Homework 2

7、作形如 $f(x)=\dfrac{x}{a+bx}$ 的拟合函数

x_i	2.1	2.5	2.8	3.2
y_i	0.6087	0.6849	0.7368	0.8111

① 预处理：（课本P54） （这里 $m=4$） 改作业的时候看到有3种Q的构造方法，我认为都是合理的

1° $y=\dfrac{x}{a+bx} \Rightarrow \dfrac{1}{y}=a\cdot\dfrac{1}{x}+b$ 令 $\tilde{y}_i=\dfrac{1}{y_i}$ $\tilde{x}_i=\dfrac{1}{x_i}$

$$Q \triangleq \sum_{i=1}^{m}\left(a\tilde{x}_i+b-\tilde{y}_i\right)^2$$

$$\begin{cases} \dfrac{\partial Q}{\partial a}=2\sum\limits_{i=1}^{m}(a\tilde{x}_i+b-\tilde{y}_i)\cdot\tilde{x}_i & (1) \\ \dfrac{\partial Q}{\partial b}=2\sum\limits_{i=1}^{m}(a\tilde{x}_i+b-\tilde{y}_i) & (2) \end{cases}$$

Hessian $H=\begin{pmatrix} \dfrac{\partial^2 Q}{\partial a^2} & \dfrac{\partial^2 Q}{\partial a\partial b} \\ \dfrac{\partial^2 Q}{\partial b\partial a} & \dfrac{\partial^2 Q}{\partial b^2}\end{pmatrix}=2\begin{pmatrix}\sum\limits_{i=1}^{m}\tilde{x}_i^2 & \sum\limits_{i=1}^{m}\tilde{x}_i \\ \sum\limits_{i=1}^{m}\tilde{x}_i & m\end{pmatrix}$

[其正定性可由定理："实对称方阵 S 正定当且仅当所有顺序主子式大于0"
只需验证 ① $\sum\limits_{i=1}^{m}\tilde{x}_i^2>0$ ② $m\left(\sum\limits_{i=1}^{m}\tilde{x}_i^2\right)-\left(\sum\limits_{i=1}^{m}\tilde{x}_i\right)^2=\left(\sum\limits_{i=1}^{m}1^2\right)\left(\sum\limits_{i=1}^{m}\tilde{x}_i^2\right)-\left(\sum\limits_{i=1}^{m}\tilde{x}_i\right)^2>0$]
（Cauchy-Schwartz 不等式）

H 对称正定 故令 (1) = (2) = 0

$$\Rightarrow \begin{cases}\left(\sum\limits_{i=1}^{m}\tilde{x}_i^2\right)a+\left(\sum\limits_{i=1}^{m}\tilde{x}_i\right)b=\sum\limits_{i=1}^{m}\tilde{x}_i\tilde{y}_i \\ \left(\sum\limits_{i=1}^{m}\tilde{x}_i\right)a+mb=\sum\limits_{i=1}^{m}\tilde{y}_i\end{cases} \Rightarrow \begin{pmatrix}0.6120 & 1.5458 \\ 1.5458 & 4\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}2.2363\\ 5.6930\end{pmatrix}$$

$b=\dfrac{1}{m}\sum\limits_{i=1}^{m}\tilde{y}_i-\left(\dfrac{1}{m}\sum\limits_{i=1}^{m}\tilde{x}_i\right)\cdot a$

Cramer $\Longrightarrow$

$$\begin{cases} a=\dfrac{m\left(\sum\limits_{i=1}^{m}\tilde{x}_i\tilde{y}_i\right)-\left(\sum\limits_{i=1}^{m}\tilde{y}_i\right)\left(\sum\limits_{i=1}^{m}\tilde{x}_i\right)}{m\left(\sum\limits_{i=1}^{m}\tilde{x}_i^2\right)-\left(\sum\limits_{i=1}^{m}\tilde{x}_i\right)\left(\sum\limits_{i=1}^{m}\tilde{x}_i\right)}=2.4867 \\ b=\dfrac{\left(\sum\limits_{i=1}^{m}\tilde{x}_i^2\right)\left(\sum\limits_{i=1}^{m}\tilde{y}_i\right)-\left(\sum\limits_{i=1}^{m}\tilde{x}_i\right)\left(\sum\limits_{i=1}^{m}\tilde{x}_i\tilde{y}_i\right)}{m\left(\sum\limits_{i=1}^{m}\tilde{x}_i^2\right)-\left(\sum\limits_{i=1}^{m}\tilde{x}_i\right)\left(\sum\limits_{i=1}^{m}\tilde{x}_i\right)}=0.4623\end{cases}$$

2° $y=\dfrac{x}{a+bx} \Rightarrow \underline{x-ay-bxy=0} \Rightarrow Q \hat{=} \sum_{i=1}^{m}(x_i-ay_i-bx_iy_i)^2$

$$\begin{cases} \dfrac{\partial Q}{\partial a} = 2\sum_{i=1}^{m}(x_i-ay_i-bx_iy_i)(-y_i) & (1) \\ \dfrac{\partial Q}{\partial b} = 2\sum_{i=1}^{m}(x_i-ay_i-bx_iy_i)(-x_iy_i) & (2) \end{cases}$$

Hessian : $H = \begin{pmatrix} \dfrac{\partial^2 Q}{\partial a^2} & \dfrac{\partial^2 Q}{\partial a\partial b} \\ \dfrac{\partial^2 Q}{\partial b\partial a} & \dfrac{\partial^2 Q}{\partial b^2} \end{pmatrix} = 2\begin{pmatrix} \sum_{i=1}^{m} y_i^2 & \sum_{i=1}^{m} x_iy_i^2 \\ \sum_{i=1}^{m} x_iy_i^2 & \sum_{i=1}^{m} x_i^2y_i^2 \end{pmatrix}$.

正定性的证明同上.

H 对称正定, 故令 (1) = (2) = 0

$$\Rightarrow \begin{cases} \left(\sum_{i=1}^{m} y_i^2\right)a + \left(\sum_{i=1}^{m} x_iy_i^2\right)b = \sum_{i=1}^{m} x_iy_i \\ \left(\sum_{i=1}^{m} x_iy_i^2\right)a + \left(\sum_{i=1}^{m} x_i^2y_i^2\right)b = \sum_{i=1}^{m} x_i^2y_i \end{cases} \Rightarrow \begin{pmatrix} 2.0404 & 5.5761 \\ 5.5761 & 15.5586 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 7.6491 \\ 21.0472 \end{pmatrix}$$

Cramer $\Rightarrow$
$$\begin{cases} a = \dfrac{\left(\sum_{i=1}^{m} x_i^2y_i\right)\left(\sum_{i=1}^{m} x_iy_i^2\right) - \left(\sum_{i=1}^{m} x_iy_i\right)\left(\sum_{i=1}^{m} x_i^2y_i^2\right)}{\left(\sum_{i=1}^{m} x_iy_i^2\right)\left(\sum_{i=1}^{m} x_iy_i^2\right) - \left(\sum_{i=1}^{m} y_i^2\right)\left(\sum_{i=1}^{m} x_i^2y_i^2\right)} = 2.5262 \\ b = \dfrac{\left(\sum_{i=1}^{m} x_iy_i^2\right)\left(\sum_{i=1}^{m} x_iy_i\right) - \left(\sum_{i=1}^{m} y_i^2\right)\left(\sum_{i=1}^{m} x_i^2y_i\right)}{\left(\sum_{i=1}^{m} x_iy_i^2\right)\left(\sum_{i=1}^{m} x_iy_i^2\right) - \left(\sum_{i=1}^{m} y_i^2\right)\left(\sum_{i=1}^{m} x_i^2y_i^2\right)} = 0.4474 \end{cases}$$

3° $y=\frac{x}{a+bx} \iff a+bx=\frac{x}{y}$, 令 $z_i=\frac{x_i}{y_i}$

(这个方法可以参考谢同学的过程)

Computational Methods: Homework 2

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7. *Solution.* 令 $z_i = \frac{x_i}{y_i}$, 则求原问题的拟合函数等价于求 z_i 关于 x_i 的拟合函数 $g(x) = ax + b$. 这样我们有:

x_i	2.1	2.5	2.8	3.2
z_i	3.4500	3.6502	3.8002	3.9453

用最小二乘法进行线性拟合, 记 $X = (\boldsymbol{x}, \mathbf{1})$, 系数向量 $\boldsymbol{\beta} = (a, b)^T$, 则此时需要极小化如下函数:

$$\min E(\boldsymbol{\beta}) = (X\boldsymbol{\beta} - \boldsymbol{z})^T(X\boldsymbol{\beta} - \boldsymbol{z})$$

这是个无约束二次优化问题, 对 $\boldsymbol{\beta}$ 求偏导, 得到最优性条件:

$$0 = \frac{\partial E}{\partial \boldsymbol{\beta}} = 2X^T(X\boldsymbol{\beta} - \boldsymbol{z})$$

, 本题中 X^TX 是正定矩阵, 得到所求的 $\boldsymbol{\beta} = (X^TX)^{-1}X^T\boldsymbol{z} = (0.45, 2.51)^T$ 故所求的拟合函数 $f(x) = \frac{x}{0.45x+2.51}$.

8、$I(\ln x)=\int_1^2 \ln x\,dx$，取 $\varepsilon=10^{-4}$，$h=1$，用 Romberg 计算直到 $|R_{k,k}-R_{k-1,k-1}|<\varepsilon$，

并做出 Romberg 积分数值表。

$$\begin{cases} R_{k,j}=R_{k,j-1}+\dfrac{R_{k,j-1}-R_{k-1,j-1}}{4^{j-1}-1} \\ R_{k,1}=\left(R_{k-1,1}+h_{k-1}\sum_{i=1}^{2^{k-2}} f(a+(2i-1)h_k)\right)/2 \end{cases}$$

（P122 T_{2n} 与 T_n 关系）

记 $h_k=\dfrac{h}{2^{k-1}}$

(P125). $R_{1,1}=\dfrac{(\ln(1)+\ln(2))\cdot h}{2}=0.34657359$

$$R_{2,1}=(R_{1,1}+h_1 f(1+h_2))/2=(R_{1,1}+1\cdot\ln(1+0.5))/2=0.376019349$$

$$R_{2,2}=R_{2,1}+\frac{R_{2,1}-R_{1,1}}{4-1}=0.385834602$$

△ $|R_{2,2}-R_{1,1}|=0.039261>10^{-4}$ 继续

$$R_{3,1}=\left(R_{2,1}+h_2\sum_{i=1}^{2} f(1+(2i-1)\cdot h_3)\right)/2=0.383699510$$

$$R_{3,2}=R_{3,1}+\frac{R_{3,1}-R_{2,1}}{4-1}=0.386259563$$

$$R_{3,3}=R_{3,2}+\frac{R_{3,2}-R_{2,2}}{4^2-1}=0.386287893$$

△ $|R_{3,3}-R_{2,2}|=4.53\times10^{-4}>\varepsilon=10^{-4}$ 继续

$$R_{4,1}=\left(R_{3,1}+h_3\sum_{i=1}^{4} f(1+(2i-1)\cdot h_4)\right)/2=0.385643910$$

$$R_{4,2}=R_{4,1}+\frac{R_{4,1}-R_{3,1}}{4-1}=0.386292043$$

$$R_{4,3}=R_{4,2}+\frac{R_{4,2}-R_{3,1}}{4^2-1}=0.386294209$$

$$R_{4,4}=R_{4,3}+\frac{R_{4,3}-R_{3,2}}{4^3-1}=0.386294310$$

△ $|R_{4,4}-R_{3,3}|=6.42\times10^{-6}<\varepsilon=10^{-4}$，停止。

Romberg 积分数值表如下：

0.346573590			
0.376019349	0.385834602		
0.383699510	0.386259563	0.386287893	
0.385643910	0.386292043	0.386294209	0.386294310

10. 用具有3阶代数精度的Gauss-Legendre积分公式计算 $\int_{-3}^{1}(x^5+x)dx$

(P130 ~ P131)

令 $2n-1=3 \Rightarrow n=2$

因此我们用2个积分节点。

经查表6.4 (P131) 可知，$x_1^{(2)}=0.5773502692$　$x_2^{(2)}=-0.5773502692$，$\alpha_1^{(2)}=\alpha_2^{(2)}=1$

由公式 $G_n(f)=\frac{b-a}{2}\sum_{i=1}^{n}\alpha_i^{(n)}f\left(\frac{(a+b)+(b-a)x_i^{(n)}}{2}\right)$，

得 $G_2(x^5+x)=\frac{1-(-3)}{2}\cdot\sum_{i=1}^{2}f\left(\frac{-2+4x_i^{(n)}}{2}\right)$

$\approx -96.888888933611 55$