Probability Theory

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Abstract

Notes of Real Analysis mainly from Abbott's *Understanding Analysis* and Rudin's *Principles of Mathematical Analysis*.

1 Real Numbers

Rational numbers have *holes*.

Theorem 1. *There is no rational number whose square is 2.*

Proof. Proof is by contradiction. Let there be a rational number p/q where p and q have no common factors and whose square is 2. Find the common factor 2 and reach the contradiction. $\square$

Further, let A be the set of all positive rational numbers q such that $q^2 < 2$ and B be the set of all rational numbers p such that $p^2 > 2$. Then, A contains no largest number and B has no smallest number. Consider

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2}.$$

$$\implies q^2 - 2 = 2\frac{p^2 - 2}{(p + 2)^2}.$$

If $p \in A$ then $q > p$ and $q \in A$. If $p \in B$ then $q < p$ and $q \in B$. Clearly, rational number system has certain holes. Interestingly between every two rational numbers r and s there is a rational $\frac{r+s}{2}$. Still the rational number system has gaps and the real system will try to fill these gaps by defining a new *irrational number* wherever there are holes.

1.1 Ordered Set

A set is a *collection* of objects called *elements* of the set. An order is a *relation* defined on a set, say S and is denoted by $<$. Order has 2 properties:

1. For $x \in S$ and $y \in S$, either $x < y$ or $x = y$ or $x > y$.
2. If $x, y, z \in S$, if $x < y$ and $y < z$ then $x < z$.

An *Ordered Set* is a set on which an order is defined.

Bound. Let S be an ordered set and $E \subset S$. If there is a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$ then E is *bounded above* and β is the *upper bound* of E. Note that β might not belong to E. Similar definition for *lower bound*.

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Least Upper bound. Let S be an ordered set and $E \subset S$ which is bounded above. Then if there is an $\alpha \in S$ such that

1. α is an upper bound of E and
2. if $\gamma < \alpha$, then γ is not an upper bound of E

then α is the *least upper bound* of E or *supremum* of E. There is at most one such number.

$$\alpha = supE.$$

Greatest Lower Bound. For the same S and E defined above, if there is a $\beta \in S$ such that (1) β is a lower bound of E and (2) if $\gamma > \beta$ then γ is not a lower bound of E then β is the *greatest lower bound* of E or *Infimum* of E.

$$\beta = infE.$$

Least Upper Bound Property. S has least upper bound property if for any $E \subset S$, if E is not empty and E is bounded above then sup E esists in S.

Theorem 2. *Suppose S has least upper bound property, then for every $B \subset S$, B is not empty and* **B** *is bounded below, inf B exists in S.*

Proof. Let L be the set of all lower bounds of B i.e. L consists of all $y \in S$ such that $y < x$ for every $x \in B$. Then L is not empty. L is bounded above by every element of B. So L must have a supremum in S, say α. α might not be in L.
Let $\gamma < \alpha$, then by definition, γ is not an upper bound of L but every element of B is an upper bound of L, so $\gamma \notin B$. This means that $\alpha \leq x$ for every $x \in B$. So α is a lower bound of $B \implies \alpha \in L$. Also, if $\beta > \alpha$ then $\beta \notin L$ since α is an upper bound of L. So, α is a lower bound of B but β is not if $\beta > \alpha$. $\implies \alpha = \inf B$. $\square$