Functional Analysis

Code2Hack

Contents

1 Metric Spaces

1.1 Metric Space

(P18) **Definition 1.1.1 (Metric space,metric)**

A *metric space* is a pair (X, d) where X is a set and $d : X \times X \to R$

1. d is real-valued, finite and nonnegative.
2. $d(x, y) = 0$ iff x=y.
3. $d(x, y) = d(y, x)$
4. $d(x, y) \leq d(x, z) + d(z, y)$ (**Triangle inequality**).

Examples

Example 1.1.6

- $x = (\xi_j)$
- $|\xi_j| \leq c_x$, where c_x is a real number may depend on x, but not on j.
- $d(x, y) = \sup_{j \in N} |\xi_j - \eta_j|$

Example 1.1.7 X is the set of all real-valued functions defined on closed interval $J = [a, b]$ and

$$d(x, y) = \max_{t \in J} |x(t) - y(t)|,$$

1.2 Further Examples of Metric Spaces

(P24)

Example 1.2.1 In contrast with 6,

1.3 Open Set, Closed Set, Neighborhood

Definition 1.3.1 (Ball and sphere)

Definition 1.3.2 (Open set, closed set)
A subset M of a metric space is *open* if it contains a ball about each of its points. A subset K is closed if $K^c = X - K$ is open.

Remark (Topological Space) For the collection of all the open subsets of X called $\mathcal{T}$:

- (T1) $\emptyset \in \mathcal{T}, X \in \mathcal{T}$.
- (T2) The union of **any** members of $\mathcal{J}$ is a member of $\mathcal{T}$.
- (T3) The intersection of **finitely** many members of $\mathcal{T}$ is a member of it.

Definition 1.3.3 (Continuous mapping)
For $X = (X, d)$ *and* $Y = (Y, \tilde{d})$, $T : X \longrightarrow Y$ is continuous at point x_0 if for every $\epsilon > 0$ there's a $\delta > 0$ such that

Theorem 1.3.4 (Continuous mapping) A mapping $T : X \longrightarrow Y$ is continuous iff **the inverse image of any open subset of Y is an open subset of X**.

Definition 1.3.5 (Dense set, separable space)
A subset M of a metric space X is *dense* in X if

$$\bar{M} = X$$

X is *separable* if it has a **countable** subset which is dense in X.

Examples (P37)

1.4 Convergence, Cauchy Sequence, Completeness

Definition 1.4.1 (Convergence of a sequence, limit)

Lemma 1.4.2 (Boundedness, limit) (a) A convergent sequence in X is bounded and its limit is unique.
(b) If $x_n \to x$ and $y_n \to y$ in X, then $d(x_n, y_n) \to d(x, y)$.