value of $f(x) = (1 + \cos x)\sin x$ is

A 3

B $3\sqrt{3}$

C 4

D $\frac{3\sqrt{3}}{4}$ ✓

$f'(x) = (1+\cos x)(\cos x) + (-\sin x)(\sin x)$

$\cos x + \cos^2 x - \sin^2 x = 0$

$\cos x + \cos^2 x - (1 - \cos^2 x) = 0$

$f'(x) = 2\cos^2 x + \cos x - 1 = 0$

$(2\cos x - 1)(\cos x + 1) = 0$

$\cos x = \frac{1}{2}$ or $\cos x = -1$

$\Rightarrow x = \pi/3$ or $x = \pi$

$f''(x) = 4\cos x(-\sin x) - \sin x$

$= -2\sin 2x - \sin x$

At $x=\pi/3$ $\quad f''(\pi/3) = -2\sin\left(\frac{2\pi}{3}\right) - \sin\frac{\pi}{3} = -2\times\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} < 0$

$\Rightarrow$ Maxima

At $x=\pi$ $\quad f''(\pi) = -2\sin 2\pi - \sin\pi$

$= 0 - 0$

$\therefore$ At $x=\pi/3$ $f(x)$ is maxima

$\sin\left(\pi - \frac{\pi}{3}\right)$

$= \sin\pi/3$

$f(\pi/3) = (1+\cos x)\sin x$

$= (1+\cos\pi/3)\sin\pi/3$

$= \left(1+\frac{1}{2}\right)\frac{\sqrt{3}}{2} = \frac{3}{2}\frac{\sqrt{3}}{2}$

$= \frac{3\sqrt{3}}{4}$

Question - 03

A greatest value of $f(x)=\dfrac{\sin 2x}{\sin\left(x+\frac{\pi}{4}\right)}$ on the interval $\left[0,\frac{\pi}{2}\right]$ is

A $\frac{1}{\sqrt{2}}$

B $\sqrt{2}$

C 1

D $-\sqrt{2}$

$$f(x) = \frac{2\sin x\cos x}{\sin x\cos\frac{\pi}{4} + \cos x\sin\frac{\pi}{4}}$$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$= \frac{2\sin x\cos x}{\frac{1}{\sqrt{2}}(\sin x + \cos x)} = \frac{2\sqrt{2}}{\frac{\sin x + \cos x}{\sin x\cos x}}$$

$$f(x) = \frac{2\sqrt{2}}{\frac{1}{\cos x} + \frac{1}{\sin x}}$$

Let D_r be $z = \sec x + \operatorname{cosec} x$

$$z = \sec x + \operatorname{cosec} x$$

$$\frac{dz}{dx} = \sec x \tan x - \operatorname{cosec} x \cot x = 0$$

$$\frac{1}{\cos x} \cdot \tan x - \frac{1}{\sin x} \cdot \frac{1}{\tan x} = 0$$

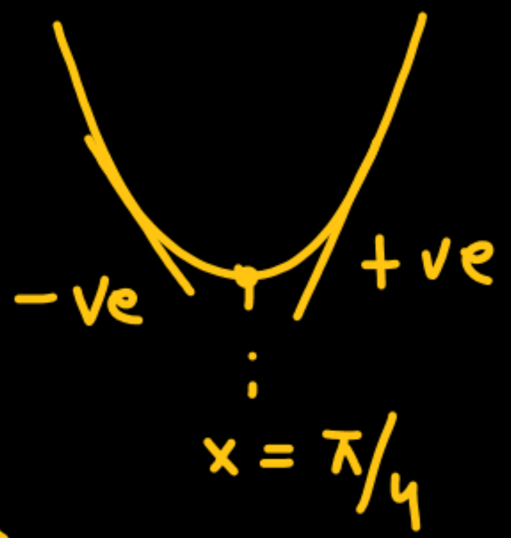
$$\frac{\sin x \tan^2 x - \cos x}{\sin x \cos x \tan x} = 0$$

$$\sin x \tan^2 x = \cos x$$

$$\tan^3 x = 1$$

$$\tan x = 1$$

$$x = \pi/4$$



Sign of dz/dx changes from −ve to +ve when x passes from $\pi/4$ $\therefore$ z is minimum. Hence $f(x)$ is maximum.

$$f(x) = \frac{\sin 2x}{\sin(x + \pi/4)} \Rightarrow f(\pi/4) = \frac{\sin \pi/2}{\sin \pi/2} = 1$$

Question - 04

If $y = a\log x + bx^2 + x$ has its extremum values at $x = -1$ and $x = 2$, then

A $a = -\frac{1}{2}, b = 2$

B $a = 2, b = -1$

C $a = 2, b = -\frac{1}{2}$

D None of these

$y = a\log x + bx^2 + x$

$y' = \frac{a}{x} + 2bx + 1 = 0$

At $x = -1 \Rightarrow -a - 2b + 1 = 0$

At $x = 2 \Rightarrow \frac{a}{2} + 4b + 1 = 0$

$a + 2b = 1$ — (1)

$a + 8b = -2$ — (2)

$-6b = 3$

$b = -\frac{1}{2}$

$\therefore a = 1 - 2\left(-\frac{1}{2}\right) = 2$

Question - 05

The co-ordinates of the point on the curve $4x^2 + 5y^2 = 20$ that is farthest from the point $(0, -2)$ are

$$\frac{x^2}{5} + \frac{y^2}{4} = 1 \quad (\text{Ellipse})$$

A $(\sqrt{5}, 0)$

Parametric point on ellipse $(\sqrt{5}\cos\theta, 2\sin\theta)$

B $(\sqrt{6}, 0)$

Dist. b/w point on ellipse & $(0, -2)$

$$D = \sqrt{(\sqrt{5}\cos\theta - 0)^2 + (2\sin\theta + 2)^2}$$

C $(0, 2)$

↳ on ellipse

Let $z = 5\cos^2\theta + 4(1 + \sin\theta)^2$

$$\frac{dz}{d\theta} = 10\cos\theta(-\sin\theta) + 8(1+\sin\theta)(\cos\theta) = 0$$

D None of these

$$= -2\cos\theta\sin\theta + 8\cos\theta = 0$$

$$= -\sin 2\theta + 8\cos\theta$$

$$2\cos\theta(-\sin\theta+4)=0$$

$$\cos\theta=0$$

$$\Rightarrow \theta=\pi/2$$

$$\frac{dz}{d\theta}=-\sin 2\theta+8\cos\theta$$

$$\frac{d^2z}{d\theta^2}=-2\cos 2\theta-8\sin\theta$$

$$\text{At } x=\pi/2 = -2\cos 2\frac{\pi}{2}-8\sin\frac{\pi}{2} = -2(-1)-8=-6<0$$

$$\Rightarrow \text{Maxima}$$

$$\therefore \text{Point } (\sqrt{5}\cos\theta, 2\sin\theta)$$

$$\left(\sqrt{5}\cos\frac{\pi}{2}, 2\sin\pi/2\right)=(0,2)$$

Question - 06

For what value of $x\left(0 \le x \le \frac{\pi}{2}\right)$, the function $y=\frac{x}{(1+\tan x)}$ has a maxima?

A $\tan x$

B 0

C $\cot x$

D $\cos x$ ✓

$$y = \frac{x}{1+\tan x}$$

$$\frac{dy}{dx} = \frac{(1+\tan x).1 - x(\sec^2 x)}{(1+\tan x)^2} = 0$$

$$1+\tan x - x\sec^2 x = 0$$

$$1+\tan x = x\sec^2 x$$

$$1+\frac{\sin x}{\cos x} = \frac{x}{\cos^2 x}$$

$$\frac{dy}{dx} \rightarrow +ve \text{ to } -ve$$

$$\cos x(\sin x + \cos x) = x$$

$$\boxed{x = \cos x}$$

$$[\because \sin x + \cos x = 1]$$

Question - 07

The co-ordinates of the point on the parabola $y = x^2 + 7x + 2$ which is closest to the straight-line $y = 3x - 3$, are

let point on parabola be (x, y)

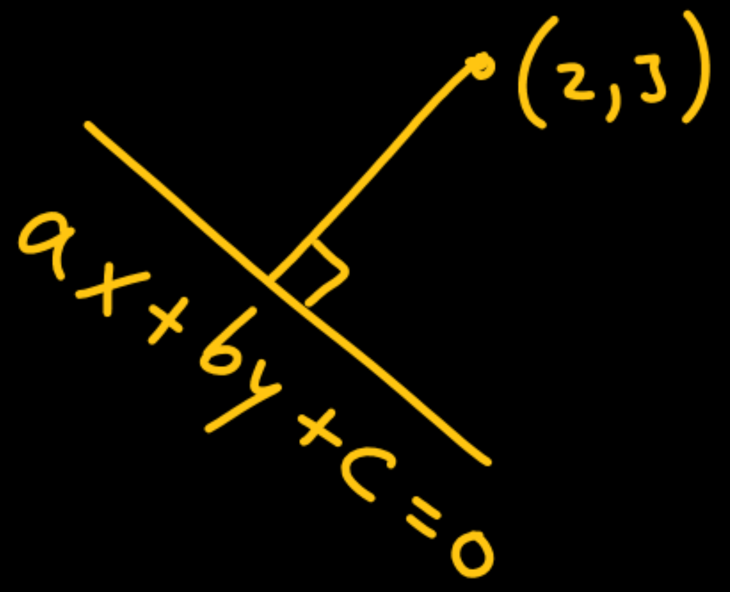
- **A** $(-2, -8)$ ✓
- **B** $(2, -8)$
- **C** $(-2, 0)$
- **D** None of these

$$D = \frac{|y - 3x + 3|}{\sqrt{1^2 + (-3)^2}}$$

$$D = \frac{|x^2 + 7x + 2 - 3x + 3|}{\sqrt{10}}$$

$$D = \frac{1}{\sqrt{10}}[(x^2 + 4x + 4) + 1]$$

$$D = \frac{1}{\sqrt{10}}[(x+2)^2 + 1]$$



$$D = \frac{|2a + 3b + c|}{\sqrt{a^2 + b^2}}$$

$$\frac{dD}{dx} = \frac{1}{\sqrt{10}}\left[2(x+2)\right] = 0$$

$$\therefore x = -2$$

$$\frac{d^2D}{dx^2} = \frac{2}{\sqrt{10}} > 0 \quad (\text{Minima})$$

At $x = -2$; $y = x^2 + 7x + 2$

$$= (-2)^2 + 7(-2) + 2$$

$$= 4 - 14 + 2 = -8$$

Parabola $(-2, -8)$ is at least dist. from line $y = 3x - 3$

Question - 08

The maximum value of $\left(\frac{1}{x}\right)^x$ is

A e

B $e^{-\frac{1}{e}}$

C $\left(\frac{1}{e}\right)^e$

D None of these

$y = \frac{1}{x^x} = x^{-x}$

$\frac{dy}{dx} = -x^{-x}(1+\log x) = 0$

$\therefore 1 + \log x = 0$

$\log x = -1$

$\boxed{x = e^{-1}} = 1/e$

$\frac{d^2y}{dx^2} = -x^{-x} \cdot \frac{1}{x} + x^x(1+\log x)^2$

$\frac{d^2y}{dx^2} = -\left(\frac{1}{e}\right)^{-1/e} \cdot \frac{1}{1/e} < 0$ (Maxima)

$\left(\frac{1}{1/e}\right)^{1/e} = e^{1/e}$

$= \left(\frac{1}{e}\right)^{-1/e}$

$y = x^{-x}$

$\log y = -x \log x$

$\frac{1}{y} \cdot \frac{dy}{dx} = -x \frac{1}{x} - 1 \cdot \log x$

$= -1 - \log x$

$\frac{dy}{dx} = -x^{-x}(1+\log x)$

Thank you

GW
Soldiers !