SNM_HW04

PB19111713钟颖康

4.6.(1)

$Crout$分解:

$$\begin{pmatrix}5 & 1 & 2\\ 1 & 3 & -1\\ 2 & 3 & 5\end{pmatrix} = \begin{pmatrix}l_{11} & & \\ l_{21} & l_{22} & \\ l_{31} & l_{32} & l_{33}\end{pmatrix}\begin{pmatrix}1 & u_{12} & u_{13}\\ & 1 & u_{23}\\ & & 1\end{pmatrix} = \begin{pmatrix}5 & & \\ 1 & 2.8 & \\ 2 & 2.6 & 5.5\end{pmatrix}\begin{pmatrix}1 & 0.2 & 0.4\\ & 1 & -0.5\\ & & 1\end{pmatrix}$$

故

$$\begin{pmatrix}5 & & \\ 1 & 2.8 & \\ 2 & 2.6 & 5.5\end{pmatrix}\begin{pmatrix}1 & 0.2 & 0.4\\ & 1 & -0.5\\ & & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}10\\ 2\\ 15\end{pmatrix}$$

解得

$$\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}1\\ 1\\ 2\end{pmatrix}$$

4.7.(1)

根据题意可列

$$\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & -2\\ 3 & -2 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}-3\\ 10\\ 7\end{pmatrix}$$

$$k=1: d_1 = a_{11} = 1, l_{21} = a_{21}/d_1 = 2, l_{31} = a_{31}/d_1 = 3$$

$$k=2: d_2 = a_{22} - l_{21}^2 d_1 = -3, l_{32} = (a_{32} - l_{31}l_{21}d_1)/d_2 = \frac{8}{3}$$

$$k=3: d_3 = a_{33} - l_{31}^2 d_1 - l_{32}^2 d_2 = \frac{40}{3}$$

$$L = \begin{pmatrix}1 & & \\ 2 & 1 & \\ 3 & \frac{8}{3} & 1\end{pmatrix}, D = \begin{pmatrix}1 & & \\ & -3 & \\ & & \frac{40}{3}\end{pmatrix}$$

由

$$LDL^T X = b, LZ = b, Z = (-3, 16, -\frac{80}{3})^T$$

$$DY = Z, Y = (-3, -\frac{16}{3}, -2)^T$$

$$L^T X = Y, X = (3, 0, -2)^T$$

5.5.(1)

易知方程组的$Gauss-Seidel$迭代格式为

$$\begin{cases} x_1^{k+1} = \frac{1}{5}x_2^{(k)} + \frac{1}{10}x_3^{(k)} \\ x_2^{k+1} = \frac{1}{5}x_1^{(k+1)} + \frac{1}{10}x_3^{(k)} - 2.1 \\ x_3^{k+1} = \frac{1}{5}x_1^{(k+1)} + \frac{2}{5}x_2^{(k+1)} - 4 \end{cases}$$

x_1	x_2	x_3	$\|\|X^{(k+1)} - X^{(k)}\|\|_\infty$
0.000000	0.000000	0.000000	
0.000000	-2.100000	-4.840000	4.840000
-0.904000	-2.764800	-5.286720	0.904000
-1.081632	-2.844998	-5.354326	0.177632
-1.104432	-2.856319	-5.363414	0.022800
-1.107605	-2.857862	-5.364666	0.003173
-1.108039	-2.858074	-5.364838	0.000434
-1.108099	-2.858103	-5.364861	0.000060

故停止迭代时$X^{(k)} = (-1.108099, -2.858103, -5.364861)^T$

5.6

- **(1)**

$$R = I - D^{-1}A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & t \\ t & 2 \end{pmatrix} = \begin{pmatrix} 0 & -t \\ -\frac{t}{2} & 0 \end{pmatrix}$$

  易知特征值为$\lambda = \pm\sqrt{\frac{t^2}{2}} = \pm\frac{t}{\sqrt{2}}$.

  因此迭代矩阵的谱半径$\rho(R) = \frac{|t|}{\sqrt{2}}$

  迭代收敛条件为$\rho(R) < 1$，即$-\sqrt{2} < t < \sqrt{2}$

- **(2)**

$$S = -(D+L)^{-1}U = \left[\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}\right]^{-1} \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -t \\ 0 & \frac{t^2}{2} \end{pmatrix}$$

  易知原始矩阵A的特征值为$\lambda_{1,2} = \frac{3\pm\sqrt{1+4t^2}}{2}$

  当$-\sqrt{2} < t < \sqrt{2}$时特征值均大于0，此时A为正定矩阵，且$Gauss-Seidel$迭代收敛。

5.7

- **(1)**

$$\begin{aligned}R_A &= I_A - D_A^{-1}A_A \\ &= \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}^{-1}\begin{pmatrix}1&2&-2\\1&1&1\\2&2&1\end{pmatrix} \\ &= \begin{pmatrix}0&-2&2\\-1&0&-1\\-2&-2&0\end{pmatrix}\end{aligned}$$

矩阵特征值为0，谱半径$0<1$，故收敛

$$\begin{aligned}S_A &= -(D_A+L_A)^{-1}U_A \\ &= \left[\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + \begin{pmatrix}0&0&0\\1&0&0\\2&2&0\end{pmatrix}\right]^{-1}\begin{pmatrix}0&2&-2\\0&0&1\\0&0&0\end{pmatrix} \\ &= \begin{pmatrix}0&-2&2\\0&2&-3\\0&0&2\end{pmatrix}\end{aligned}$$

矩阵特征值为$0,2,2$，谱半径$2>1$，显然不收敛

- **(2)** 由**(1)**同理可得

$$R_B = \begin{pmatrix}0&\frac{1}{2}&-\frac{1}{2}\\-1&0&-1\\\frac{1}{2}&\frac{1}{2}&0\end{pmatrix}$$

矩阵特征值为$0,\pm\frac{\sqrt{5}}{2}i$，谱半径$\frac{\sqrt{5}}{2}>1$，故不收敛

$$S_B = \begin{pmatrix}0&-2&2\\-1&0&-1\\-2&-2&0\end{pmatrix}$$

矩阵特征值为$0,-0.5,-0.5$，谱半径为$0.5<1$，故收敛