Mathematical Logic

Code2Hack

2020 年 1 月 23 日

目录

1 Introduction

1.1 An Example from Group Theory

Axiom 1.1.1: *Group theory*

A *group* is a triple $(G, \circ^G, e^G)$ which satisfies:

- (G1) For all $x, y, z:\quad (x \circ y) \circ z = x \circ (y \circ z)$
- (G2) For all $x:\quad x \circ e = x$
- (G3) For every x there is a y such that $x \circ y = e$

1.2 An Example from the Theory of Equivalence Relations

Axiom 1.2.1: *Theory of equivalence relations*

The pair (A, R^A):

- (E1) xRx.
- (E2) If xRy then yRx.
- (E3) If xRy and yRz then xRz.

1.3 A Preliminary Analysis

A few important points:

1. Is every proposition ψ which **follows from** Φ also **provable from** Φ?

   Remark: *Godel's Completeness Theorem.*

   If not, the axioms are not complete/sufficient?

   Yes, this is exactly the content of *Godel's Completeness Theo-*

rem.

2. The concept *follows from(the Consequence Relation)*.

3. The concept *proof*: *inferences* from the axioms to the proposition.

   - *Connectives.*
   - *Quantifiers.*
   - *Rules.*

1.4 Preview

- Godel's Completeness Theorem forms a bridge between syntax and semantics.(Chapter VI)
- The consistancy of mathematics or a justification of rules of inference.(Chapter VII X)
- Proofs by computers.(Chapter X)
- *Logic programming*: the starting point of AI.(Chapter XI)

2 Syntax of First-Order Languages

2.1 Alphabets

A^* denotes the set of all strings over A.

Lemma 2.1.1: *Countability*

For a nonempty set M, the following are equivalent:

- (a) M is at most countable.
- (b) There is a surjective map $\alpha : \mathbf{N} \to M$.
- (c) There is an injective map $\beta : M \to \mathbf{N}$.

Proof

Hints. p.20

1. (a)→(b): by the definition of countability.
2. (b)→(c): the least index.
3. (c)→(a): by ranking of the images under β.

Remark

- The subset of a countable set is also countable.
- The union of two countable sets is also countable.
- Alphabet could be finite, countable or uncountable(VII.4 p.116).

Lemma 2.1.2: *Countability of $\mathcal{A}^*$*

If alphabet $\mathcal{A}$ is at most countable, then $\mathcal{A}^*$ is countable.

Proof. p.21

1. Define a map $\beta : \mathcal{A}^* \to \mathbf{N}$.
2. Prove β is injective by *the foundamental theorem of arithmetic.*

2.2 The Alphabet of a First-Order Language

First, review the three important concepts:

- Connectives.
- Quantifiers.

- Equality relation.

Definition 2.2.1: *Alphabet of first-order language*

The alphabet of first-order language is $\mathcal{A}_S := \mathcal{A} \cup S$.

The set $\mathcal{A}$:

- (a) $v_0, v_1, \ldots$ (variables);
- (b) $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$;
- (c) $\forall, \exists$;
- (d) $\equiv$;
- (e)),(;

The *symbol set* S, which determines the first-order language.(???)

1. *n-ary relation symbols.*
2. *n-ary function symbols.*
3. a set of constants.

Remark

A first-order language is determined by the symbol set S because S determines the way that terms, formulas are formed(syntax)

2.3 Terms and Formulas in First-Order Languages

Definition 2.3.1: *S-term* *p.23*

- (T1) Every variable.
- (T2) Every constant.
- (T3) If the strings $t_1, \ldots, t_n$ are $S - terms$, then $ft_1 \ldots t_n$ is also an $S - term$.

We denote the set of $S - terms$ by T^S.

Definition 2.3.2: *S-formulas* *p.24*

- (F1) $t_1 \equiv t_2$.
- (F2) $Rt_1 \dots t_n$.
- (F3) If ψ is an $S - formula$, so is $\neg\psi$.
- (F4) If ψ and ϕ are $S - formulas$, so are $(\psi \wedge \phi), (\psi \vee \phi), (\psi \rightarrow \phi), (\psi \leftrightarrow \phi)$.
- (F5) If ψ is an $S - term$ and x is a variable, then $\forall x\psi$ and $\exists x\psi$ are $S - formulas$.

We denote the set of $S - formulas$ by L^S.

Remark: Difference between $\equiv$ and binary relation$R$

Why isn't $\equiv$ included in the binary relations?

Lemma 2.3.3: *Countability of T^S and L^S*

If S is at most countable, then T^S and L^S are countable.

2.4 Induction in the Calculus of Terms and in the Calculus of Formulas

p.27

Definition 2.4.1: *Induction over $\mathfrak{C}$*

Let S be a symbol set and let $Z \subset \mathcal{A}_S^*$ be a set of strings over $\mathcal{A}_S$. To describe the elements of Z by means of a calculus $\mathfrak{C}$, we formally write the rules as:

$$\frac{\zeta_1, \dots, \zeta_n}{\zeta}$$

Example: Rewrite the definition of $S-term$ 2.3.1

Here the set $Z = T^s$.

- (T1) $\frac{}{x}$;
- (T2) $\frac{}{c}$ if $c \in S$;
- (T3) $\frac{t_1,\dots,t_n}{ft_1\dots t_n}$ if $f \in S$ and f is n-ary.

Proof by induction on terms p.28

Proof by induction on formulas p.28

Example

- (a) For all symbol sets S, the empty string $\square$ is neighter an $S-term$ nor an $S-formula$.
- (b) (1) $\circ$ is not an $S_{gr}-term$. (2) $\circ\circ v_1$ is not an $S_{gr}-term$.
- (c) For all symbol sets S, every $S-formula$ contains the same number of the left and right parentheses.

Proof. *p.28*

- (a) Let P be the property on $\mathcal{A}_S^*$ which holds for a string ζ iff ζ is nonempty.
- (b) P on $\mathcal{A}_S^*$ holds for a string ζ over $\mathcal{A}_S$ iff ζ is distinct from (1) $\circ$; (2) $\circ \circ v_1$.
- (c)

Lemma 2.4.2: *Initial segment*

- (a) For all terms t and t', t is not a proper initial segment of t'.
- (b) For all formulas φ and φ', φ is not an initial segement of φ'.

Proof. **p.29**

Proper P holds for a string η iff for all terms t', t' is not a proper initial segement of η and η is not an initial segment of t'.

1. $t = x$ or $t = c$.
2. $t = ft_1 \dots t_n$ and P holds for $t_1, \dots, t_n$.

Question

Is this second-order language? $\forall t \forall t' ((Pt \wedge Pt') \to \neg(Rtt' \vee Rt't)))$

Lemma 2.4.3

- (a) If $t_1, \dots, t_n = t'_1, \dots, t'_m$, then $n = m$ and $t_i = t'_i$ for $1 \le i \le n$.

- (b) The same as (a) for formulas.

Theorem 2.4.4: *Unique Decomposition* *p.30*

Definition 2.4.5: *Two important functions: Var and SF* *p.31*

Var for *variable*

$$\begin{aligned} var(x) &:= \{x\} \\ var(c) &:= \emptyset \\ var(ft_1 \ldots t_n) &:= var(t_1) \cup \ldots var(t_n) \end{aligned}$$

SF for *subformulas*:

$$\begin{aligned} SF(t_1 \equiv t_2) &:= \{t_1 \equiv t_2\} \\ SF(Rt_1 \ldots t_n) &:= \{Rt_1 \ldots t_n\} \\ SF(\neg\epsilon) &:= \{\neg\varphi\} \cup SF(\varphi) \end{aligned}$$

2.5 Free Variables and Sentences

p.32

Definition 2.5.1: *Free variables in an $S-term$*

$$\begin{aligned} free(t_1 \equiv t_2) &:= var(t_1) \cup var(t_2) \\ free(Pt_1 \ldots t_n) &:= var(t_1) \cup \ldots var(t_n) \\ free(\neg\varphi) &:= free(\varphi) \\ free((\varphi * \psi)) &:= free(\varphi) \cup free(psi) for * = \wedge, \neg, \rightarrow, \leftrightarrow \\ free(\forall x\varphi) &:= free(\varphi)\ x \\ free(\exists x\varphi) &:= free(\varphi)\ x \end{aligned}$$

Definition 2.5.2: *Sentences*

Formulas without free variables are called *sentences*

We denote:

$$L_n^S := \{\varphi | \varphi \in L^S \textit{ and } free(\varphi) \subset \{v_0, \dots, v_{n-1}\}\}$$

In particular L_0^S is the set of $S-sentences$.

3 Semantics of First-Order Languages

3.1 Structures and Interpretations

Definition 3.1.1: *S-Structure* $\mathfrak{A}$ *p.36*

An S-structure is a pair $\mathfrak{A} = (A, \mathfrak{a})$:

- A is a nonempty set, the *domain* of $\mathfrak{A}$.
- $\mathfrak{a}$ is a map defined on S satisfying:
  1. for every n-ary relation symbol R on S, $\mathfrak{a}(R)$ is an n-ary relation on A.
  2. for every n-ary function symbol f on S, $\mathfrak{a}(f)$ is an n-ary function on A>
  3. for every constant c in S, $\mathfrak{a}(f)$ is an element of A.

Example:

In arithmetic, two symbol sets:

$$S_{ar} := \{+, \cdot, 0, 1\}$$
$$S_{ar}^< := \{+, \cdot, 0, 1, <\}$$

are just symbols without meaning.

Until we define the domain and the meaning of the symbols and build the structures:

$$\mathfrak{N} := \{\mathbf{N}, +^{\mathbf{N}}, \cdot^{\mathbf{N}}, 0^{\mathbf{N}}, 1^{\mathbf{N}}\}$$
$$\mathfrak{N}^< := \{\mathbf{N}, +^{\mathbf{N}}, \cdot^{\mathbf{N}}, 0^{\mathbf{N}}, 1^{\mathbf{N}}, <^{\mathbf{N}}\}$$

Definition 3.1.2: *Assignment* β *p.36*

An *assignment* in an S-structure is a map $\beta : \{v_n \mid n \in \mathbf{N}\} \to A$.

Definition 3.1.3: *S-interpretation* $\mathfrak{I}$

An S-interpretation is a pair $(\mathfrak{A}, \beta)$.

$\beta\frac{a}{x}$ and $\mathfrak{I}\frac{a}{x}$ p.37

Given $a \in A, x \in S, y \in S$, and β is an assignment in $\mathfrak{A}$. Let $\beta\frac{a}{x}$ be the assignment in $\mathfrak{A}$ which maps x to a and agrees with β on all variables distinct from x:

$$\beta\frac{a}{x}(y) := \begin{cases} \beta(y) & if\ y \neq x \\ a & if\ y = x \end{cases}$$

For $\mathfrak{I} = (\mathfrak{A}, \beta)$ let $\mathfrak{I}\frac{a}{x} := (\mathfrak{A}, \beta\frac{a}{x})$

3.2 Standardization of Connectives

p.38

3.3 The Satisfaction Relation

Definition 3.3.1: $\mathfrak{I}(t)$

- (a) For a variable x, $\mathfrak{I}x := \beta(x)$.
- (b) For a constant c, $\mathfrak{I}(c) := c^{\mathfrak{A}}$.
- (c) For n-ary function symbol $f \in S$, and terms $t_1, \dots, t_n$.

$$\mathfrak{I}(ft_1 \dots t_n) := f^{\mathfrak{A}}(\mathfrak{I}(t_1) \dots \mathfrak{I}(t_n))$$

Definition 3.3.2: *Satisfaction Relation* $\models$

Satisfaction relation is a binary relation between S-interpretation and S-formula, which is a property in induction.
Also we say the left side is the *model* of the right side.

$$\begin{aligned} \mathfrak{I} \models t_1 \equiv t_2 &\ :iff\ \mathfrak{I}(t_1) \equiv \mathfrak{I}(t_2) \\ \mathfrak{I} \models Rt_1 \dots t_n &\ :iff\ R^{\mathfrak{I}} t_1 \dots t_n \\ \mathfrak{I} \models \neg\varphi &\ :iff \text{ not } \mathfrak{I} \models \mathfrak{I}(\varphi) \\ \mathfrak{I} \models \forall x \varphi &\ :iff \text{ for all } a \in A, \mathfrak{I}\frac{a}{x} \models \varphi \end{aligned}$$

Given a set   of formulas, $\mathfrak{I}$ is a model of   iff for all $\varphi \in$, $\mathfrak{I} \models \varphi$.

3.4 The Consequence Relation

Definition 3.4.1: *Consequence Relation* *p.41*

A relation between a set of formulas and one formula. We say φ is a consequence of Φ:

$$\Phi \models \varphi\ :iff$$

Every interpretation which satisfies Φ is also a model of φ.

Definition 3.4.2: *Valid* *p.42*

A formula φ is *valid* ($\models \varphi$) :iff $\emptyset \models \varphi$.

Definition 3.4.3: *Satisfiable*

A formula φ is satisfiable(written Sat φ) :iff there is an interpretation which is a model of φ.
A set of formulas Φ is satisfiable: Sat Φ iff there is an interpretation which is a model of all the formulas in Φ.

Lemma 3.4.4

For all Φ and all φ. $\Phi \vDash \varphi$ *iff not* Sat $\Phi \cup \{\neg\varphi\}$.

Definition 3.4.5: *Logically equivalent*

Logically equivalant :iff $\varphi \vDash \psi$ and $\psi \vDash \varphi$

Example: Logically equivalant

$$\begin{aligned} \varphi \wedge \psi \quad & \textit{and} \quad \neg(\neg\varphi \vee \neg\psi) \\ \varphi \rightarrow \psi \quad & \textit{and} \quad \neg\varphi \vee \psi \\ \varphi \leftrightarrow \psi \quad & \textit{and} \quad \neg(\varphi \wedge \psi) \vee \neg(\neg\varphi \vee \neg\psi) \\ \forall x\varphi \quad & \textit{and} \quad \neg\exists x \neg\varphi \end{aligned}$$

We can dispense with the connectives $\wedge, \rightarrow, \leftrightarrow, \forall$ with a map * by induction on formulas, which associates each formula φ with an equivalent formula φ^* which does not contain$\wedge, \rightarrow, \leftrightarrow, \forall$. p.42

Lemma 3.4.6: *Coincidence Lemma* *p.43*

Let $\mathfrak{I}_1 = (\mathfrak{A}_1, \beta_1)$ be an $S_1 - interpretation$, let $\mathfrak{I}_2 = (\mathfrak{A}_2, \beta_2)$ be an $S_2 - interpretation$, both with the same domain $A_1 = A_2 = A$. Consider the symbol set $S = S_1 \cap S_2$.

1. Let t be an $S{-}term$. If $\mathfrak{I}_1$ and $\mathfrak{I}_2$ agree on the $S{-}symbols(f, g, c)$ and the variables occurring in t, then $\mathfrak{I}_1(t) = \mathfrak{I}_2(t)$.

2. Let φ be an $S - formula$. If $\mathfrak{I}_1$ and $\mathfrak{I}_2$ agree on the $S - symbols$ and the variables occurring free in φ, then $\mathfrak{I}_1 \vDash \varphi \; iff \; \mathfrak{I}_2 \vDash \varphi$.

Proof p.44

For $\varphi = \exists x\psi, \mathfrak{I}_1 \models \varphi$

iff there is an $a \in A_1$ such that $\mathfrak{I}_1\frac{a}{x} \models \psi$

Because $\mathfrak{I}_1$ and $\mathfrak{I}_2$ agree on $free(\varphi)$ and, $free(\psi) \subset free(\varphi) \cup \{x\}$. Thus $\mathfrak{I}_1\frac{a}{x}$ and $\mathfrak{I}_2\frac{a}{x}$ agree on S and $free(\psi)$.

iff there is an $a \in A_2$ such that $\mathfrak{I}_2\frac{a}{x} \models \psi$.

iff $\mathfrak{I}_2 \models \exists x\varphi$.

Definition 3.4.7: *S-reduct and expansion* *p.45*

Let S and S' be symbol sets such that $S \subset S'$. Let $\mathfrak{A} = (A, \mathfrak{a})$ be an $S - structure$ and $\mathfrak{A}' = (A', \mathfrak{a}')$ be an $S' - structure$. We say:

$\mathfrak{A}$ is a *reduct* of $\mathfrak{A}'$, or conversely *expansion* iff

$A = A'$ and $\mathfrak{a}$ and $\mathfrak{a}'$ agree on S. Writen

$$\mathfrak{A} = \mathfrak{A}'|_S$$

Example

$$\mathfrak{R} = \mathfrak{R}^{<}|_{S_{ar}}$$

Check 3.1.1

Remark

With the *coincidence lemma 3.4.6*, the symbol set S need not be fixed when discussing interpretation, consequence and satisfiablility.

Theorem 3.4.8: *p.45*

Φ is satisfiable with respect to S iff Φ is satisfiable with respect to S'.

3.5 Two Lemmas on the Satisfaction Relation

Definition 3.5.1: *Isomorphism* *p.47*

Let $\mathfrak{A}$ and $\mathfrak{b}$ be the $S-structures$. A map $\pi : A \to B$ is called an *isomorphism* of $\mathfrak{A}$ onto $\mathfrak{B}$ (written: $\mathfrak{A} \cong \mathfrak{B}$)

1. π is a bijection of A onto B.
2. For n-ary $R \in S$ and $a_1, \dots, a_n \in A$.

$$R^{\mathfrak{A}} a_1 \dots a_n \quad iff \quad R^{\mathfrak{B}} \pi(a_1) \dots \pi(a_n)$$

3. For n-ary $f \in S$ and $a_1, \dots, a_n \in S$.

$$\pi(f^{\mathfrak{A}}(a_1, \dots, a_n)) = f^{\mathfrak{B}}(\pi(a_1), \dots, \pi(a_n))$$

4. For $c \in S$, $\pi c^{\mathfrak{A}} = c^{\mathfrak{B}}$.

$\mathfrak{A}$ and $\mathfrak{B}$ are said to be *isomorphic* iff there is an isomorphism $\pi : \mathfrak{A} \cong \mathfrak{B}$.

Example

The $S_{gr}-structure$ $(\mathbf{N}, +, 0)$ is isomorphic to the $S_{gr}-structure$ $(G, +^G, 0)$, where G consists of the even natural numbers. The map $\pi : \mathbf{N} \to G$ with $\pi(n) = 2n$ is an isomporphism. Check 1.1.1

Lemma 3.5.2: *Isomorphism Lemma* *p.47*

If $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic $S-structures$, with every assignment β in $\mathfrak{A}$ we associate the assignment $\beta^\pi := \pi \circ \beta$ in $\mathfrak{B}$, and for the corresponding interpretations $\mathfrak{I} = (\mathfrak{A}, \beta)$ and $\mathfrak{I}^\pi = (\mathfrak{B}, \beta^\pi)$, we have:

1. For every $S-term$ t,

$$\pi(\mathfrak{I}(t)) = \mathfrak{I}^\pi(t)$$

2. For every $S-formula$ φ,

$$\mathfrak{I} \models \varphi \quad iff \quad \mathfrak{I}^\pi \models \varphi$$

In particular, for all $S-formulas$ φ

$$\mathfrak{A} \models \varphi \quad iff \quad \mathfrak{B} \models \varphi.$$

Remark

To discuss the *structure* differences(assignment not considered),

- The *coincidence lemma 3.4.6*: same domain A, different S, with the same $\mathfrak{a}$ and β(agreement on the intersection of S);
- The *Isomorphism lemma 3.5.2*: different domain A and B, the same S, also with the different $\mathfrak{a}$ and $\mathfrak{b}$. The bridge is isomorphism.

Both focus on the topic of *satisfaction, validity and consequence.*

Corollary 3.5.3: *p.48*

If $\pi : \mathfrak{A} \cong \mathfrak{B}$, then for $\varphi \in L_n^S$, *and* $a_0, \ldots, a_{n-1}$

$$\mathfrak{A} \vDash \varphi[a_0, \ldots, a_{n-1}] \quad iff \quad \mathfrak{B} \vDash \varphi[\pi(a_0), \ldots, \pi(a_{n-1})]$$

Isomorphic structures cannot be disguinshed in L_0^S. Conversely,

Example

The $S_{ar} - structure$

$$(\mathbf{Q}, +, 0) \vDash \forall v_0 \exists v_1 \; v_1 + v_1 \equiv v_0$$

While in the integers this sentence no longer holds

$$not \quad (\mathbf{Z}, +, 0) \vDash \forall v_0 \exists v_1 \; v_1 + v_1 \equiv v_0$$

j In this case, sentences might not hold when passing to substructures.

Definition 3.5.4: *Substructure* *p.49*

Let $\mathfrak{A}$ and $\mathfrak{B}$ be $S - structures$. We say $\mathfrak{A} \subset \mathfrak{B}$ if
(a) $A \subset B$. (b)

1. For n-ary $R \in S$, for all $a_1, \ldots, a_n \in A$, $R^{\mathfrak{A}} a_1 \ldots a_n \quad iff \quad R^{\mathfrak{B}} a_1 \ldots a_n$. Written as $R^{\mathfrak{A}} = R^{\mathfrak{B}} \cap A^n$.

2. For n-ary $f \in S$, $f^{\mathfrak{A}}$ is the restriction of $f^{\mathfrak{B}}$ to A^n.

3. For $c \in S, c^{\mathfrak{A}} = c^{\mathfrak{B}}$.

Remark: *S-closed*

If $\mathfrak{A} \subset \mathfrak{B}$, then A is $S-closed$;
Conversely, every $S-closed$ subset $X \subset B$ uniquely generates a substructure named $[X]^{\mathfrak{B}}$.

Lemma 3.5.5: *p.49*

Let $\mathfrak{A}$ and $\mathfrak{B}$ be $S-structures$ with $\mathfrak{A} \subset \mathfrak{B}$. Let $\beta : \{v_n \mid n \in \mathbf{N}\} \to A$ be an assignment in $\mathfrak{A}$.
Then for every $S-term\, t$:

$$(\mathfrak{A}, \beta)(t) = (\mathfrak{B}, \beta)(t);$$

For every quantifier-free $S-formula\, \varphi$:

$$(\mathfrak{A}, \beta) \vDash \varphi \quad iff \quad (\mathfrak{B}, \beta) \vDash \varphi.$$

Definition 3.5.6: *Universal formulas*

- (i) $\frac{}{\varphi}$, if φ is quantifier-free.
- (ii) $\frac{\varphi,\psi}{\varphi*\psi}$. Where $* = \wedge, lor$.
- (iii) $\frac{\varphi}{\forall x \varphi}$.

Lemma 3.5.8: *Substructure Lemma p.50*

Let $\mathfrak{A}$ and $\mathfrak{B}$ be $S-structures$ with $\mathfrak{A} \subset \mathfrak{B}$, and let $\varphi \in L_n^S$ be universal. Then the following holds for all $a_0, \dots, a_{n-1} \in A$:

$$\mathfrak{B} \vDash \varphi[a_0, \dots, a_{n-1}] \quad then \quad \mathfrak{A} \vDash \varphi[a_0, \dots, a_{n-1}]$$

Corollary 3.5.8

If $\mathfrak{A} \subset \mathfrak{B}$, then for every universal sentence φ:

$$If \quad \mathfrak{B} \vDash \varphi \quad then \quad \mathfrak{A} \vDash \varphi.$$

3.6 Some Simple Formalizations

Definition 3.6.1: *Equivalance Relations p.52*

$$\forall x Rxx,$$
$$\forall x \forall y (Rxy \to Ryx),$$
$$\forall x \forall y \forall z ((Rxy \wedge Ryz) \to Rxz).$$

See 1.2.1

Example: continuity p.52

Example: Cardinality Statements p.53

Example:The Theory of Orderings p.53

A structure $\mathfrak{A} = (A, <^{\mathfrak{A}})$ is called an *ordering* if:

$$\mathfrak{A} \vDash \Psi_{ord} \begin{cases} \forall x \neg x < x, \\ \forall x \forall y \forall z ((x < y) \wedge (y < z) \to (x < z)), \\ \forall x \forall y (x < y \vee x \equiv y \vee y < x) \end{cases}$$

Partially defined ordering:

The Theory of Fields p.54

The Theory of Graphs

3.7 Some Remarks on Formalizability

Partial Functions p.55

e.g. division in $\mathbf{R}$

Many-Sorted Structures p.55

e.g. the structure with two domains: scalar and vector space.

Limits of Formalizability p.57

1. *Torsion Groups.*

$$\forall x(x \equiv e \vee x \circ x \equiv e \vee (x \circ x) \circ x \equiv e \vee \dots)$$

   This infinite formula cannot be formed by first-order language.

2. *Peano's Axioms*

Dedekind's Theorem

3.8 Substitution

Definition 3.8.1: *Substitution for Terms* *p.60*

1. Variables.

$$x\frac{t_0 \dots t_r}{x_0 \dots x_r} := \begin{cases} x & if\ x \neq x_0, \dots, x_r \\ t_i & if\ x = x_i \end{cases}$$

2. Constants.

$$c\frac{t_0 \dots t_r}{x_0 \dots x_r} := c.$$

3. Functions.

Definition 3.8.2: *Substitution for Formulas* *p.60*

(e) Suppose $x_{i_1}, \dots, x_{i_s}$ are exactly the variables x_i among $x_0, \dots, x_r$, such that

$$x_i \in free(\exists x\varphi) \text{ and } x_i \neq t_i$$

then set

$$[\exists\varphi]\frac{t_0, \dots, t_r}{x_0, \dots, t_r} := \exists u\frac{t_{i_1} \dots t_{i_s} u}{x_{i_1} \dots x_{i_s} x}$$

Where u is the variable x if x does not occur in $t_{i_1} \dots t_{i_s}$; Otherwise u is the first variable in $v_0, \dots$ which does not occur in $\varphi, t_{i_1}, \dots, t_{i_s}$.

Definition 3.8.3: $\mathfrak{I}\frac{a}{x}$

Let $\mathfrak{I} = (\mathfrak{A}, \beta)$ be an interpretation, $x_0, \dots, x_r$ be pairwise distinct and $a_0, \dots, a_r \in A$.

$$\beta\frac{a_0 \dots a_r}{x_0 \dots x_r}(y) := \begin{cases} \beta(y) & if\ y \neq x_i (0 \leq i \leq r) \\ a_i & if\ y = x_i \end{cases}$$

$$\mathfrak{I}\frac{a_0 \dots a_r}{x_0 \dots x_r} := (\mathfrak{A}, \beta a_0 \dots a_r x_0 \dots x_r)$$

Lemma 3.8.4: *Substitution Lemma* *p.61*

(a) For every term t:

$$\mathfrak{I}\left(t\frac{t_0 \dots t_r}{x_0 \dots x_r}\right) = \mathfrak{I}\frac{\mathfrak{I}(t_0) \dots \mathfrak{I}(t_r)}{x_0 \dots x_r}(t)$$

(b) For every formula φ:

$$\mathfrak{I} \models \varphi\frac{t_0 \dots t_r}{x_0 \dots x_r} \quad iff \quad \mathfrak{I}\frac{\mathfrak{I}(t_0) \dots \mathfrak{I}(t_r)}{x_0 \dots x_r} \models \varphi$$

Remark

语法替换和语义替换是等价的?

Lemma 3.8.5: *p.62*

(a) For every permutation π of $0, \dots, r$

$$\varphi\frac{t_0 \dots t_r}{x_0 \dots x_r} = \varphi\frac{t_{\pi(0)} \dots t_{\pi(r)}}{x_{\pi(0)} \dots x_{\pi(r)}}$$

(b)

(c)

Corollary 3.8.6: *p.63*

Suppose $free(\varphi) \subset \{x_0, \dots, x_r\}$, and $x_0, \dots, x_r$ are distinct. Then, for terms $t_0, \dots, t_r$ that $var(t_i) \subset \{v_0, \dots, v_{n-1}\}$, we have

$$\varphi\frac{t_0 \dots t_r}{x_0 \dots x_r} \in L_n^S.$$

In particular, $\varphi\frac{c_0 \dots c_r}{x_0 \dots x_r}$ is a sentence.

Definition 3.8.7: *Rank of a formula*

$$
\begin{aligned}
rk(\varphi) &:= 0 \quad if \varphi \text{ is atomic.}\\
rk(\neg\varphi) &:= rk(\varphi) + 1\\
rk(\varphi \vee \psi) &:= rk(\varphi) + rk(\psi) + 1\\
rk(\exists x \varphi) &:= rk(\varphi) + 1
\end{aligned}
$$

Lemma 3.8.8: *Substitution and rank* *p.64*

$$
rk\left(\varphi \frac{t_0 \dots t_r}{x_0 \dots x_r}\right) = rk(\varphi)
$$

4 A Sequent Calculus

p.65

Some basic concepts

If S is a symbol set and Φ is a set of $S - sentences$(axioms?).

1. We note $\Phi^{\models}$ as the set of S-sentences which are consequences of Φ.
2. Whether every sentence in $\Phi^{\models}$ can be proved from the axioms in Φ?
3. Formal proofs can be regarded as syntactic operations on strings of symbols. Thus obtaining a *calculus* $\mathfrak{S}$.
4. Formally provable(syntactic) v.s. Consequence(semantic).

4.1 Sequent Rules

Terminologies p.66

- *Sequent: a nonempty set of formulas* $\varphi_1, \dots \varphi_n$*. Abbreviated as* $\Gamma, \Delta, \dots$.
- *Antecedent:* $\varphi_1, \dots, \varphi_n$.
- *Succedent:* φ.
- *Sequent calculus* $\mathfrak{S}$*: rules to prove.*
- *Derivable: formally provable.* $\vdash \Gamma\varphi$.

Definition 4.1.1: *Derivable*

A formula φ is *derivable(formally provable)* from a set Φ of formulas(written: $\Phi \vdash \varphi$)
iff
There are finitely many formulas $\varphi_1, \dots, \varphi_n$ in Φ that $\vdash \varphi_1 \dots \varphi_n \varphi$.

Derivable v.s. Correct

A sequent $\Gamma\varphi$ is correct if $\Gamma \vDash \varphi$, which is semantic and different from derivable.

4.2 Structural Rules and Connective Rules

Structure Rules p.68

Antecedent Rule(Ant)

$$\frac{\Gamma \quad \varphi}{\Gamma' \quad \varphi} \quad if \quad \Gamma \subset \Gamma'$$

Assumption Rule(Assm)

$$\frac{}{\Gamma \quad \varphi} \quad if \quad \varphi \in \Gamma$$

Connective Rules

Proof by Cases Rule(PC)

$$\frac{\begin{array}{lll}\Gamma & \psi & \varphi \\ \Gamma & \neg\psi & \varphi\end{array}}{\begin{array}{lll}\Gamma & & \varphi\end{array}}$$

Contradiction Rule(Ctr)

$$\frac{\begin{array}{lll}\Gamma & \neg\varphi & \psi \\ \Gamma & \neg\varphi & \neg\psi\end{array}}{\begin{array}{lll}\Gamma & & \varphi\end{array}}$$

$\vee$-Rule for the Antecedent($\vee$A)

$$\frac{\begin{array}{lll}\Gamma & \varphi & \chi \\ \Gamma & \psi & \chi\end{array}}{\begin{array}{lll}\Gamma & (\varphi \vee \psi) & \chi\end{array}}$$

$\vee$-Rules for the Succedent($\vee$S)

(a) $\dfrac{\Gamma \quad \varphi}{\Gamma \quad (\varphi \vee \psi)}$

(b) $\dfrac{\Gamma \quad \varphi}{\Gamma \quad (\psi \vee \varphi)}$

4.3 Derivable Connective Rules

Derivable Connective Rules **p.70**

Second Contradiction Rule(**Ctr'** p.70)

$$\frac{\begin{array}{ll}\Gamma & \psi \\ \Gamma & \neg\psi\end{array}}{\begin{array}{ll}\Gamma & \varphi\end{array}}$$

Chain Rule(Ch p.70)

$$\frac{\begin{array}{lll}\Gamma & & \varphi \\ \Gamma & \varphi & \psi\end{array}}{\begin{array}{lll}\Gamma & & \psi\end{array}}$$

Contraposition Rules(Cp p.70)

$$\frac{\begin{array}{lll}\Gamma & \varphi & \psi\end{array}}{\begin{array}{lll}\Gamma & \neg\psi & \neg\varphi\end{array}}$$

Noname? p.71

$$\frac{\begin{array}{ll}\Gamma & (\varphi \vee \psi) \\ \Gamma & \neg\varphi\end{array}}{\begin{array}{ll}\Gamma & \psi\end{array}}$$

"Modus ponens".

$$\frac{\begin{array}{ll}\Gamma & (\varphi \to \psi) \\ \Gamma & \varphi\end{array}}{\begin{array}{ll}\Gamma & \psi\end{array}}$$

4.4 Quantifier and Equality Rules

Quantifier Rules

$\exists$**S**(p.72)

$$\frac{\begin{array}{ll}\Gamma & \varphi\frac{t}{x}\end{array}}{\begin{array}{ll}\Gamma & \exists x\varphi\end{array}}$$

Correctness.

3.8.4(substitution lemma)

3.3.2(definition of satisfaction)

$\exists$**A**(p.72)

$$\frac{\begin{array}{lll}\Gamma & \varphi\frac{y}{x} & \psi\end{array}}{\begin{array}{lll}\Gamma & \exists x\varphi & \psi\end{array}}$$

If y is not free in $\Gamma\ \exists x\varphi\ \psi$.

Correctness.

3.4.6(Coincidence Lemma)

Equality Rules p.73

Reflexivity Rule for Equality($\equiv$).

$$\frac{}{\Gamma \quad t \equiv t}$$

Substitution Rule for Equality(Sub)

$$\frac{\Gamma \qquad \varphi\frac{t}{x}}{\Gamma \quad t \equiv t' \quad \varphi\frac{t'}{x}}$$

4.5 Further Derivable Rules and Sequents

p.74

$$(a)\frac{\Gamma \quad \varphi}{\Gamma \quad \exists x\varphi}$$

$$(b)\frac{\Gamma \quad \varphi \quad \psi}{\Gamma \quad \exists x\varphi \quad \psi}$$

if x is not free in $\Gamma\psi$