第四次作业.

P95: 6(1)

$$\begin{bmatrix} 5 & 1 & 2 \\ 1 & 3 & -1 \\ 2 & 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ 15 \end{bmatrix}$$ by crout factorization.

$$\left[\begin{array}{ccc|ccc} 5 & 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & -1 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & \frac{1}{5} & \frac{2}{5} \\ 1 & 14/5 & -7/5 & 0 & 1 & 0 \\ 2 & 13/5 & 21/5 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & \frac{1}{5} & \frac{2}{5} \\ 1 & 14/5 & 0 & 0 & 1 & -\frac{1}{2} \\ 2 & 13/5 & \frac{11}{2} & 0 & 0 & 1 \end{array}\right]$$

$A=LU$, 先解 $LY=b$, 得: $y=\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$.

再解 $UX=Y$, 得: $x=\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

P95: 7(1): Use LDL^T factorization to solve:

$$\begin{bmatrix} -6 & 3 & 2 \\ 3 & 5 & 1 \\ 2 & 1 & 6 \end{bmatrix} x = \begin{bmatrix} -4 \\ 11 \\ -8 \end{bmatrix}.$$

解: $d_1=-6$, $L_{21}=a_{21}/d_1=-\frac{1}{2}$, $L_{31}=\frac{a_{31}}{d_1}=-\frac{1}{3}$.

$d_2=a_{22}-L_{21}^2d_1=\frac{13}{2}$, $L_{32}=(a_{32}-L_{31}L_{21}d_1)/d_2=\frac{4}{13}$.

$d_3=a_{33}-L_{31}^2d_1-L_{32}^2d_2=6-\frac{1}{9}(-6)-\frac{16}{169}\cdot\frac{13}{2}=6+\frac{2}{3}-\frac{8}{13}=\frac{236}{39}$.

则: $L=\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{3} & \frac{4}{13} & 1 \end{bmatrix}$. $D=\mathrm{diag}(-6, 13/2, 236/39)$.

① 求 $Lz=b$, 得: $z=\begin{bmatrix} -4 \\ 9 \\ -\frac{472}{39} \end{bmatrix}$.

② $Dy=z$: $y=\begin{bmatrix} \frac{2}{3} \\ 18/13 \\ -2 \end{bmatrix}$

③ $L^Tx=y$: $x=\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

$LDL^Tx=b$

令 $L^Tx=y \Leftrightarrow LDy=b$.

令 $Dy=z \Leftrightarrow Lz=b$

最后用代回法即可。

P108: 5(1): Use Gauss-Seidel solving:

$$\begin{bmatrix}10 & -2 & -1\\ -2 & 10 & -1\\ -1 & -2 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ -21\\ -20\end{bmatrix}.$$ 当 $\|X^{(k+1)}-X^{(k)}\|_\infty<10^{-4}$ 时停止.

解: $Ax=b \Leftrightarrow (L+U+D)x=b \Leftrightarrow (D+L)x=-Ux+b \Rightarrow x=-(D+L)^{-1}Ux+(D+L)^{-1}b$

~~则~~ $(D+L)^{-1}=\begin{bmatrix}10 & 0 & 0\\ -2 & 10 & 0\\ -1 & -2 & 5\end{bmatrix}^{-1}=\begin{bmatrix}\frac{1}{10} & 0 & 0\\ \frac{1}{50} & \frac{1}{10} & 0\\ \frac{7}{250} & \frac{1}{25} & \frac{1}{5}\end{bmatrix}$

则 $(D+L)^{-1}U=\begin{bmatrix}0 & -\frac{1}{5} & -\frac{1}{10}\\ 0 & -\frac{1}{25} & -\frac{3}{25}\\ 0 & -\frac{7}{125} & -\frac{17}{250}\end{bmatrix}$, $(D+L)^{-1}\vec{b}=\begin{bmatrix}0\\ -\frac{21}{10}\\ -\frac{121}{25}\end{bmatrix}$

则: 迭代格式为: $X^{(k+1)}=\begin{bmatrix}0 & \frac{1}{5} & \frac{1}{10}\\ 0 & \frac{1}{25} & \frac{3}{25}\\ 0 & \frac{7}{125} & \frac{17}{250}\end{bmatrix}X^{(k)}+\begin{bmatrix}0\\ -\frac{21}{10}\\ -\frac{121}{25}\end{bmatrix}$

~~取初值为~~ (具体计算略去).

~~$X^{(0)}=\vec{0}$, $X^{(1)}=$~~ Rmk: 实际计算时, 我们不显式形成迭代阵与右端项.

P108: 6: $\begin{bmatrix}1 & t\\ t & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}b_1\\ b_2\end{bmatrix}$.

(1). 求 Jacobi 迭代阵, 并讨论.

(2) 求 GS 迭代阵, 并讨论.

(1): $G=-D^{-1}(L+U)=-\begin{bmatrix}1 & 0\\ 0 & \frac{1}{2}\end{bmatrix}\cdot\begin{bmatrix}0 & t\\ t & 0\end{bmatrix}=\begin{bmatrix}0 & -t\\ -\frac{t}{2} & 0\end{bmatrix}$

$\rho(G)=\frac{|t|}{\sqrt{2}} \Rightarrow$ 收敛 $\Leftrightarrow t\in(-\sqrt{2},\sqrt{2})$.

(2). $G=-(D+L)^{-1}U=-\begin{bmatrix}1 & 0\\ t & 2\end{bmatrix}^{-1}\begin{bmatrix}0 & t\\ 0 & 0\end{bmatrix}=-\begin{bmatrix}1 & 0\\ -\frac{t}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}0 & t\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & -t\\ 0 & \frac{t^2}{2}\end{bmatrix}$

$\rho(G)=\frac{t^2}{2} \Rightarrow$ 收敛 $\Leftrightarrow t\in(-\sqrt{2},\sqrt{2})$.

P108:7: $A=\begin{bmatrix}1&2&-2\\1&1&1\\2&2&1\end{bmatrix}$, $B=\begin{bmatrix}2&-1&1\\1&1&1\\1&1&-2\end{bmatrix}$.

Jacobi: $G_A=\begin{bmatrix}0&-2&2\\-1&0&-1\\-2&-2&0\end{bmatrix}$, $G_B=\begin{bmatrix}0&\frac{1}{2}&-\frac{1}{2}\\-1&0&-1\\\frac{1}{2}&\frac{1}{2}&0\end{bmatrix}$.

$\rho(G_A)=0$

$\rho(G_B)=\frac{\sqrt{5}}{2}>1$.

GS: $G_A=\begin{bmatrix}0&-2&2\\0&+2&-3\\0&0&2\end{bmatrix}$. $G_B=\begin{bmatrix}0&\frac{1}{2}&-\frac{1}{2}\\0&-\frac{1}{2}&-\frac{1}{2}\\0&0&-\frac{1}{2}\end{bmatrix}$

$\rho(G_A)=2$. $\rho(G_B)=\frac{1}{2}$.

补充:

P183: 2(2): $B=\begin{bmatrix}5&3\\-2&0\end{bmatrix}$, $B^{-1}=\begin{bmatrix}0&-\frac{1}{2}\\\frac{1}{3}&\frac{5}{6}\end{bmatrix}$.

$X^{(0)}=\begin{bmatrix}1\\0\end{bmatrix}$, $X^{(1)}=\begin{bmatrix}0\\\frac{1}{3}\end{bmatrix}\to\begin{bmatrix}0\\1\end{bmatrix}$

$X^{(2)}=\begin{bmatrix}-\frac{1}{2}\\\frac{5}{6}\end{bmatrix}\to\begin{bmatrix}-\frac{3}{5}\\1\end{bmatrix}$, $X^{(4)}=\begin{bmatrix}-\frac{1}{2}\\\frac{19}{30}\end{bmatrix}\to\begin{bmatrix}-\frac{15}{19}\\1\end{bmatrix}$

$X^{(5)}=\begin{bmatrix}-\frac{1}{2}\\\frac{65}{114}\end{bmatrix}\to\begin{bmatrix}-\frac{57}{65}\\1\end{bmatrix}$, $X^{(6)}=\begin{bmatrix}-\frac{1}{2}\\\frac{211}{390}\end{bmatrix}\to\begin{bmatrix}-\frac{195}{211}\\1\end{bmatrix}$.

取λ为第1个元之比:

$\lambda=\{0, +\infty, 6/5, 30/19, 114/65, \cdots\}$.

P183: 3(4) $D=\begin{bmatrix}2&-1&3\\-1&5&0\\3&0&1\end{bmatrix}$, 用Jacobi方法求特征值.

(用Givens旋转).

$\left[\begin{array}{ccc|ccc}1&0&0&2&-1&3\\0&1&0&-1&5&0\\0&0&1&3&0&1\end{array}\right]$.

Givens旋转: $\begin{bmatrix}c&+s\\-s&c\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}ca+sb\\-sa+cb\end{bmatrix}=\begin{bmatrix}r\\0\end{bmatrix}$.

(直接消去).

取 $c=\frac{a}{\sqrt{a^2+b^2}}$, $s=\frac{b}{\sqrt{a^2+b^2}}$

$r=\sqrt{a^2+b^2}$.