ENGINEERING MATHEMATICS

ALL BRANCHES

Calculus

Integration & Definite integration and its application

DPP-09 Solution

By- CHETAN SIR

Question 1

If $F(a)=\frac{1}{\log a}, a>1$ and $F(x)=\int a^x dx+K$ is equal to

A $\frac{1}{\log a}(a^x-a^a+1)$

B $\frac{1}{\log a}(a^x-a^a)$

C $\frac{1}{\log a}(a^x+a^a+1)$

D $\frac{1}{\log a}(a^x+a^a-1)$

$$F(x)=\frac{a^x}{\log a}+K$$

$$F(a)=\frac{a^a}{\log a}+K=\frac{1}{\log a}$$

$$\therefore K=\frac{1}{\log a}-\frac{a^a}{\log a}$$

$$\therefore F(x)=\frac{a^x}{\log a}+\frac{1}{\log a}-\frac{a^a}{\log a}$$

$$F(x)=\frac{1}{\log a}(a^x-a^a+1)$$

Question 2

$\int \frac{dx}{1+\sin x}$ is equal to

A $-\cot x + \operatorname{cosec} x + c$

B $\cot x + \operatorname{cosec} x + c$

C $\tan x - \sec x + c$ ✓

D $\tan x + \sec x + c$

$$\int \frac{dx}{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}$$

$$= \int \frac{\sec^2 x/2 \; dx}{\tan^2 x/2 + 1 + 2\tan x/2}$$

$$= \int \frac{\sec^2 x/2}{(1+\tan x/2)^2}\, dx$$

$$= \int \frac{2\,dt}{t^2}$$

$$= -\frac{2}{t} + c$$

$$-\frac{2}{1+\tan x/2}+c$$

$$= -\frac{2\cos x/2}{\cos x/2+\sin x/2}+c$$

$$= -\frac{2\cos x/2}{\cos x/2+\sin x/2}\times\frac{\cos x/2-\sin x/2}{\cos x/2-\sin x/2}+c$$

$$= -\frac{2\cos^2 x/2+2\sin x/2\cos x/2}{\cos x}+c$$

$$= -\frac{(1+\cos x)+\sin x}{\cos x}+c$$

$$-\sec x-1+\tan x+c$$

$$\tan x-\sec x+c'$$

Let $1+\tan\frac{x}{2}=t$

$$\frac{1}{2}\boxed{\sec^2\frac{x}{2}\,dx}=dt$$

$$\cos^2 x/2-\sin^2 x/2=\cos x$$

Question 3

$\int \frac{(3x+1)}{2x^2-2x+3} dx$ equal to

A $\frac{3}{4} log(2x^2 - 2x + 3) + \frac{\sqrt{5}}{2} tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)$

B $\frac{4}{3} log(2x^2 - 2x + 3) + \sqrt{5}\, tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)$

C $\frac{4}{3} log(2x^2 - 2x + 3) + \frac{2}{\sqrt{5}} tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)$

D $\frac{3}{4} log(2x^2 - 2x + 3) + \frac{2}{\sqrt{5}} tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)$

$$3x+1 = p(4x-2)+q \qquad \therefore\ p = 3/4,\ q = 5/2$$

$$4p = 3$$
$$-2p+q = 1$$

$$\Rightarrow \int \frac{p(4x-2)}{2x^2-2x+3}dx + \int \frac{q}{2x^2-2x+3}dx$$

$$p\left[\log(2x^2-2x+3)\right] + \frac{q}{2\left[x^2-x+3/2\right]}dx$$

$$\text{''} \qquad + \frac{q}{2\left[\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{5}}{2}\right)^2\right]}$$

$$p\left[\log(2x^2-2x+3) + \frac{q}{2}\,\frac{1}{\sqrt{5}/2}\tan^{-1}\frac{x-\frac{1}{2}}{\frac{\sqrt{5}}{2}} \Rightarrow \frac{3}{4}\log(2x^2-2x+3) + \frac{\sqrt{5}}{2}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)\right.$$

Question 4

$\int \frac{2x+3}{\sqrt{x^2+x+1}} dx$ is equal to

A $2\sqrt{x^2+x+1} + 2\sinh^{-1}\frac{2x+1}{\sqrt{3}}$

B $\sqrt{x^2+x+1} + 2\sinh^{-1}\frac{2x+1}{\sqrt{3}}$

C $2\sqrt{x^2+x+1} + \sinh^{-1}\frac{2x+1}{\sqrt{3}}$

D $2\sqrt{x^2+x+1} - \sinh^{-1}\frac{2x+1}{\sqrt{3}}$

$$\int \frac{2x+1}{\sqrt{x^2+x+1}}\,dx + \int \frac{2}{\sqrt{x^2+x+1}}\,dx$$

$$2\sqrt{x^2+x+1} + \int \frac{2}{\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$+ \; 2 \sinh^{-1} \frac{x+1/2}{\sqrt{3}/2}$$

$$\Rightarrow 2\sqrt{x^2+x+1} + 2\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

Question 5

The value of $\int e^x \left(\frac{1+\sin x}{1+\cos x}\right) dx$ is

$$e^x \left[\frac{1 + 2\sin x/2 \cos x/2}{2\cos^2 x/2}\right]$$

$1+\cos 2x = 2\cos^2 x$

$1-\cos 2x = 2\sin^2 x$

$\sin 2x = 2\sin x/2 \cos x/2$

A. $e^x \tan\frac{x}{2} + c$

$$\int \underbrace{e^x}_{\mathrm{I}} \frac{1}{2} \underbrace{\sec^2\frac{x}{2}}_{\mathrm{II}} + \int e^x \cdot \tan\frac{x}{2}$$

B. $e^x \cot\frac{x}{2} + c$

C. $e^x \tan x + c$

$$\frac{1}{\cancel{2}} e^x \frac{\tan\frac{x}{2}}{1/\cancel{2}} - \cancel{\int \frac{e^x}{\cancel{2}} \frac{\tan\frac{x}{2}}{1/\cancel{2}}} + \cancel{\int \frac{e^x}{2} \tan x/2}$$

D. $e^x \cot x + c$

$e^x \tan x/2 + c.$

$$\int e^x [f(x) + f'(x)] = e^x f(x)$$

Question 6

$\int_0^{\pi/2} \frac{e^x}{2}\left(sec^2\frac{x}{2} + 2\,tan\frac{x}{2}\right) dx$ is equal to

A e^{π}

B $e^{\pi/2}$

C e

D $e^{\pi/4}$

$$e^x\left[\tan\frac{x}{2} + \frac{1}{2}\sec^2 \cancel{x}x/2\right]$$

$$\int e^x\left[f(x) + f'(x)\right]$$

$$\left[e^x f(x)\right]_0^{\pi/2} = \left[e^x \tan x/2\right]_0^{\pi/2}$$

$$= e^{\pi/2}\cdot 1 - 0$$

$$= e^{\pi/2}$$

Question 7

$\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\,dy\,dx$ is equal to

A 7/60

B 3/35

C 4/49

D None of these

$$\int_{x=0}^{x=1}\left(\int_{y=x}^{y=\sqrt{x}}(x^2+y^2)\,dy\right)dx$$

$$\int_0^1 \left[x^2y + \frac{y^3}{3}\right]_x^{\sqrt{x}} dx$$

$$\int_0^1 x^{5/2} + \frac{x^{3/2}}{3} - x^3 - \frac{x^3}{3}\,dx$$

$$\left[\frac{x^{7/2}}{7/2} + \frac{x^{5/2}}{3\cdot 5/2} - \frac{x^4}{4} - \frac{x^4}{3\times 4}\right]_0^1$$

$$\frac{2}{7} + \frac{2}{15} - \frac{1}{4} - \frac{1}{12} = \frac{3}{35}$$

Question 8

$\int_{-1}^{1} \frac{|x|}{x} dx$ is equal to

A 2

B 0

C 1

D 1/2

$$\int_{-1}^{0} -\frac{x}{x}\, dx + \int_{0}^{1} \frac{x}{x}\, dx$$

$-x \;; x<0$

$x \;; x>0$

$$[-x]_{-1}^{0} + [x]_{0}^{1}$$

$$-1 + 1 = 0$$

Question 9

$\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)dy\,dx\,dz$ is equal to

$$\left(\int_{z=-1}^{z=+1}\left(\int_{x=0}^{x=z}\left(\int_{y=x-z}^{y=x+z}(x+y+z)\,dy\right)dx\right)dz\right)$$

$$\left[\frac{(x+y+z)^2}{2}\right]_{x-z}^{x+z}$$

$$\frac{(2x+2z)^2}{2}-\frac{(2x)^2}{2}$$

$$2\int_{-1}^{+1}\left(\int_{0}^{z}(x+z)^2-x^2\,dx\right)dz$$

$$\left[\frac{(x+z)^3}{3}-\frac{x^3}{3}\right]_0^z$$

A 4

B −4

C 0

D None of these

$$\frac{2}{3}\int_{-1}^{+1}\left\{(2z)^3 - z^3 - z^3\right\} dz$$

$$\frac{2}{3} \times 0 = 0$$

Question 10

$\int_0^{\pi} \cos^m x \sin^n x \, dx$ is equal to zero, if

$$\int_0^{2a} f(x)\,dx$$

A m is even

B n is even

C m is odd

D n is odd

$f(x) = \cos^m x \ \sin^n x$

$f(\pi - x) = \cos^m(\pi - x)\ \sin^m(\pi - x)$

$= (-\cos x)^m (\sin x)^n$

$f(\pi - x) = -\cos^m x \ \sin^n x$

$\therefore$ m is odd.

① $f(x) = f(2a - x)$

$\Rightarrow 2\int_0^a f(x)\,dx$

② $f(x) = -f(2a - x)$

$\Rightarrow 0$

Question 11

The area bounded by the curve $r = \theta \cos\theta$ and the lines $\theta = 0$ and $\theta = \frac{\pi}{2}$ is given by

A $\frac{\pi}{4}\left(\frac{\pi^2}{16} - 1\right)$

B $\frac{\pi}{16}\left(\frac{\pi^2}{6} - 1\right)$

C $\frac{\pi}{16}\left(\frac{\pi^2}{16} - 1\right)$

D None of these

$$A = \int_{\theta_1}^{\theta_2}\int_0^{r} r\, dr\, d\theta$$

$$A = \int_{\theta_1}^{\theta_2} \frac{r^2}{2}\, d\theta = \int_0^{\pi/2} \frac{\theta^2 \cos^2\theta}{2}\, d\theta$$

$$\frac{1}{4}\int_0^{\pi/2} \theta^2 [1 + \cos 2\theta] = \frac{1}{4}\int \theta^2 + \theta^2 \cos 2\theta$$

$$\frac{1}{4}\left[\frac{\theta^3}{3} + \theta^2 \frac{\sin 2\theta}{2} - \int \cancel{2}\theta \frac{\sin 2\theta}{\cancel{2}}\right]$$

$$\frac{1}{4}\left[\frac{\theta^3}{3} + \theta^2 \frac{\sin 2\theta}{2} + \theta \frac{\cos 2\theta}{2} - \int 1 . \frac{\cos 2\theta}{2}\right]$$

$$\frac{1}{4}\left[\frac{\theta^3}{3}+\theta^2\frac{\sin 2\theta}{2}+\theta\frac{\cos 2\theta}{2}-\frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$=\frac{\pi}{16}\left(\frac{\pi^2}{16}-1\right)$$

Question 12

The volume of the cylinder $x^2 + y^2 = a^2$ bounded below by $z = 0$ and bounded above by $z = h$ is given by

A rah

B $\pi a^2 h$

C $\frac{1}{3}\pi a^3 h$

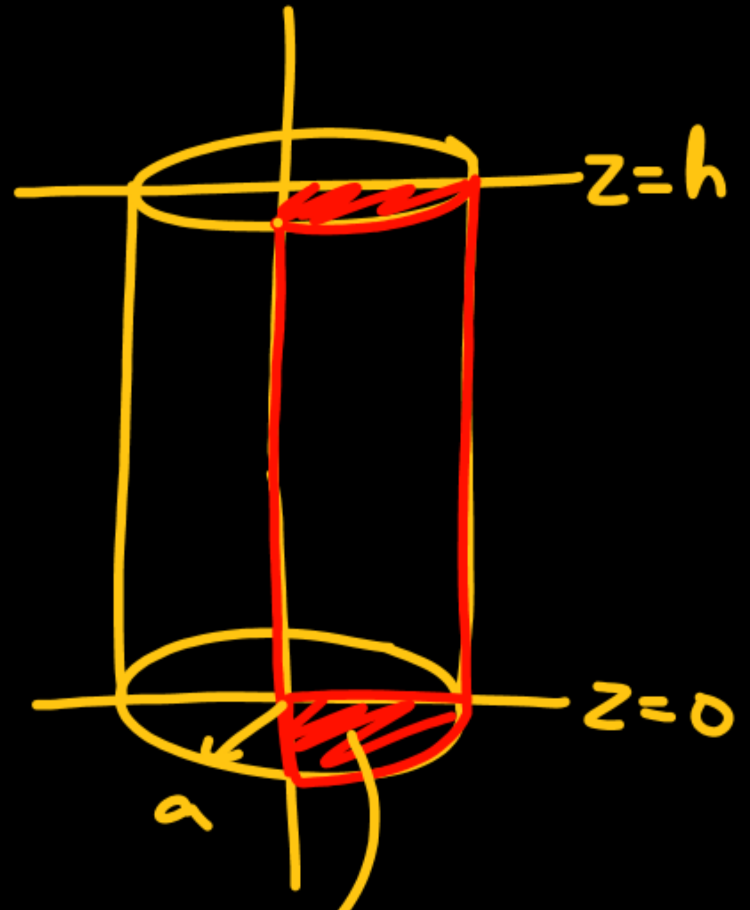
D None of these

$$V = 4\int_A z\,dx\,dy$$

$$= 4\int_{x=0}^{x=0}\int_{y=0}^{y=\sqrt{a^2-x^2}} z\,dx\,dy$$

$$V = 4h\boxed{\int\int dx\,dy}$$

$$\cancel{4}h\,\frac{\pi a^2}{\cancel{4}} = \pi a^2 h$$



$\pi a^2 h$

Question 13

Consider the triangular region P shown in the figure.

What is $\iint_P xy\,dx\,dy$?

$\frac{x}{2}+\frac{y}{1}=1$

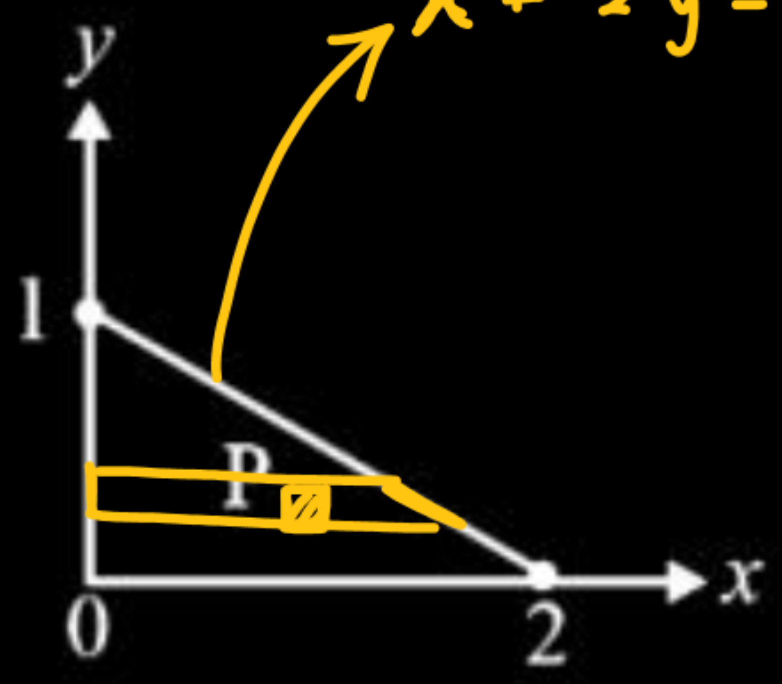
$x+2y=2$

A 1/6

B 2/9

C 7/16

D 1



$$\int_{y=0}^{y=1}\int_{x=0}^{x=2-2y} xy\,dx\,dy$$

$$\int_0^1 y\left[\frac{x^2}{2}\right]_0^{2-2y} dy$$

$$2\int_0^1 (y+y^3-2y^2)\,dy$$

$$2\left[\frac{y^2}{2}+\frac{y^4}{4}-2\frac{y^3}{3}\right]_0^1$$

$$2\left(\frac{1}{2}+\frac{1}{4}-\frac{2}{3}\right)=$$

$$1+\frac{1}{2}-\frac{4}{3}=\frac{1}{6}$$

Question 14

The area enclosed between the curves $y^2 = 4x$ and $x^2 = 4y$ is

A 16/3

B 8

C 32/3

D 16

$y^2 = 4x \quad x^2 = 4y$

$y^2 = 4\sqrt{4y}$

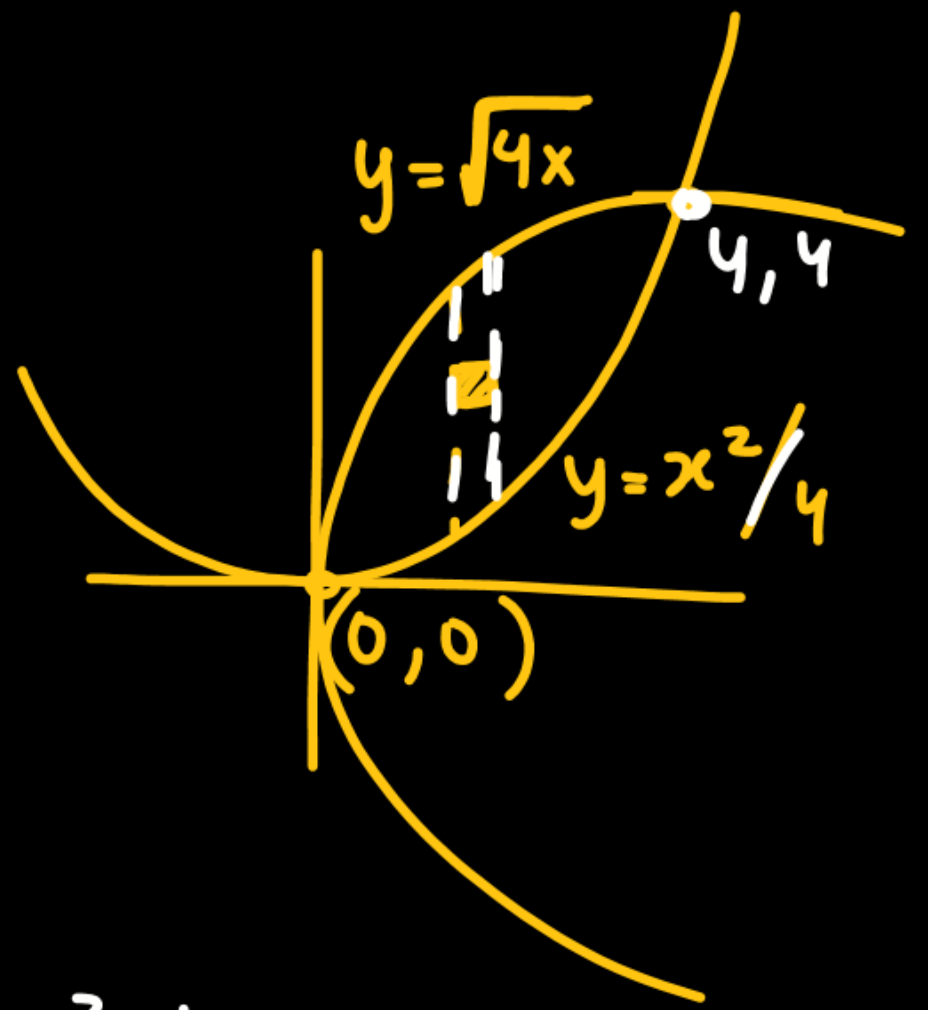
$y^4 = 16 \times 4y$

$y^4 - 64y = 0$

$y(y^3 - 64) = 0$

$y = 0, 4$

$x = 0, 4$



$$A = \int_{x=0}^{x=4} (y_1 - y_2)\,dx = \int_0^4 \sqrt{4x} - \frac{x^2}{4}\,dx$$

$$= \left[2\frac{x^{3/2}}{3/2} - \frac{x^3}{3 \times 4}\right]_0^4 = \frac{16}{3}$$

Thank you

GW Soldiers !