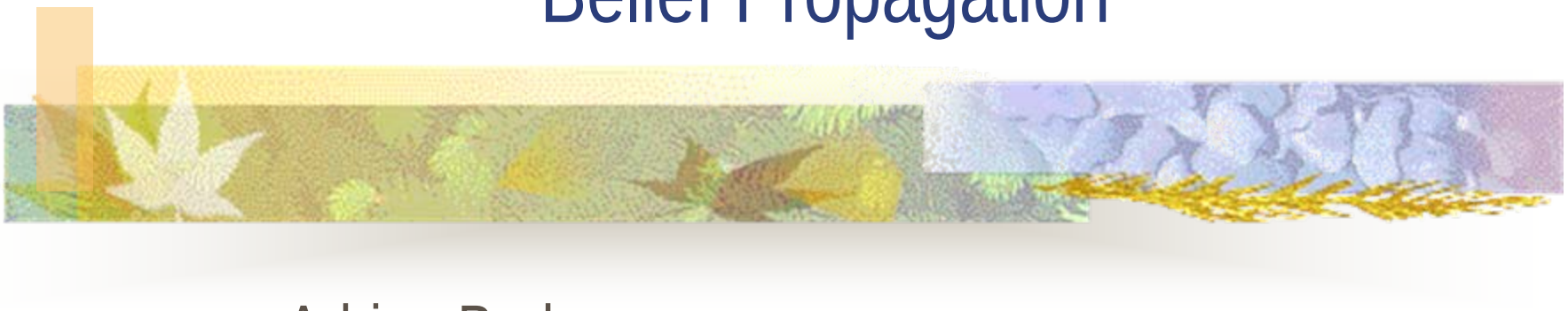
Belief Propagation



Adrian Barbu

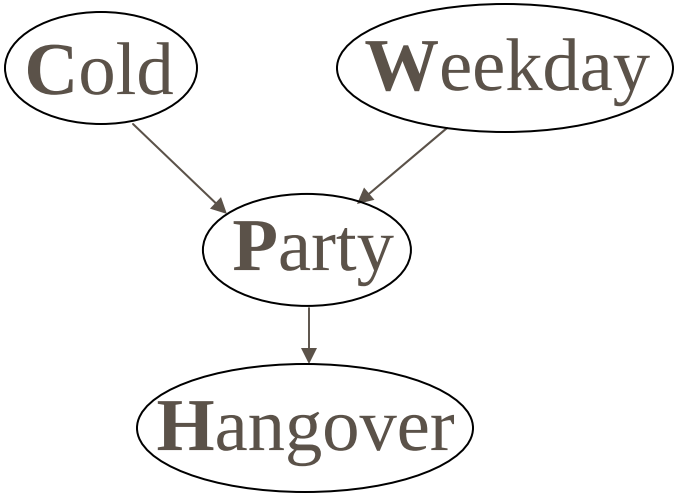
Belief Propagation

- Approximate inference in MRF or Directed Graphical Models
- General form of the probability function

$$P(\mathbf{x}) = P(x_1, ..., x_N) = \frac{1}{Z} \prod_{c \in C} \psi_c(\mathbf{x}_c)$$

- Message Passing Algorithm
- Perform Message Passing
  - Run until convergence
  - Not always converges

Example



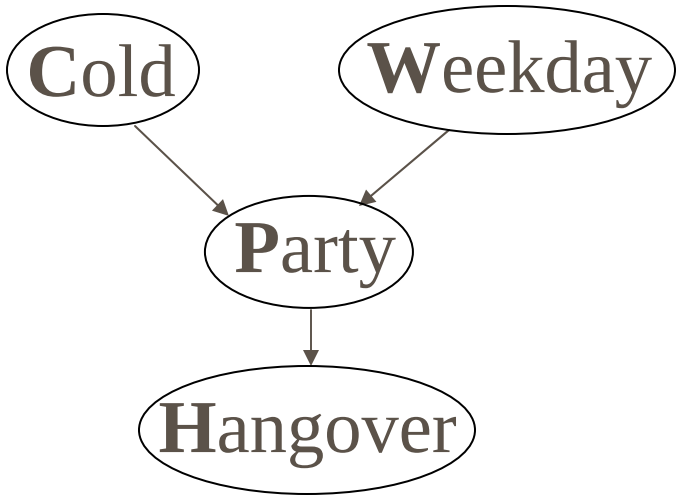
Joint Probability: $p(x_H, x_P, x_C, x_W) =$

$$p(x_H \mid x_P)\, p(x_P \mid x_C, x_W)\, p(x_C)\, p(x_W)$$

Marginal Probability: $p(x_H) = \sum_{x_P, x_C, x_W} p(x_H, x_P, x_C, x_W)$

8-sum

Example (cont.)



Marginal Probability:

$$p(x_H) = \sum_{x_P, x_C, x_W} p(x_H, x_P, x_C, x_W)$$

8-sum

Localize probabilities:

$$p(x_P) = \sum_{x_C, x_W} p(x_P \mid x_C, x_W) p(x_C) p(x_W) \leftarrow 4\text{-sum}$$

$$p(x_H) = \sum_{x_P} p(x_H \mid x_P) p(x_P) \leftarrow 2\text{-sum}$$

Approach

- GM: Define variables and connections

$$x_C, x_W, x_P, x_H; p(x_C), p(x_W), p(x_P \mid x_C, x_W), p(x_H \mid x_P)$$

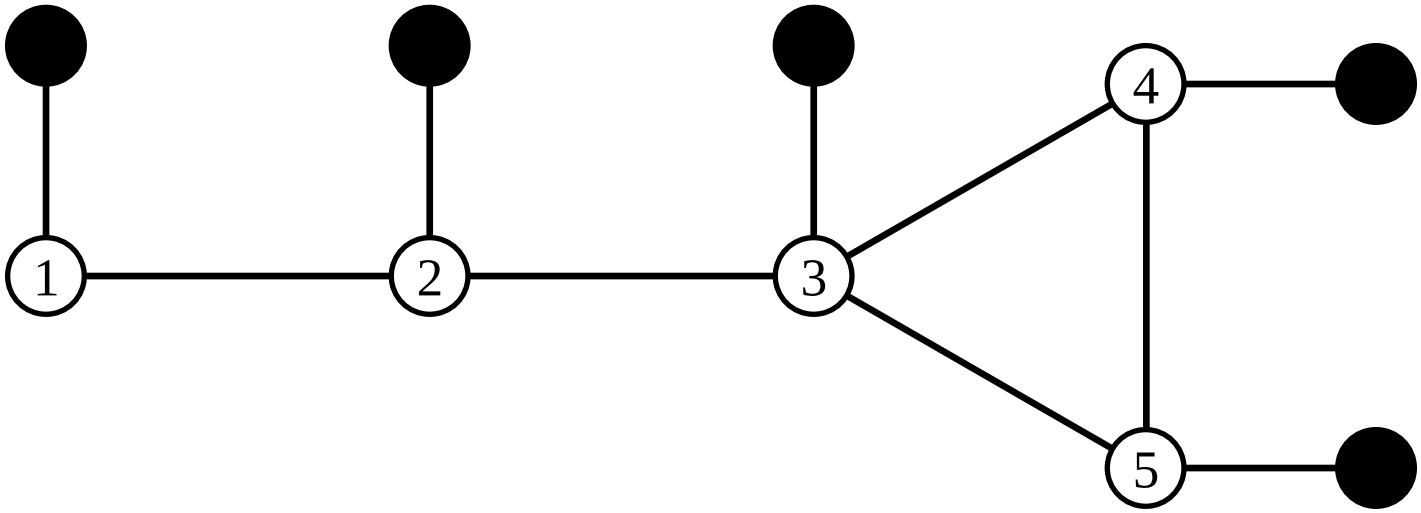
- Calculate marginal probabilities efficiently

$$p(x_H) = \sum_{x_C, x_W, x_P} p(x_C, x_W, x_P, x_H)$$

- Find most likely configuration

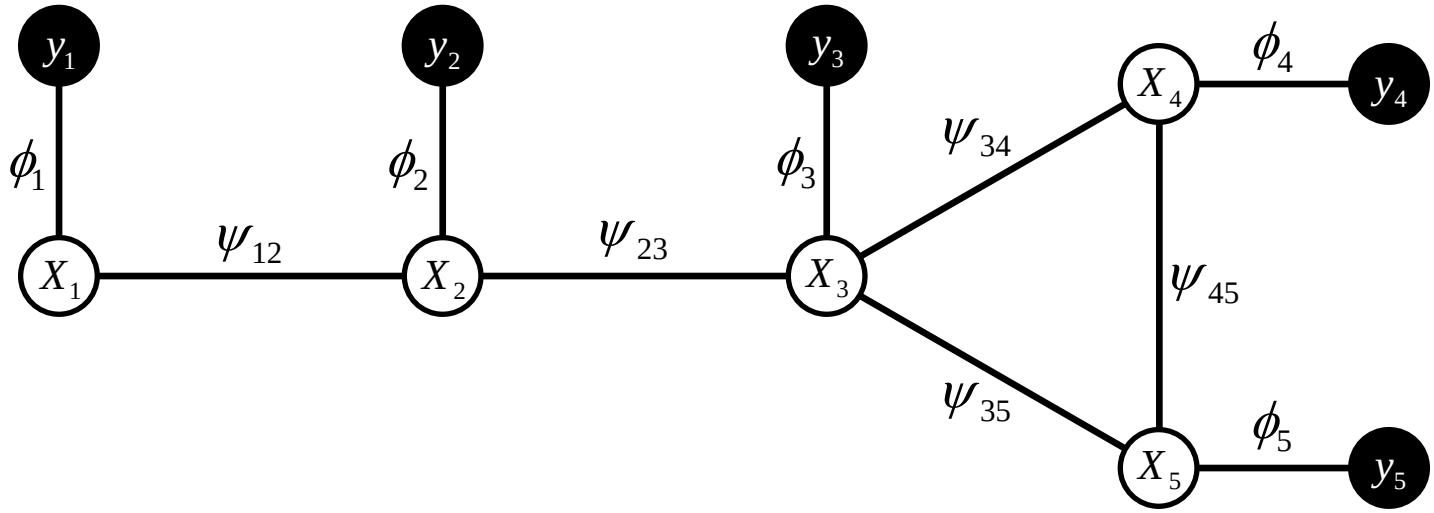
$$\arg\max_{x_H} p(x_H)$$

Pairwise Markov Random Field



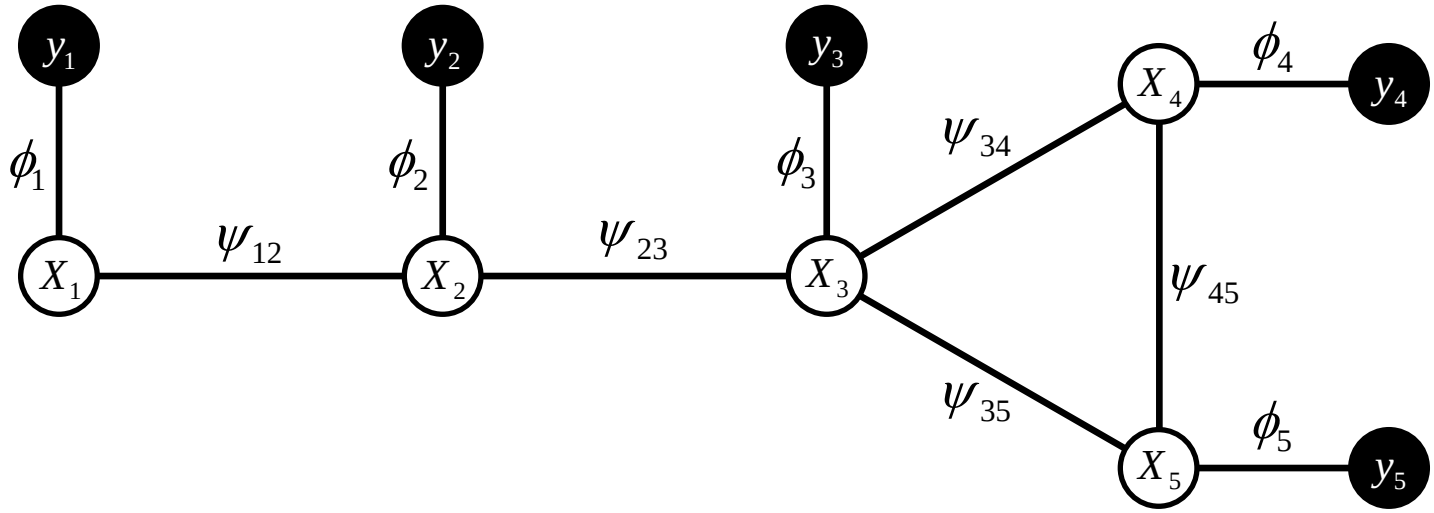
- Basic structure: vertices, edges

Pairwise Markov Random Field



- Basic structure: vertices, edges
- Vertex *i* has set of possible states X_i and observed value y_i
- Compatibility between states and observed values, $\phi_i(x_i, y_i)$
- Compatibility between neighboring vertices *i* and *j*, $\psi_{ij}(x_i, x_j)$

Pairwise MRF: Probabilities



- Joint probability: $$p(x_1,\ldots,x_5)=\frac{1}{Z}\prod_{i=1}^{5}\phi_i(x_i,y_i)\prod_{(ij)}\psi_{ij}(x_i,x_j)$$
- Marginal probability: $$p_i(x_i)=\sum_{x_j\in X_j,1\le j\le 5,j\neq i}p(x_1,\ldots,x_5)$$
  - Advantage: allows average over ambiguous states
  - Disadvantage: complexity exponential in number of vertices

BP on a Chain

$$p(\mathbf{x}) = \frac{1}{Z}\exp(-E(\mathbf{x})), E(\mathbf{x}) = \sum_i \theta_i(x_i) + \sum_{i,j} \theta_{ij}(x_i, x_j)$$

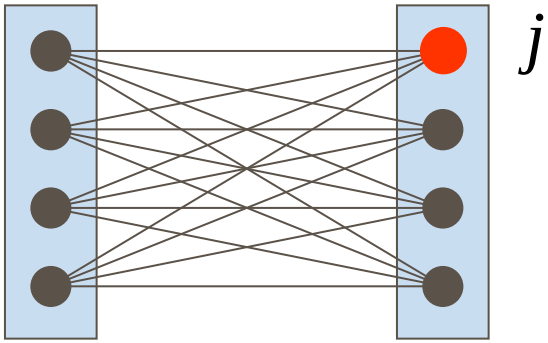
- Dynamic programming: global minimum of H in linear time
- BP:
  - Inward pass (dynamic programming)
  - Outward pass
  - Gives min-marginals

[Pearl'88]

$$\min_{\mathbf{x}} \{E(\mathbf{x}) | x_q = j\}$$

Inward Pass (Dynamic Programming)

$$\theta_p(x_p) + \theta_{pq}(x_p, x_q)$$

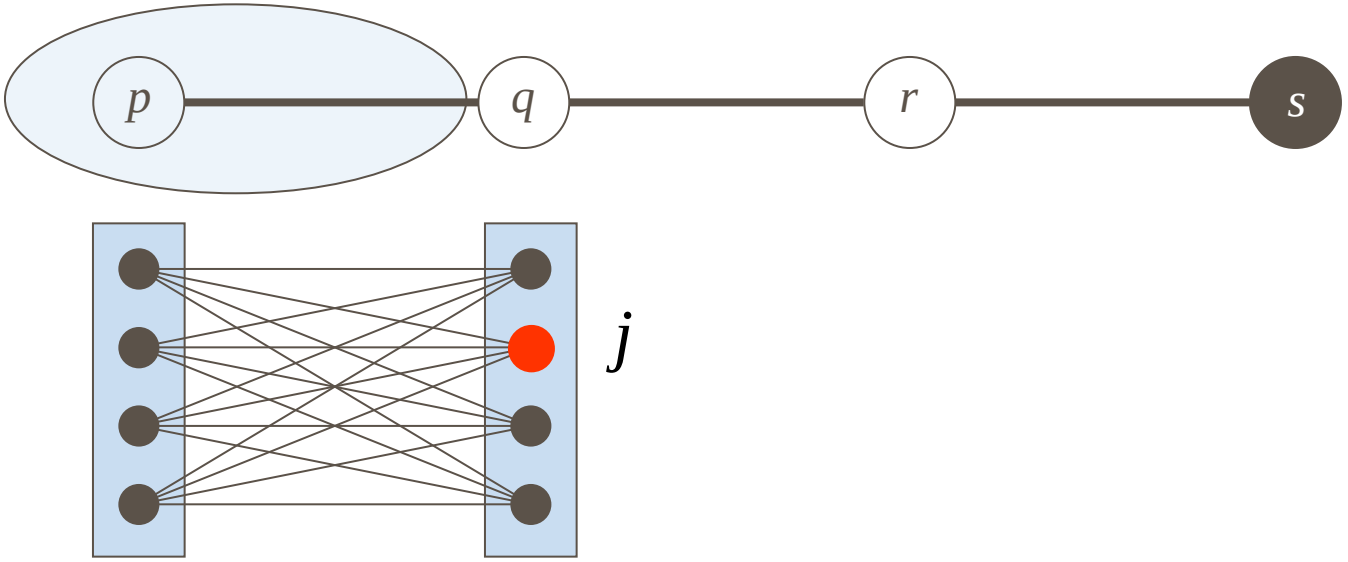


$$M_{pq}(j) = \min_i \left\{ \theta_p(i) + \theta_{pq}(i, j) \right\}$$

Slide credit: V. Kolmogorov

Inward Pass (Dynamic Programming)

$$\theta_p(x_p) + \theta_{pq}(x_p, x_q)$$



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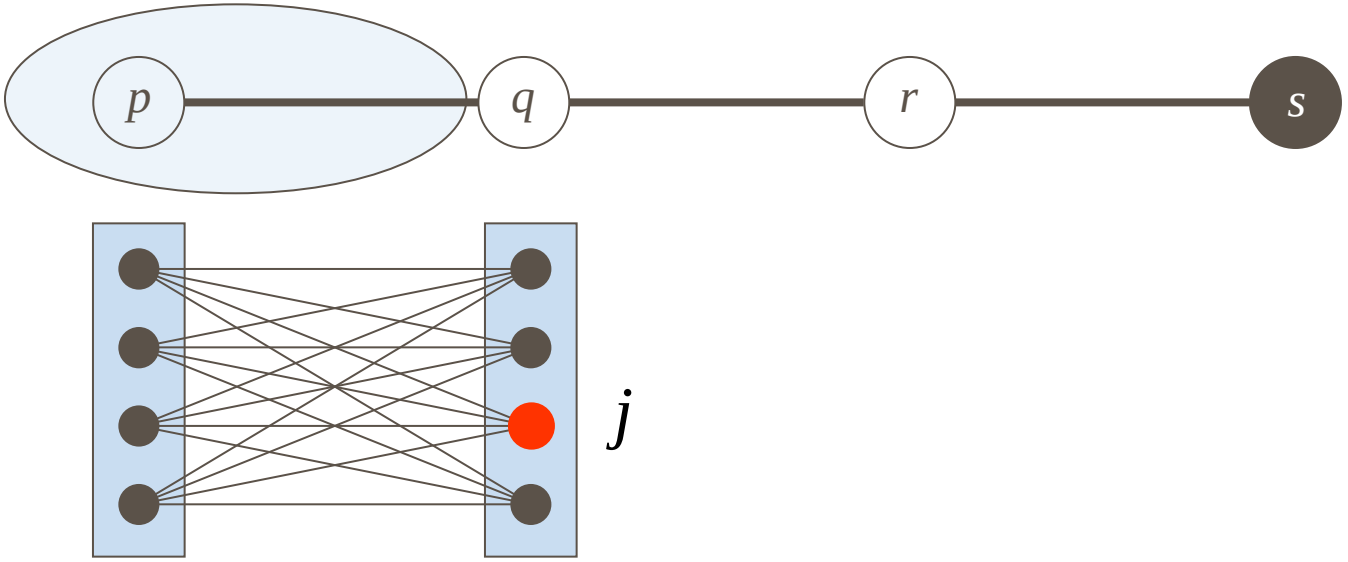

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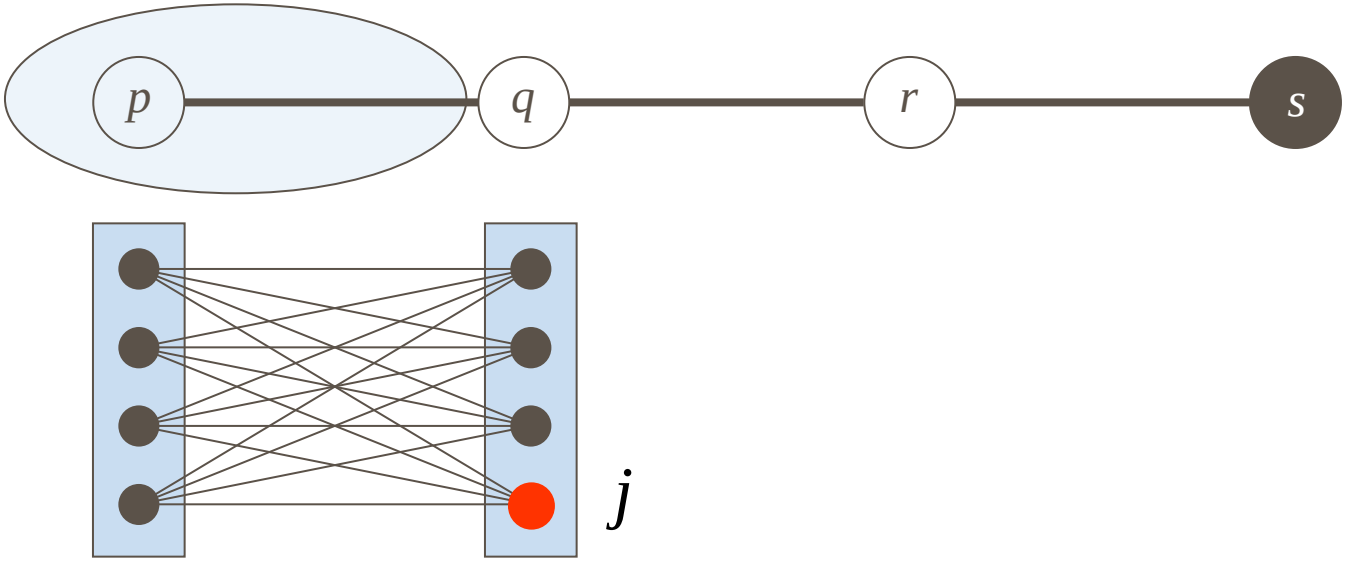

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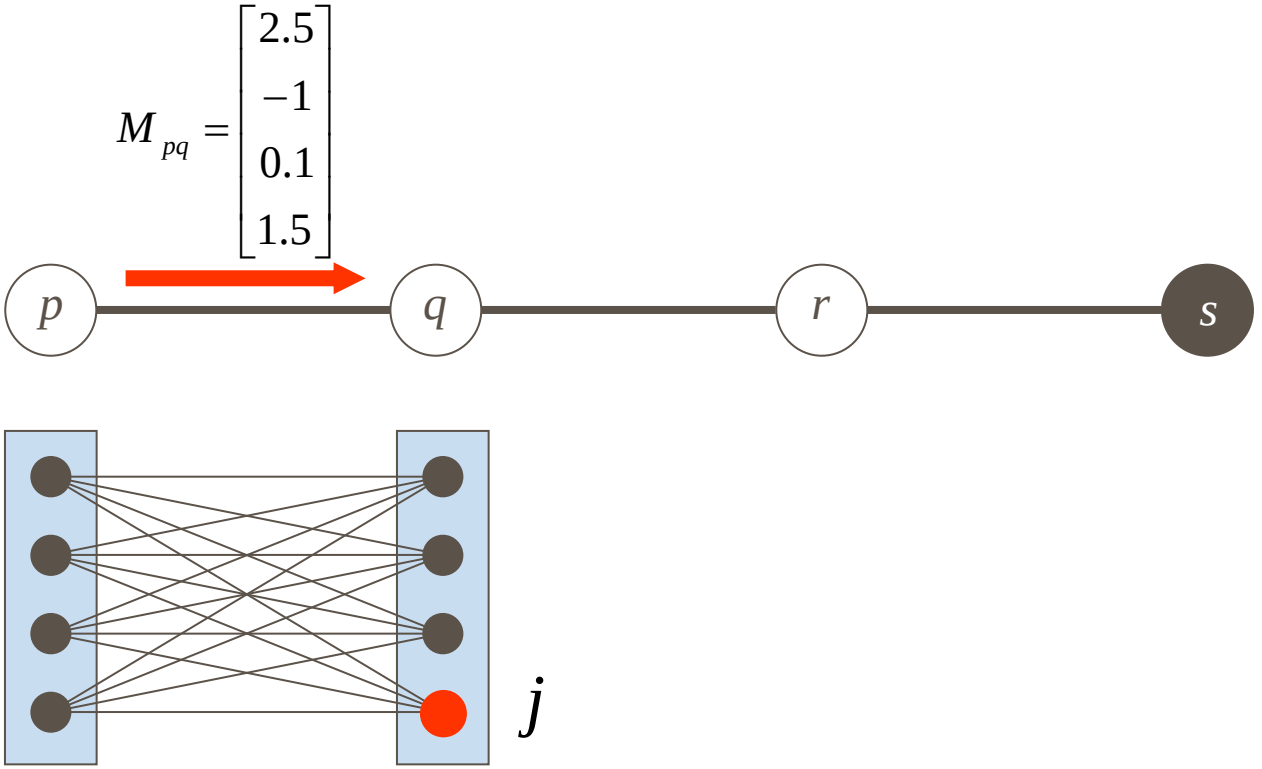

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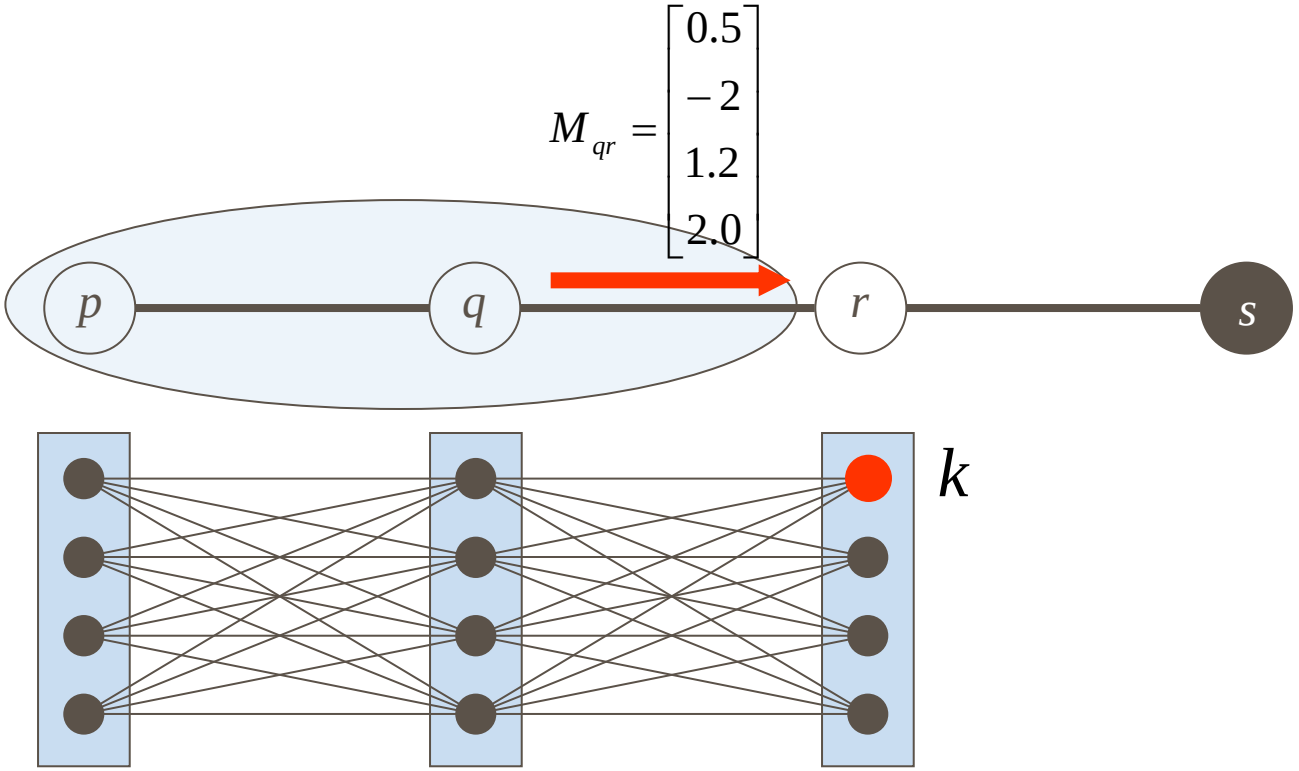
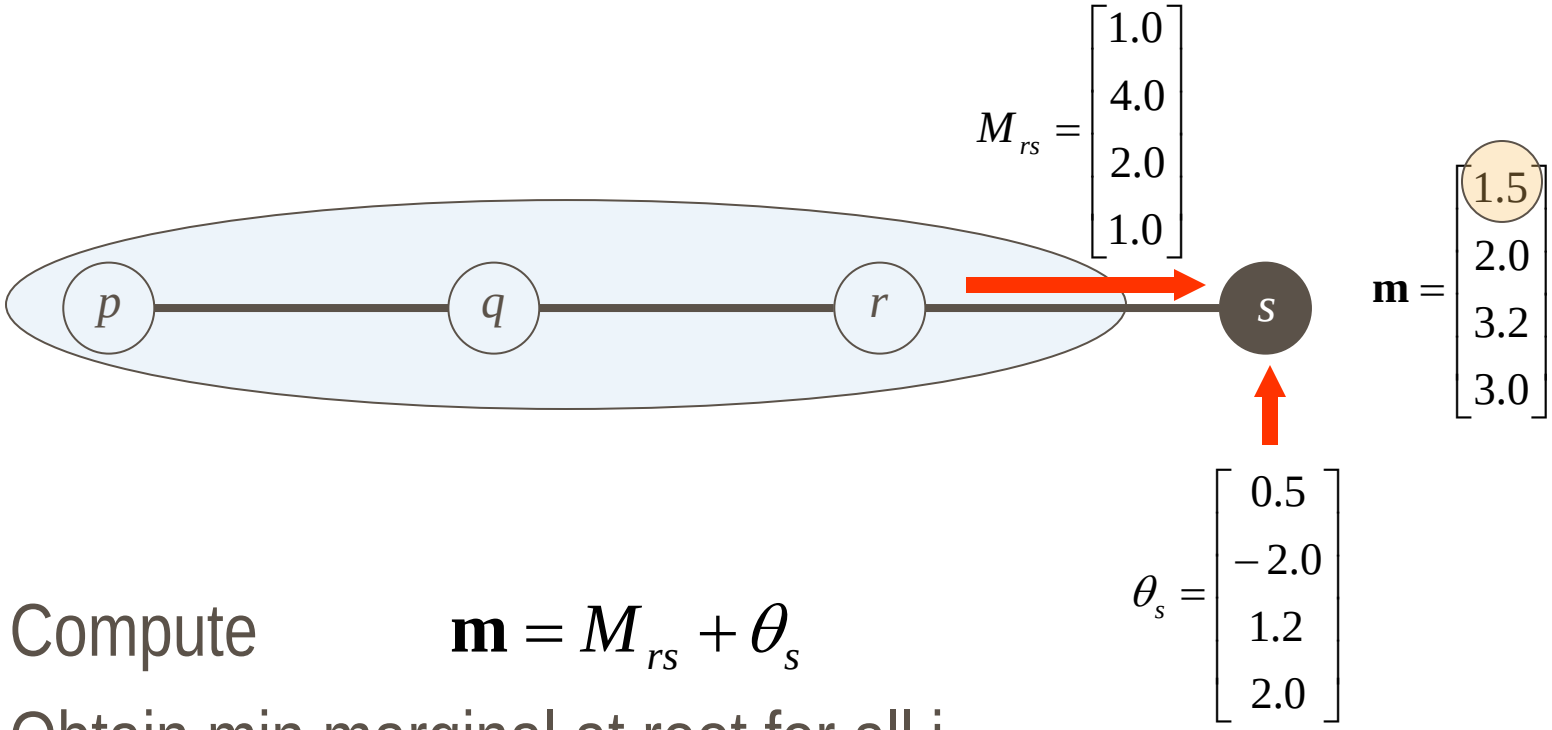


$$M_{qr}(k) = \min_j \left\{ \left(\theta_q(j) + M_{pq}(j) \right) + \theta_{qr}(j,k) \right\}$$

Slide credit: V. Kolmogorov

Inward Pass (Dynamic Programming)



- Compute $\mathbf{m} = M_{rs} + \theta_s$
- Obtain min marginal at root for all j

$$m_j = \min_{\mathbf{x}} \{ \mathrm{E}(\mathbf{x}) \mid x_s = j \}$$

- Find j with minimum E: s*=1

Outward Pass

$$r^* = \arg\min_j [\theta_r(j) + M_{qr}(j) + \theta_{rs}(j, s^*)]$$

Slide credit: V. Kolmogorov

Outward Pass

$$q^* = \arg\min_j [\theta_q(j) + M_{pq}(j) + \theta_{qr}(j, r^*)]$$

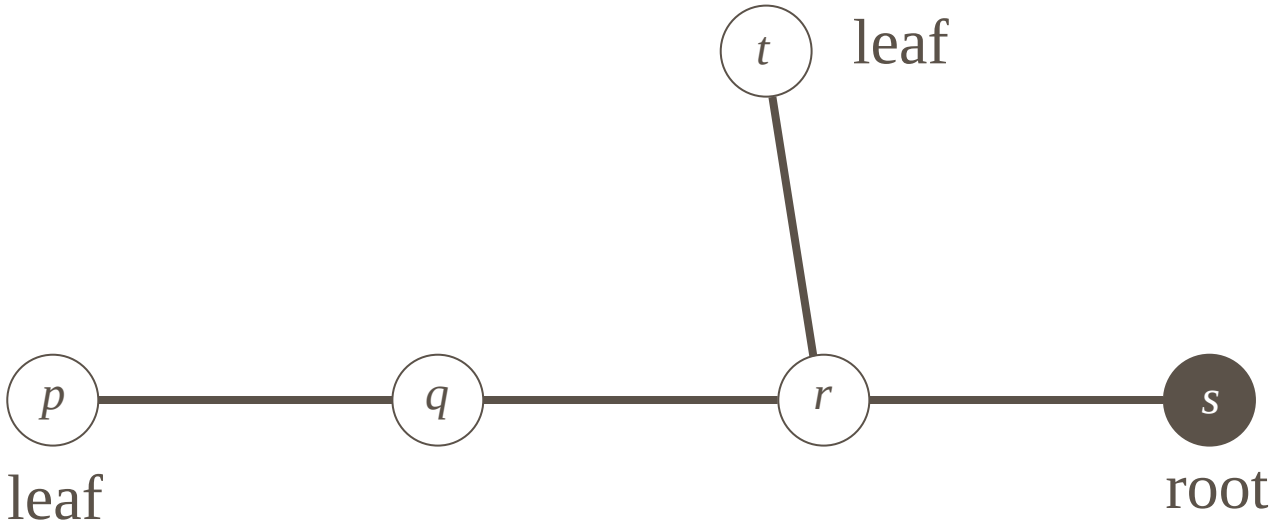
Slide credit: V. Kolmogorov

Outward Pass

$$p^* = \arg\min_j [\theta_p(j) + \theta_{pq}(j, q^*)]$$

Slide credit: V. Kolmogorov

BP on a Tree



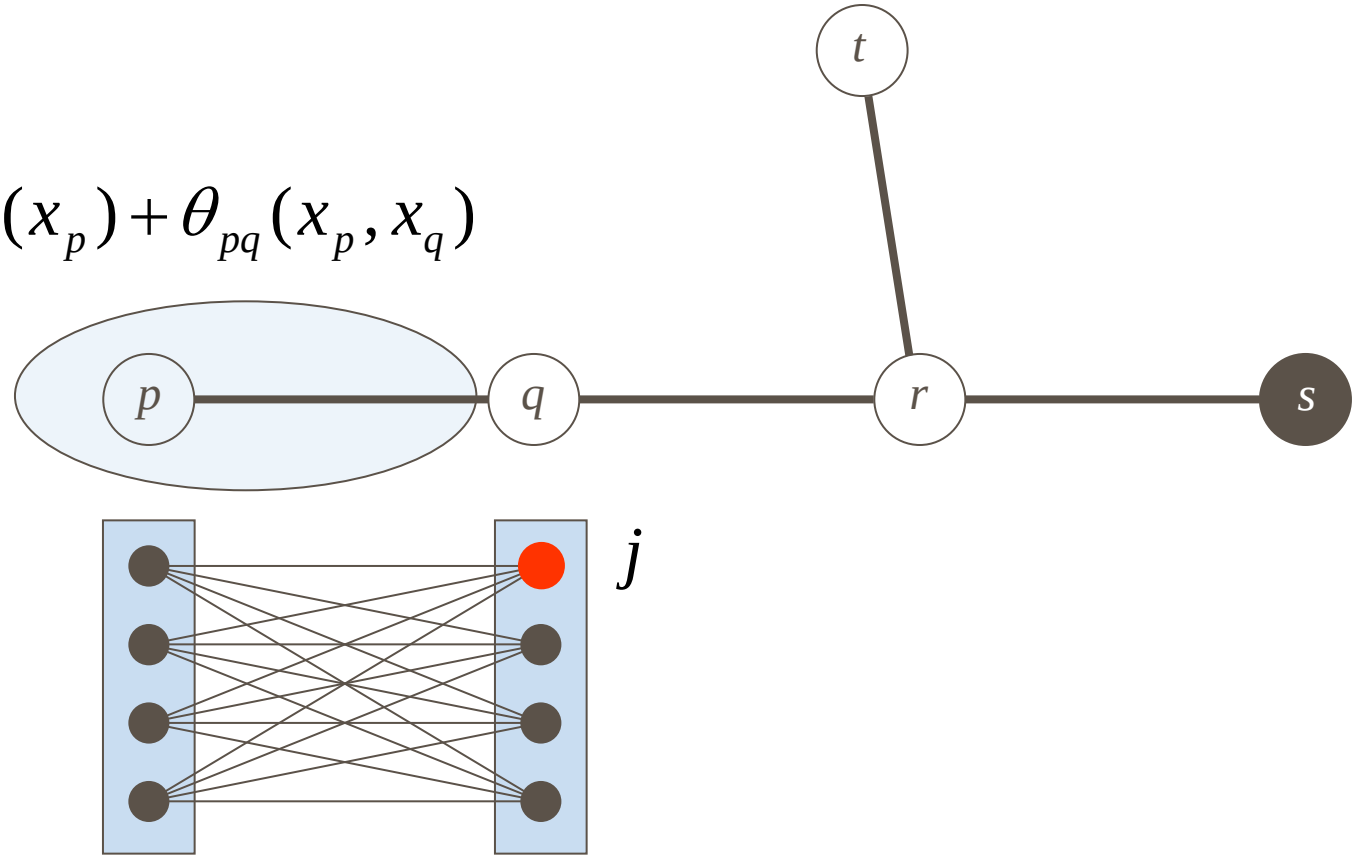
- Dynamic programming: global minimum in linear time
- BP:
  - Inward pass (dynamic programming)
  - Outward pass
  - Gives min-marginals

$$\min_{\mathbf{x}} \{E(\mathbf{x}) | x_q = j\}$$

[Pearl'88]

Inward Pass (Dynamic Programming)

$$\theta_p(x_p) + \theta_{pq}(x_p, x_q)$$



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Slide credit: V. Kolmogorov

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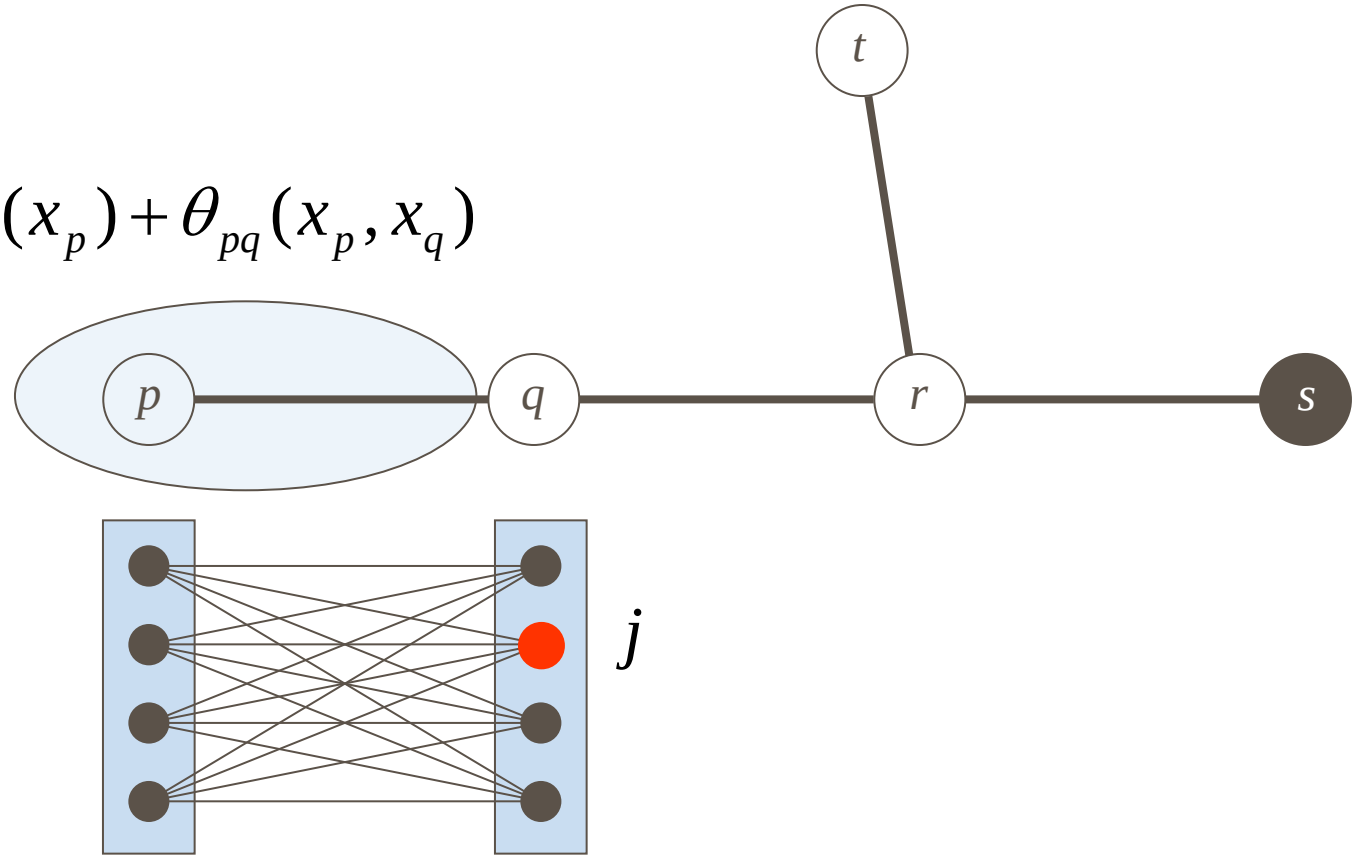

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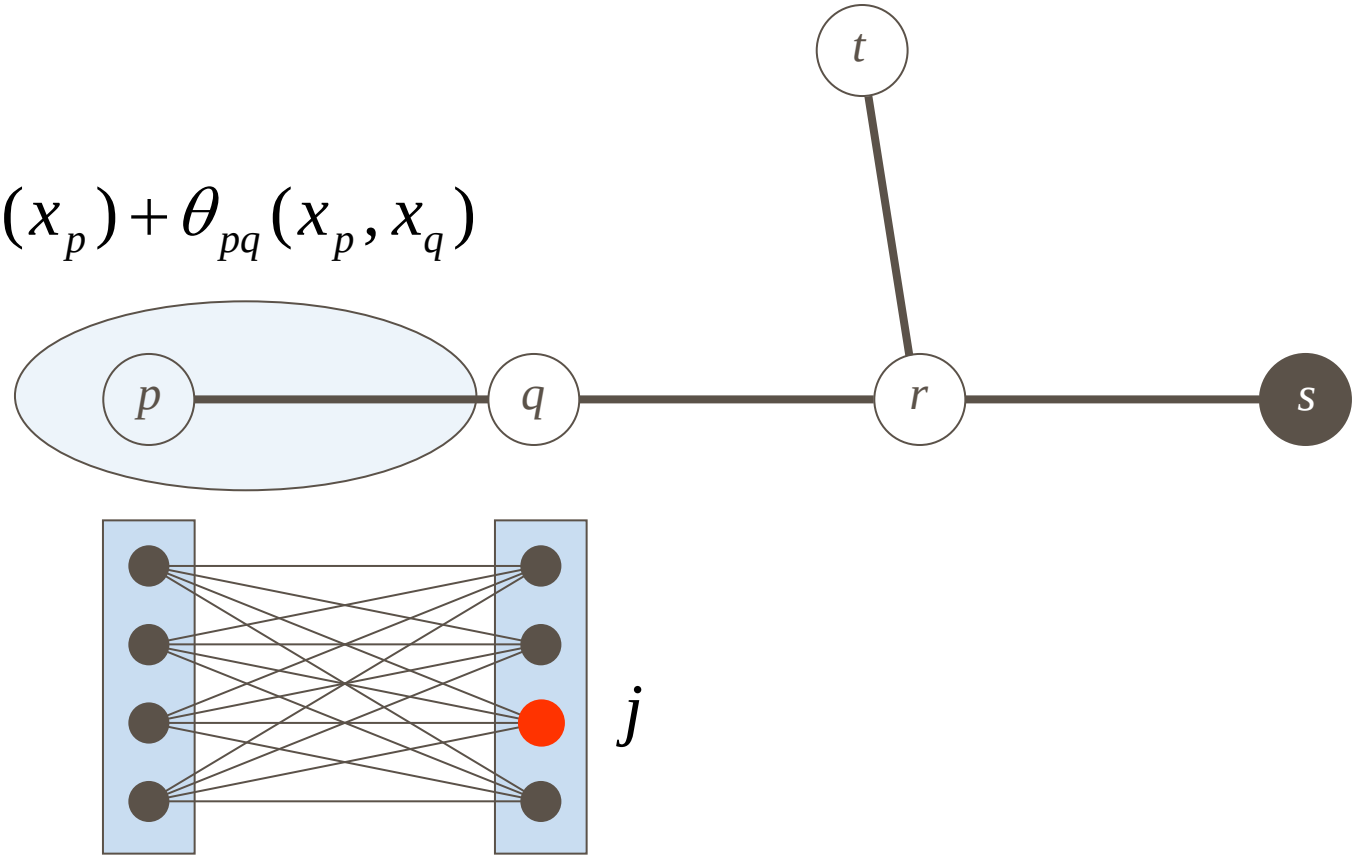

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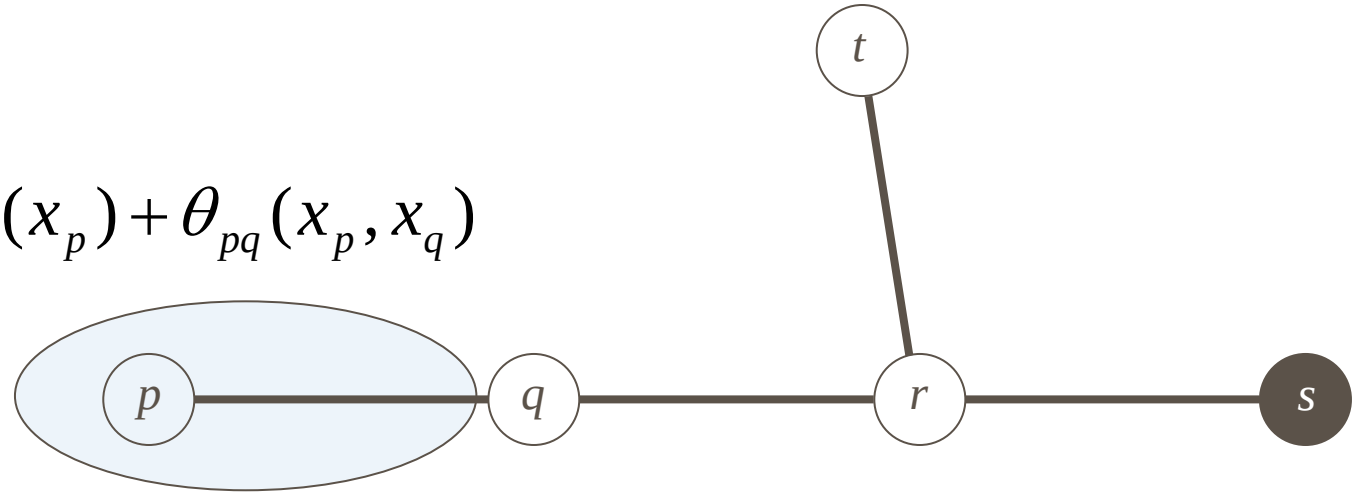

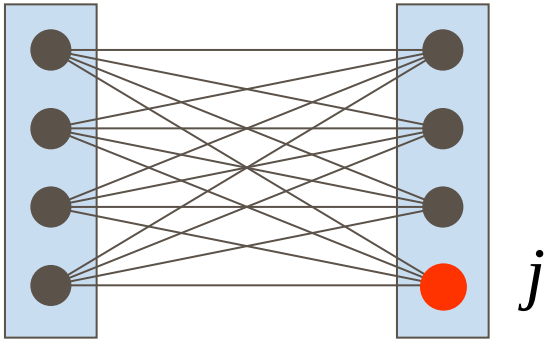

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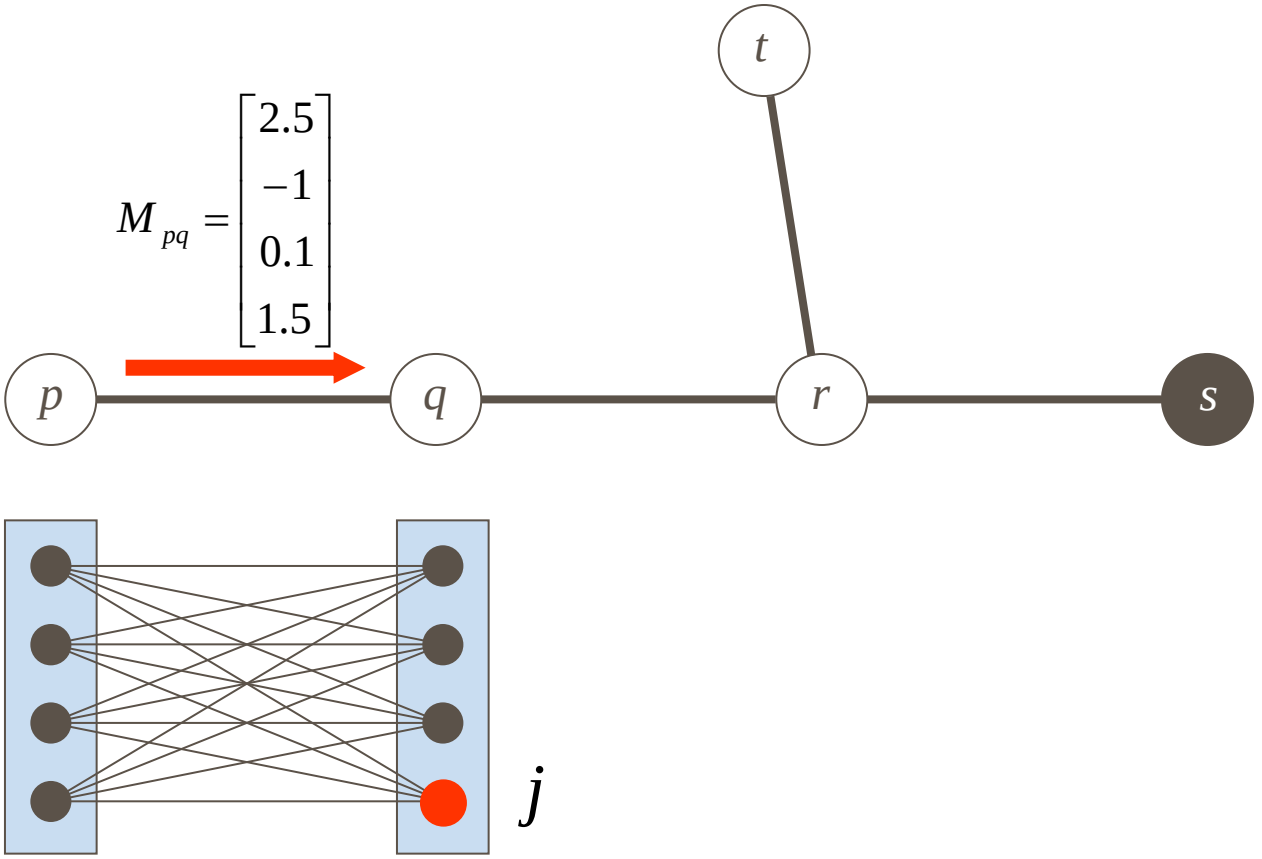
Slide credit: V. Kolmogorov

Inward Pass (Dynamic Programming)



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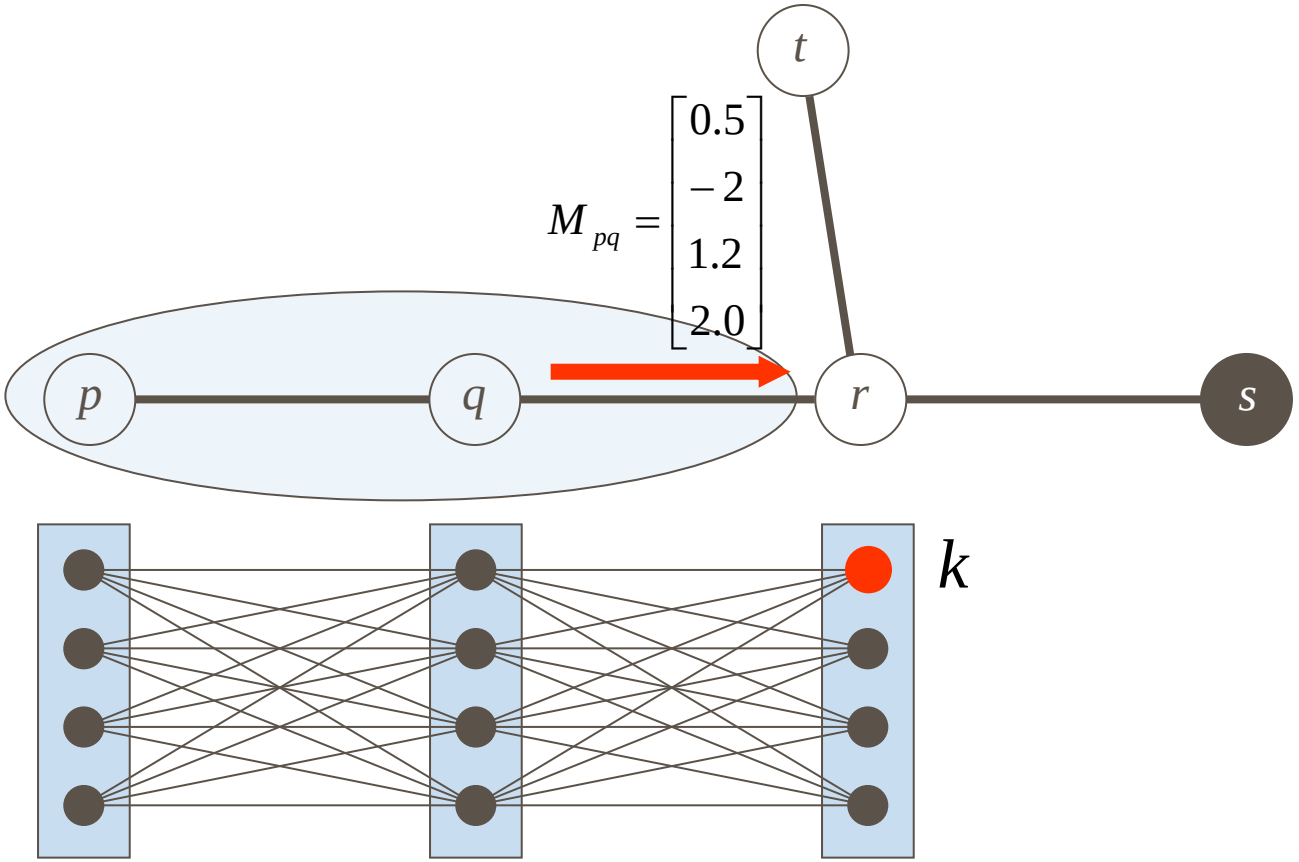
Inward Pass (Dynamic Programming)



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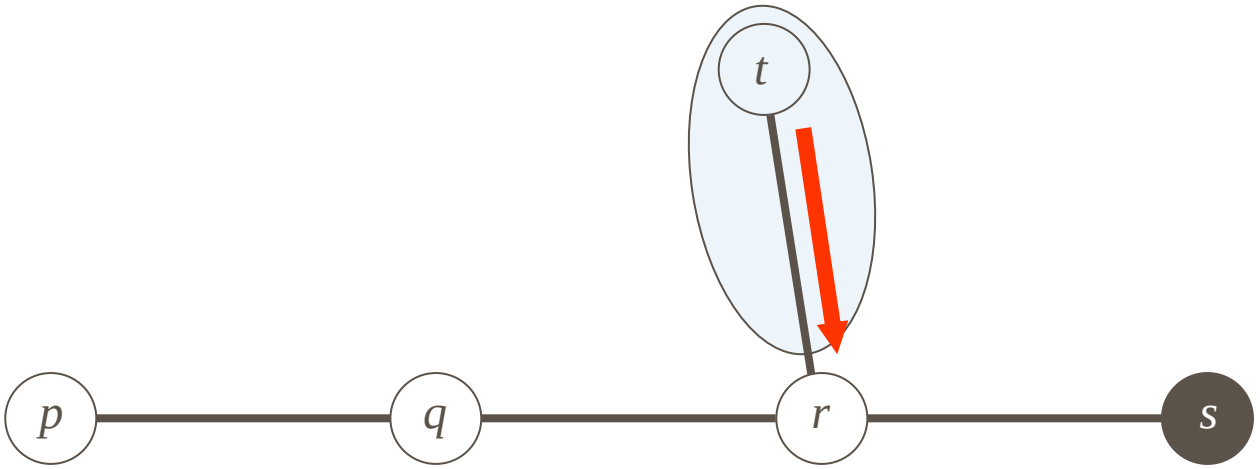

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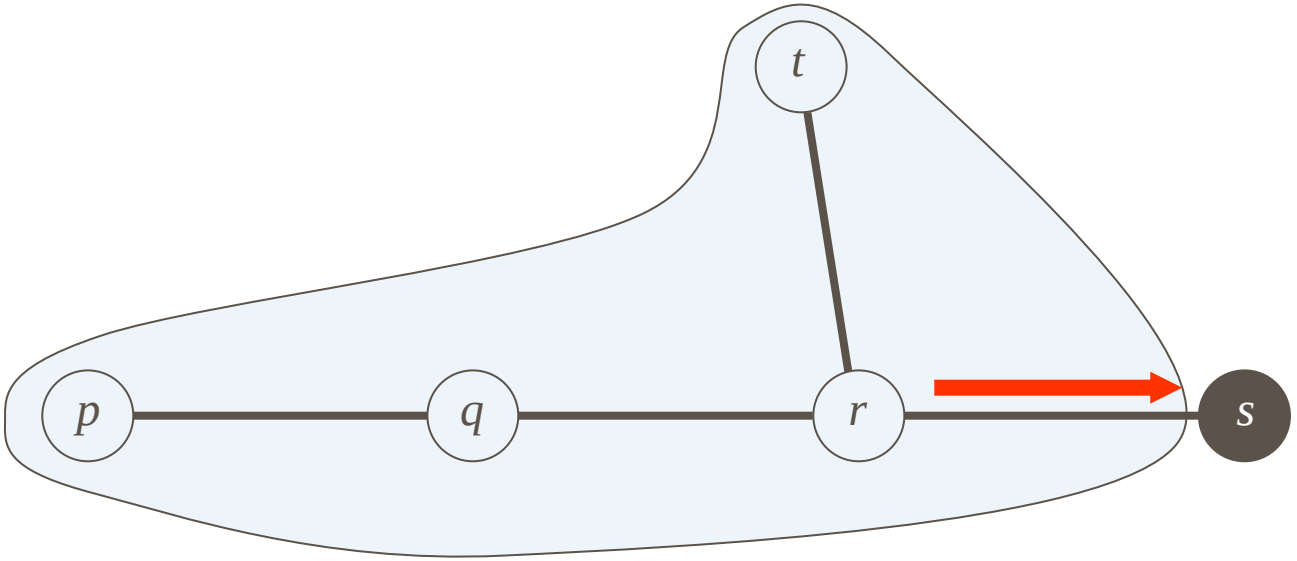
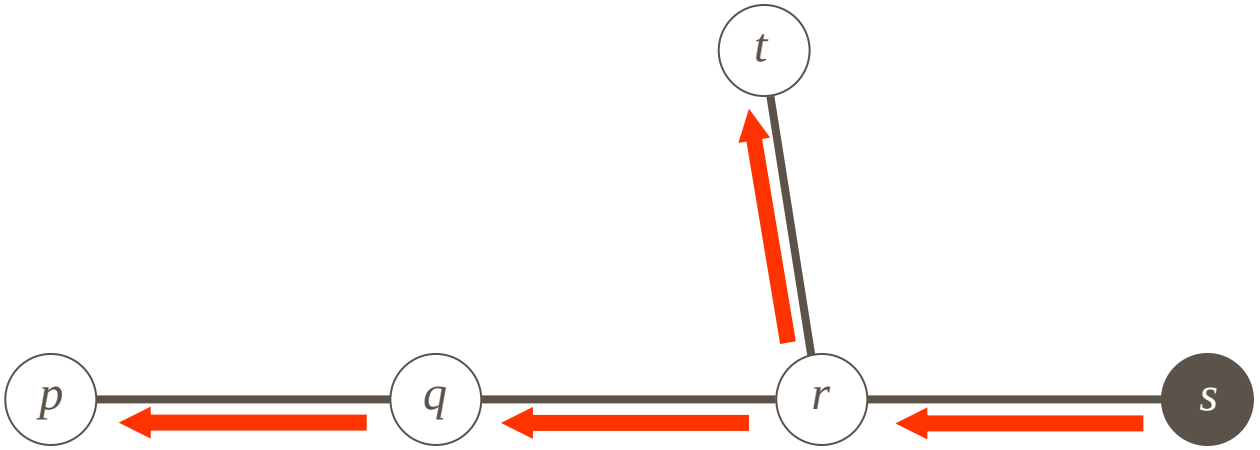


$$M_{rs}(k) = \min_j [\theta_r(j) + M_{qr}(j) + M_{tr}(j) + \theta_{rs}(j,k)]$$

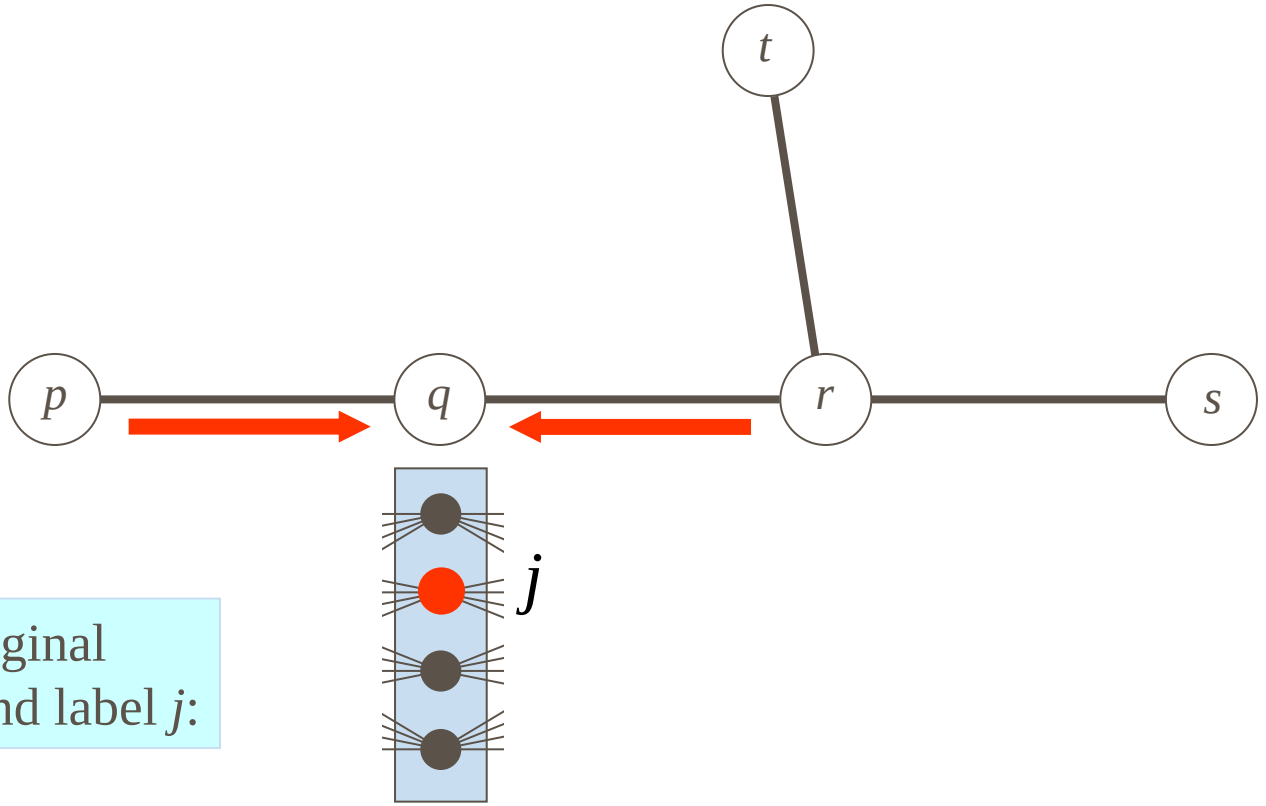
Slide credit: V. Kolmogorov

Outward Pass



Slide credit: V. Kolmogorov

BP on a Tree: Min-marginals



Min-marginal
for node q and label j:

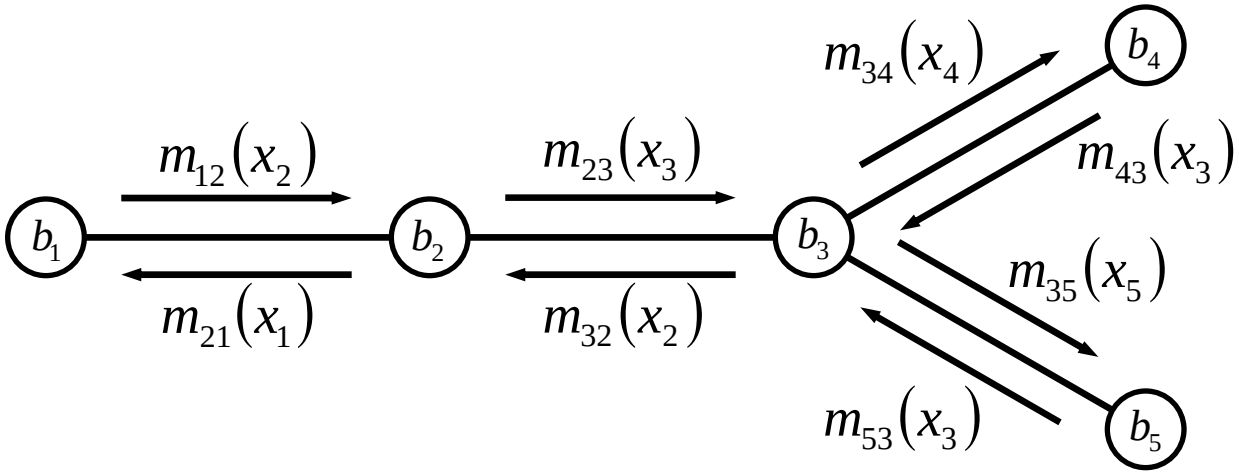
$$\min_{\mathbf{x}} \left\{ E(\mathbf{x}) \;\middle|\; x_q = j \right\} = \theta_q(j) + M_{pq}(j) + M_{rq}(j)$$

Slide credit: V. Kolmogorov

Other BP Versions

- Key property: min(a+b,a+c) = a+min(b,c)
- BP can be generalized to any operators satisfying the above property
- E.g., instead of (min,+), we could have:
  - (max,*)
    Resulting algorithm is called max-product.
    What does it compute?
    - The state with max posterior probability
  - (+,*)
    Resulting algorithm is called sum-product.
    What does it compute?
    - Marginal probabilities

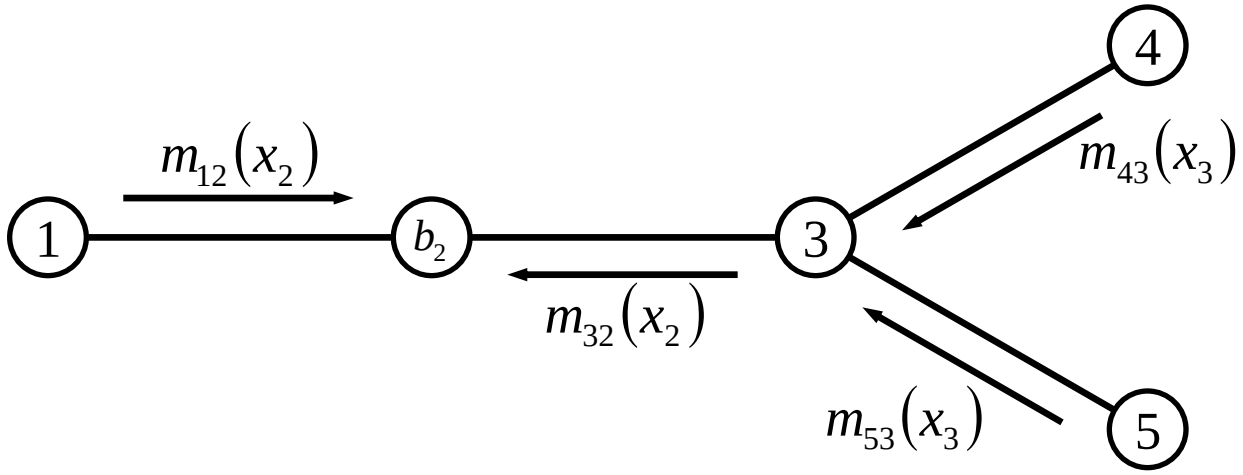
Belief Propagation



- Beliefs replace probabilities: $b_i(x_i) = \frac{1}{z_i}\phi_i(x_i, y_i)\prod_{j\in N(i)} m_{ji}(x_i)$
- Messages propagate information:

$$m_{ji}(x_i) = \sum_{x_j \in X_j} \phi_j(x_j, y_j)\psi_{ji}(x_j, x_i)\prod_{k\in N(j)\setminus i} m_{kj}(x_j)$$

Belief Propagation Example



$$b_2(x_2) = \frac{1}{z_2}\phi_2(x_2, y_2)\prod_{j\in N(2)} m_{j2}(x_2) = \frac{1}{z_2}\phi_2(x_2, y_2)m_{12}(x_2)m_{32}(x_2)$$

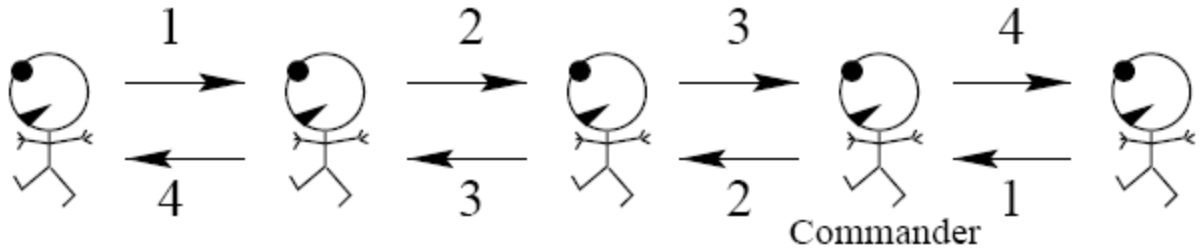
$$m_{12}(x_2) = \sum_{x_1\in X_1}\phi_1(x_1, y_1)\psi_{12}(x_1, x_2)$$

$$m_{32}(x_2) = \sum_{x_3\in X_3}\phi_3(x_3, y_3)\psi_{32}(x_3, x_2)m_{43}(x_3)m_{53}(x_3)$$

$$m_{43}(x_3) = \sum_{x_4\in X_4}\phi_4(x_4, y_4)\psi_{43}(x_4, x_3); m_{53}(x_3) = \sum_{x_5\in X_5}\phi_5(x_5, y_5)\psi_{53}(x_5, x_3)$$

BP = Distributive Algorithm

- BP works in a distributive manner
  - (as a result, it can be parallelized)
- Essentially BP is a decentralized algorithm
- Global results through local exchange of information
- Simple example: counting soldiers

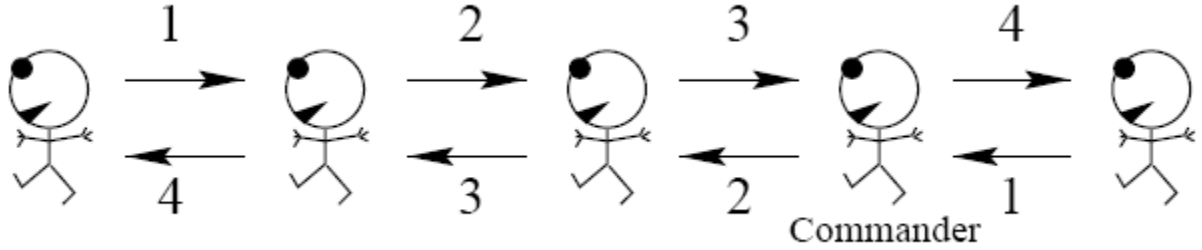


From David MacKay's book "Information Theory, Inference, and Learning"

Counting Soldiers

Each soldier follows these rules:

1. If you are the front soldier in the line, say the number `one' to the soldier behind you.
2. If you are the rearmost soldier in the line, say the number `one' to the soldier in front of you.
3. If a soldier ahead of or behind you says a number to you, add one to it, and say the new number to the soldier on the other side.



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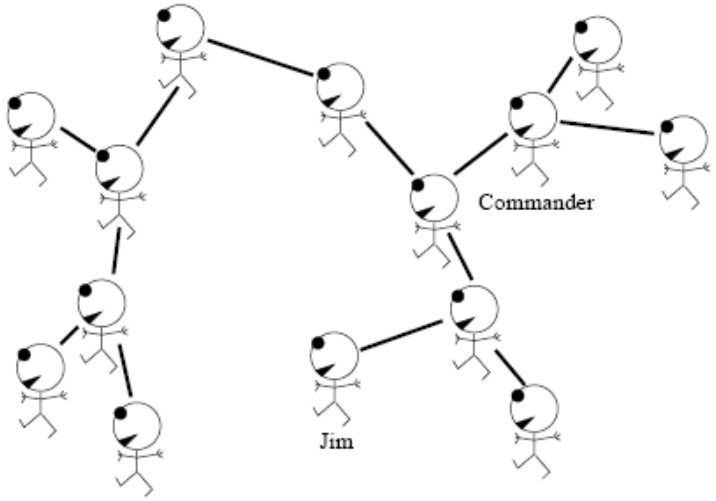
Counting Soldiers



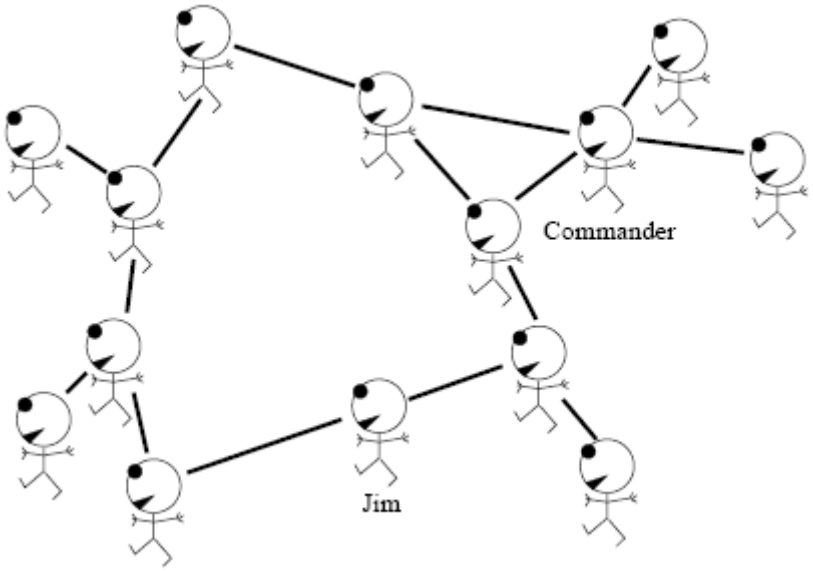
Can be extended to a tree:

1. Count your number of neighbors, N.
2. Keep count of the number of messages you have received from your neighbors, m, and of the values v_1, v_2, …, v_N of each of those messages. Let V be the running total of the messages you have received.
3. If the number of messages you have received, m, is N–1, identify the neighbor who has not sent you a message and tell him the number V+1.
4. If the number of messages you have received is equal to N, then:
   (a) the number V + 1 is the required total.
   (b) to each neighbor n say the number V+1–v_n.

From David MacKay's book "Information Theory, Inference, and Learning"

Counting Soldiers

- Localized approach does not work in graphs with loops



BP: Questions

- When can we calculate beliefs exactly?
- When do beliefs equal probabilities?
- When is belief propagation efficient?

Answer: Singly-Connected Graphs (SCG's)

- Graphs without loops
- Messages terminate at leaf nodes
- Beliefs equal probabilities
- Complexity in previous example reduced from $13S^5$ to $24S^2$

BP on Loopy Graphs

- Messages do not terminate
- Possible approximate solutions
  - Standard belief propagation
  - Generalized belief propagation

BP with Two Graphs: Goals

- Utilize advantages of SCG's
- Be accurate and efficient on loopy graphs

BP with Two Graphs: SCG's

- Consider loopy graph with *n* vertices
- Select two sets of SCG's that approximate the graph

$$\{G_1, \ldots, G_n\}$$

$$\{H_1, \ldots, H_n\}$$

- Calculate beliefs on each set of SCG's:
  - $b_i^G(x_i)$ and $b_i^H(x_i)$
- Select maximum beliefs from both sets
  - $b_i(x_i) = \max\left(b_i^G(x_i), b_i^H(x_i)\right)$

BP-TwoGraphs: Vision SCG's

- Rectangular grid of pixel vertices
- H_i: horizontal graphs
- G_i: vertical graphs

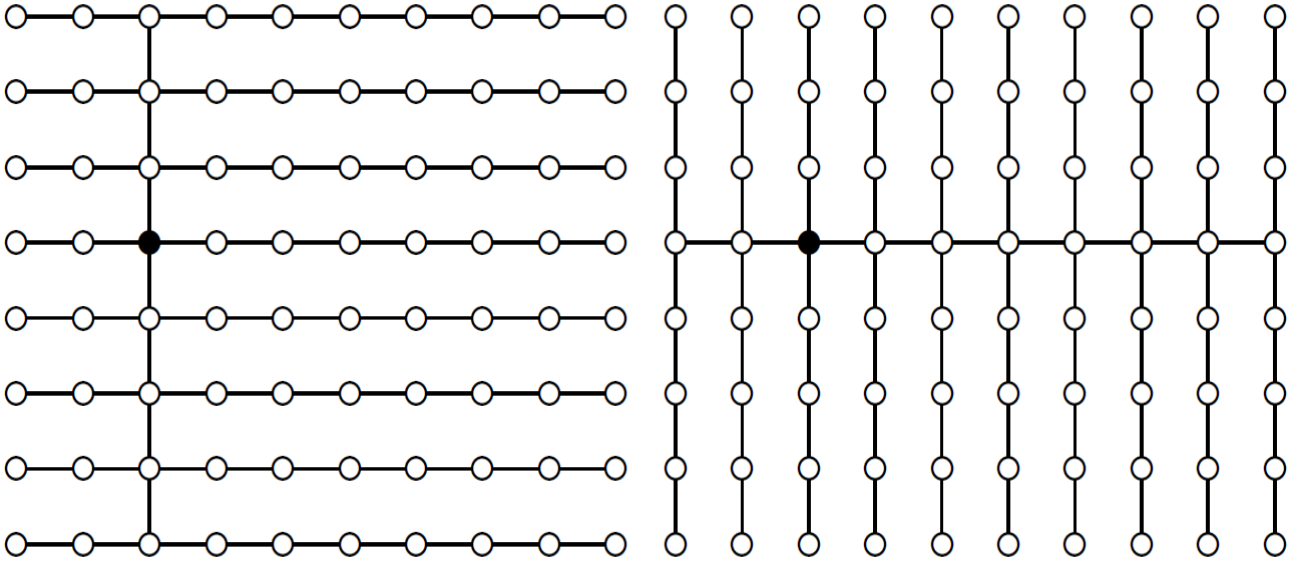


Image Segmentation

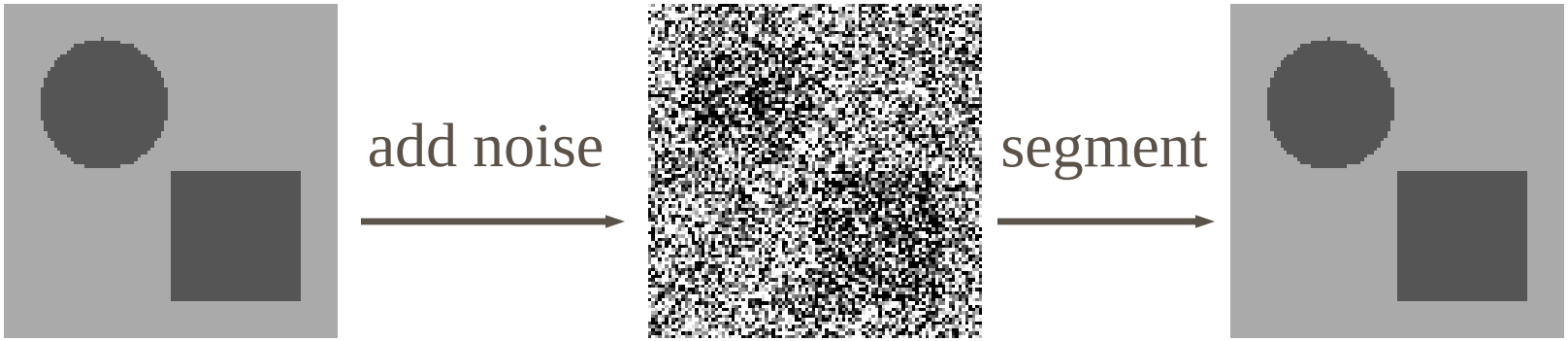
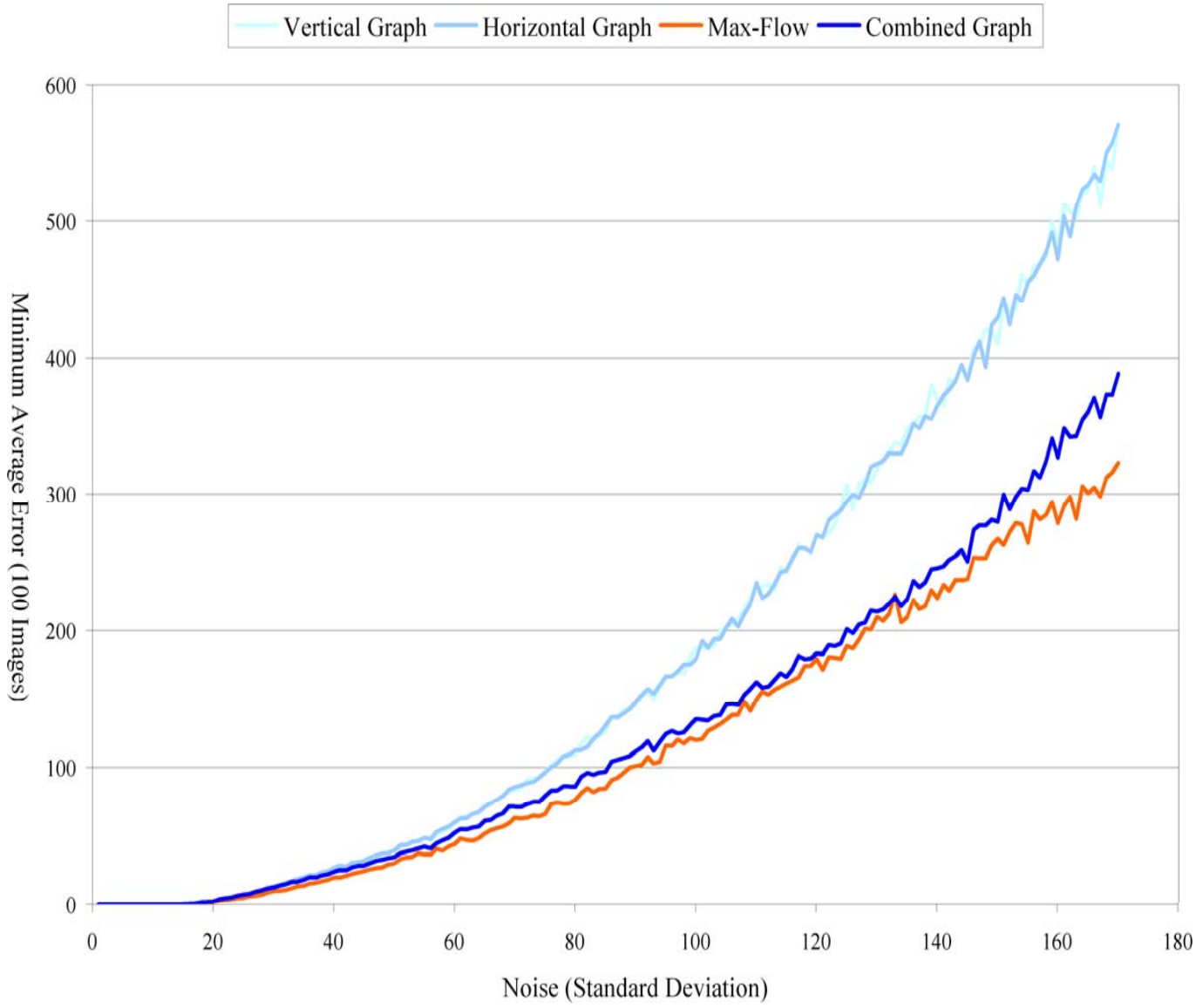
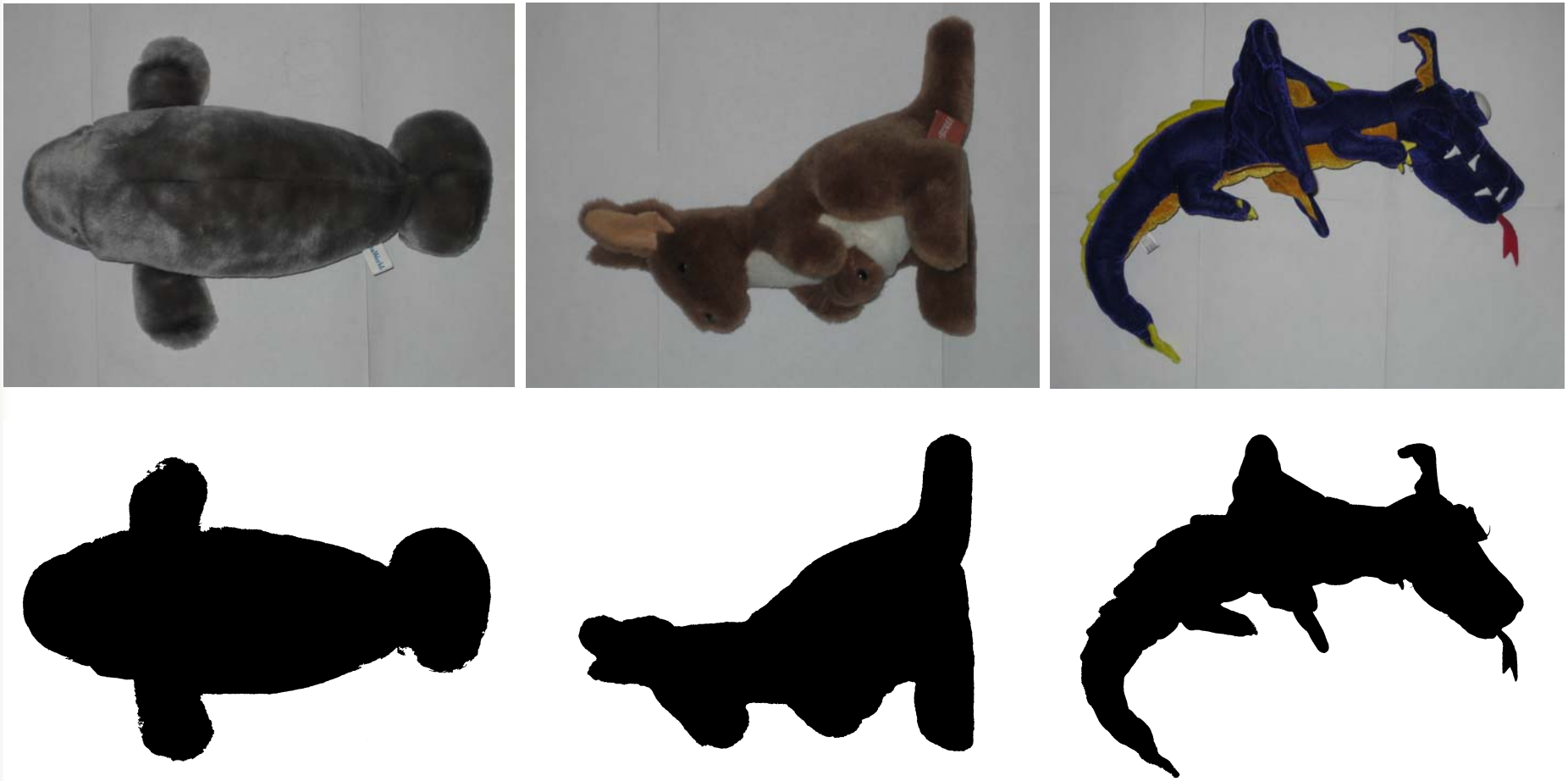


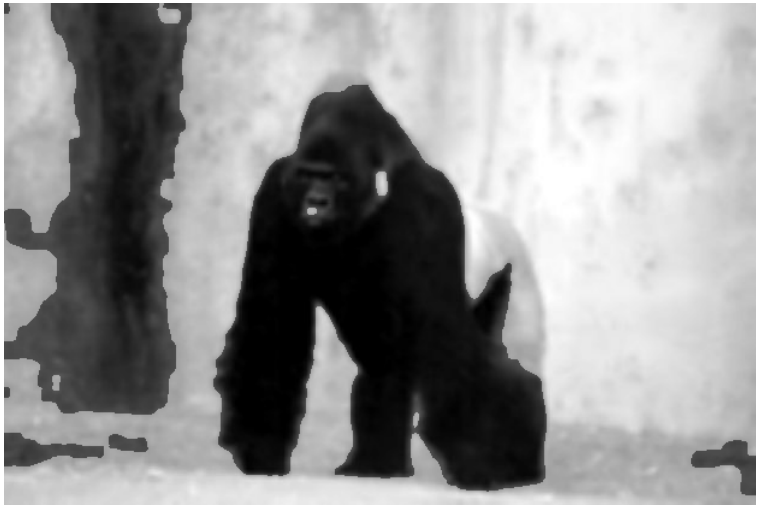
Image Segmentation: Results



Real Image Segmentation: Training



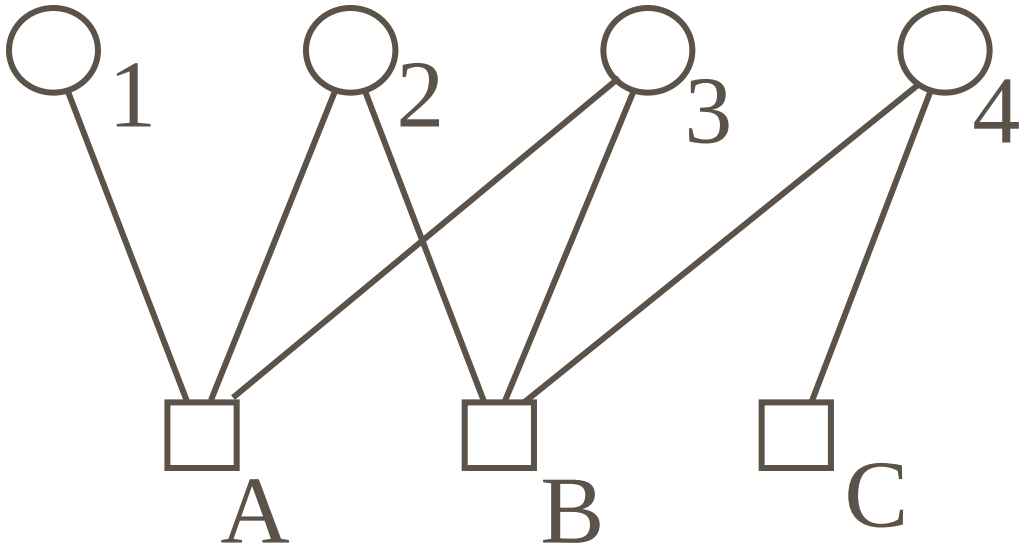
Real Image Segmentation: Results



Generalized BP: Factor Graphs

- Explicitly add nodes for each MRF clique c

$$P(\mathbf{x}) = P(x_1, ..., x_N) = \frac{1}{Z} \prod_c \psi_c(\mathbf{x}_c)$$



$$P(x_1, x_2, x_3, x_4) = \frac{1}{Z} \psi_A(x_1, x_2, x_3) \psi_B(x_2, x_3, x_4) \psi_C(x_4)$$

Computing Marginal Probabilities

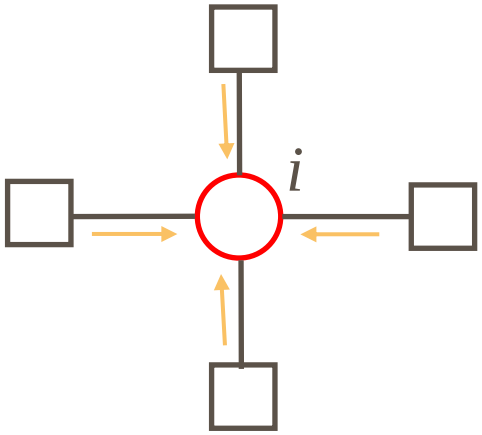
$$p_S(\mathbf{x}_S) = \sum_{\mathbf{x}_{-s}} p(\mathbf{x})$$

Fundamental for

- Decoding error-correcting codes
- Inference in Bayesian networks
- Computer Vision or Medical Imaging
- Statistical physics of magnets

Non-trivial because of the exponential number of terms in the sum.

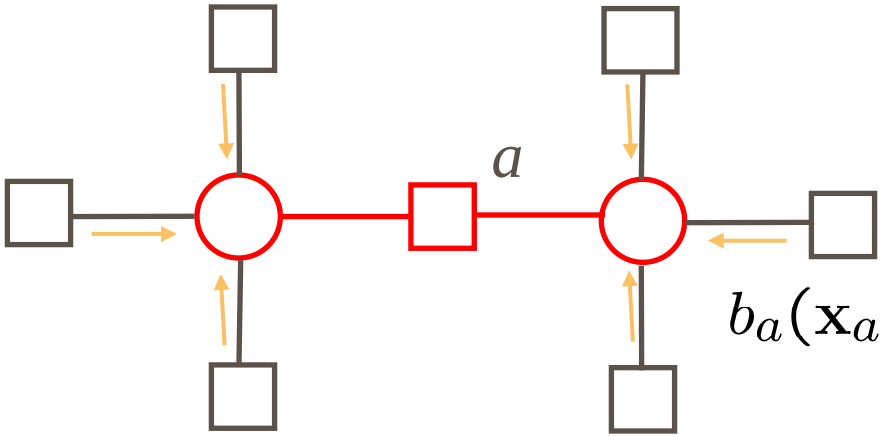
Standard Belief Propagation



$$b_i(x_i) \propto \prod_{a \in N(i)} m_{a \to i}(x_i)$$

"beliefs" "messages"

The "belief" is the BP approximation of the marginal probability.



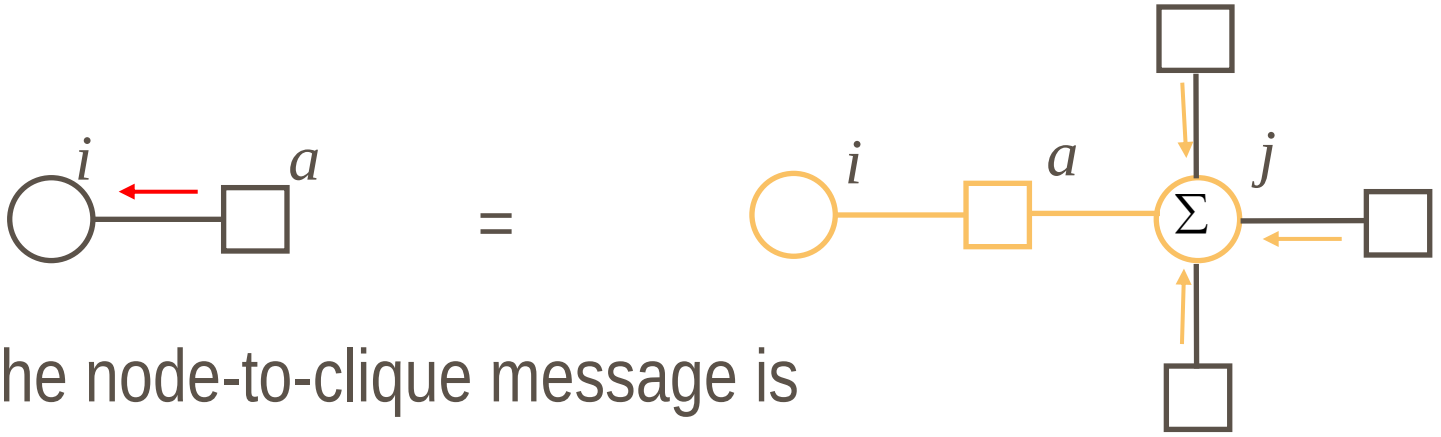
$$b_a(\mathbf{x}_a) \propto \psi_a(\mathbf{x}_a) \prod_{i \in N(a)} \prod_{b \in N(i) \setminus a} m_{b \to i}(x_i)$$

BP Message Update Rules

- Using $b_i(x_i) = \sum_{\mathbf{x}_{a-i}} b_a(\mathbf{x}_a)$

we get the clique to node message

$$m_{a\to i}(x_i) = \sum_{\mathbf{x}_{a-i}} \psi_a(\mathbf{x}_a) \prod_{j\in N(a)\setminus i} \prod_{b\in N(j)\setminus a} m_{b\to j}(x_j)$$



- The node-to-clique message is

$$m_{i\to a}(x_i) = \prod_{b\in N(i)\setminus a} m_{b\to i}(x_i)$$

Message Update

- Denoting the set of messages between all nodes as

$$\vec{M} = (m_{a \to i})_{a,i}$$

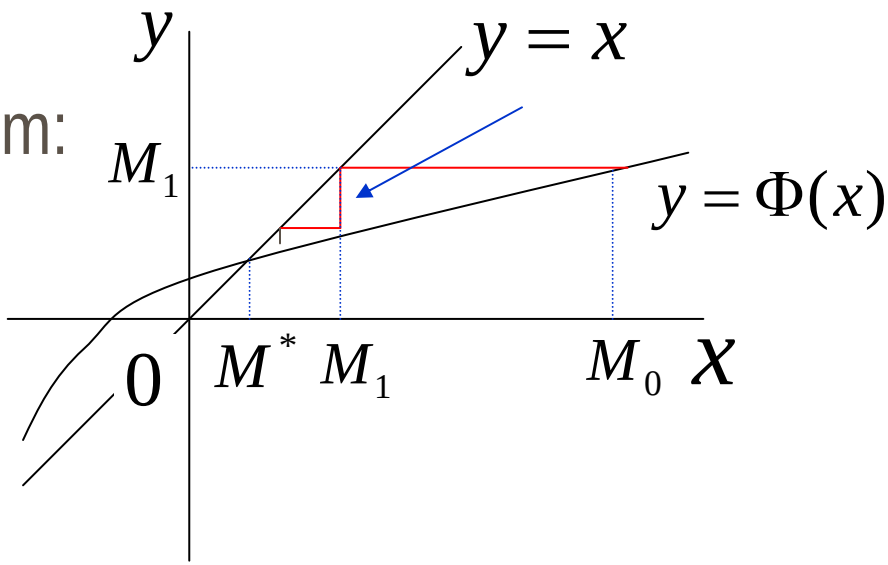
- The message update rules can be summarized as

$$\vec{M} = \Phi(\vec{M})$$

- This yields a fixed-point algorithm:
  - Initialize $\vec{M}_0$
  - Repeat

$$\vec{M}_{t+1} = \Phi(\vec{M}_t)$$

  - Until convergence



- Not guaranteed to converge, but it does in practice

Challenges in BP

- Message update:

$$m_{a\to i}(x_i) = \sum_{\mathbf{x}_{a-i}} \psi_a(\mathbf{x}_a) \prod_{j\in N(a)\setminus i} \prod_{b\in N(j)\setminus a} m_{b\to j}(x_j)$$

  - Need to sum over all other variables from x_a except x_i
  - Unfeasible for large cliques
  - BP usually used for pairwise cliques
- Memory demands
  - Need to memorize all messages
  - Each message is a probability
  - Can be huge for large graphs and many entries in the probability
- Convergence
  - Not always converges

Max-Product BP

- Finds (approximate) most probable configuration

$$\mathbf{x} = \arg\max P(\mathbf{x})$$

- Use the exponential form of the MRF

$$P(\mathbf{x}) = P(x_1, ..., x_N) = \frac{1}{Z} \exp(-\sum_c \phi_c(\mathbf{x}_c))$$

- Min-sum (aka Max-Product) BP:
  - Replace the sum by a min and the product by a sum :

$$m_{a \to i}(x_i) = \min_{\mathbf{x}_{a-i}} [\phi_a(\mathbf{x}_a) + \sum_{j \in N(a) \setminus i} m_{j \to a}(x_j)]$$

$$m_{i \to a}(x_i) = \sum_{b \in N(i) \setminus a} m_{b \to i}(x_i)$$

Some Facts About BP

- BP is exact on trees.
- If BP converges it reaches a local minimum of an objective function = the Bethe free energy (Yedidia et.al '00 , Heskes '02)
  - often good approximation of the global minimum
- When it converges, convergence is fast near the fixed point.
- Many exciting applications:
  - Error correcting decoding (MacKay, Yedidia, McEliece, Frey)
  - Computer Vision (Freeman, Weiss)
  - Bioinformatics (Weiss)
  - Constraint satisfaction problems (Dechter)
  - Game theory (Kearns)

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- http://www.csd.uoc.gr/~komod/ICCV07_tutorial/ICCV07_tutorial_vnk.ppt