Game Theory and Mechanism Design

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Problem Sets

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Contents

Assignment 1

Introduction to Game Theory

1.1 Warm-up

1. Define a game as you have understood (3 lines). What is the difference between a non-cooperative game and a cooperative game (3 lines)
2. What is the difference between game theory and mechanism design (2 lines). Give an example of a mechanism other than what is provided in the class (4 lines)
3. What is intelligence (3 lines) and how is it different from rationality (3 lines)
4. What is your predicted outcome for the Matching Pennies problem (2 lines). Why? (4 lines)

1.2 Workhorse

1. In the Braess paradox example with n=1000 given in the book, compute the minimum delay and maximum delay that could be incurred by a vehicle in Case 1 (without link AB) and in case 2 (with link AB).
2. There are n players. Each player announces a number in the set 1,2, ..., m where m is a fixed positive integer. A prize of One Rupee is split equally between all the people whose number is closes to two thirds of the average number. Formulate this as a strategic form game.
3. Develop the strategic form game for the Pigou network game (page 52) for n=4.
4. In the cake cutting problem, what would be your solution if there were three kids instead of two.

1.3 Thought Provoking

1. Why would the Vickrey auction make the players bid their true valuations without having to worry about the bids of the other players.
2. Why would the strategy profile ((1/11, 10/11), (1/11, 10/11)) constitute an equilibrium of the IISc-MG Road game?
3. Braess Paradox - Why would adding extra capacity make the network perform worse in terms of time delays?

4. Common knowledge example - Give your own intuitive explanation as to why all the five mothers cry only on the fifth day (6 lines max)

Assignment 2

Dominant Strategy Equilibria

2.1 Warm-up

1. Given a strategic form game and a player, show that a strongly dominant strategy, if one exists, must be unique.

2. Show that every dominant strategy equilibrium (strong or weak or very weak) is also a pure strategy Nash equilibrium.

3. Give an example of a simple game (two players, two strategies each) having a pure strategy Nash equilibrium that is not a very weakly dominant strategy equilibrium.

4. Transform the (IISc, MGRoad) example to have a strongly dominant strategy equilibrium.

2.2 Work horse

1. There are n departments in an organization. Each department can try to convince the central authority (of the organization) to get a certain budget allocated. If h_i is the number of hours of work put in by a department to make the proposal, let $c_i = w_i * h_i^2$ be the cost of this effort to the department, where w_i is a constant. When the effort levels of the departments are $(h_1, h_2, ..., h_n)$, the total budget that gets allocated to all the departments is:

$$\alpha \sum_{i=1}^{n} h_i + \beta \prod_{i=1}^{n} h_i$$

   where α and β are constants.
   Consider a game where the departments simultaneously and independently decide how many hours to spend on this effort. Show that a strongly dominant strategy equilibrium exists if and only if $\beta = 0$. Compute this equilibrium.

2. We have seen that a two player symmetric strategic form game is one in which $S_1 = S_2$ and $u_1(s_1, s_2) = u_2(s_2, s_1)\ \forall s_1 \in S_1\ \forall s_2 \in S_2$. Show in such a game that the strategy profile (s_1^*, s_2^*) is a pure strategy Nash equilibrium if and only if the profile (s_2^*, s_1^*) is also a pure strategy Nash equilibrium.

3. Compute strongly or weakly dominant strategy equilibria of the Braess paradox game when the number 25 is replaced by the number 20 (Example 5.5) from 'Game Theory and Mechanism Design' by Y.Narahari.

4. Consider the following instance of the prisoners' dilemma problem.

	NC	C
NC	-4,-4	-2,-x
C	-x,-2	-x,-x

   Find the values of x for which:

   (a) the profile (C, C) is a strongly dominant strategy equilibrium.

   (b) the profile (C, C) is a weakly dominant strategy equilibrium but not a strongly dominant strategy equilibrium.

   (c) the profile (C, C) is a not even a weakly dominant strategy equilibrium.

   In each case, say whether it is possible to find such an x. Justify your answer in each case.

2.3 Thought Provoking

1. Can there exist multiple WDSE in a strategic form game? Prove or disprove.

2. Do you think $(v_1, ...v_n)$ is the unique WDSE for the Vickrey auction game?

3. We showed that the strategy profile (AB, AB, ..., AB) is a SDSE for the Braess Paradox game. Try to derive conditions under which this strategy profile will not be a SDSE.

4. We showed for Vickrey auction that the profile $(v_1, v_2, ..., v_n)$ satisfies the first condition for a weakly dominant strategy equilibrium. Prove the second condition.

Assignment 3

Pure Strategy Nash Equilibria

3.1 Warm-up

1. A strategic form game has 2 players having 3 strategies each. What is the minimum number and maximum number of pure strategy Nash equilibria for such a game.

2. Give examples of two player pure strategy games for the following situations:

   (a) The game has a unique Nash equilibrium which is not a weakly dominant strategy equilibrium.

   (b) The game has a unique Nash equilibrium which is a weakly dominant strategy equilibrium but not a strongly dominant strategy equilibrium.

   (c) The game has one strongly dominant or one weakly dominant strategy equilibrium and a second one which is only a Nash equilibrium

3. Let S be any finite set with n elements. Show that the set $\Delta(S)$, the set of all probability distributions over S, is a convex set.

4. Give an example of a game with two players having 2 strategies in which only one among the four strategy profiles is not a PSNE.

3.2 Work horse

1. A strategic form game has 2 players having 3 strategies each. What is the minimum number and maximum number of mixed strategy Nash equilibria for such a game.

2. Braess paradox game: For the version without the link AB, we showed that any strategy profile with 500 players playing strategy A and 500 players plyaing strategy B is a PSNE. Are there any other PSNEs?

3. Braess paradox game: For the version with link AB, we showed that the profile where all vehicles follow the strategy AB is an SDSE. Are there any other equilibria for the game.

4. First Price Auction: Assume two bidders with valuations v_1 and v_2 for an object. Their bids are in multiples of some unit (that is, discrete). The bidder with higher bid wins the auction and pays the amount that he has bid. If both bid the same amount, one of them gets the object with equal probability $\frac{1}{2}$. In this game, compute a pure strategy Nash equilibrium of the game.

3.3 Thought Provoking

1. In the Pigou's network routing game, derive all pure strategy Nash equilibria for the cases $n = 3$ and $n = 4$. Generalize these findings to determine all pure strategy Nash equilibria for a general value of n.

2. Compute a Nash equilibrium for the two person game with $S_1 = \{0, 1\}$, $S_2 = \{3, 4\}$
   $u_1(x, y) = -u_2(x, y) = |x - y| \ \forall \ (x, y) \in \{0, 1\} \times \{3, 4\}$

3. Consider a strategic form game with $N = \{1, 2\}$; $S_1 = S_2 = [a, b] \times [a, b]$ where a and b are positive real numbers such that $a < b$. That is, each player picks simultaneously a point in the square $[a, b] \times [a, b]$. Define the payoff functions:

$$u_1(s_1, s_2) = -u_2(s_1, s_2) = d(s_1, s_2)$$

   where $d(s_1, s_2)$ is the Euclidean distance between the points s_1 and s_2 . For this above game, compute all pure strategy Nash equilibria.

4. Consider the following strategic form game (network formation game). The nodes of the network are the players: $N = \{1, 2, ..., n\}$. The strategy set S_i of player i is the set of all subsets of $N \setminus \{i\}$. A strategy of a node is to decide on with which other nodes it would like to have links. A strategy profile corresponds to a particular network or graph. Assume that δ where $0 < \delta < 1$ is the benefit that accrues to each node of a link while $c > 0$ is the cost to each node of maintaining the link. Further, assume that δ_k is the benefit that accrues from a k-hop relationship, where, k is the length of a shortest path between the two involved nodes. A link is formed under mutual consent while it can be broken unilaterally. Given a graph g formed out of a strategy profile, let the utility $u_i(g)$ of node i be given by

$$u_i(g) = \sum_{j \neq i} \delta^{l_{ij}(g)} - c.d_i(g)$$

   where $l_{ij}(g)$ is the number of links in a shortest path between i and j and $d_i(g)$ is the degree of node i. Call a network efficient if it maximizes the sum of utilities of all nodes among all possible networks. Call a network pairwise stable if there is no incentive for any pair of unlinked nodes to form a link between them and there is no incentive for any node to delete any of its links. For this setting, prove the following two results.

   (a) If $c < \delta - \delta^2$, the unique efficient network is the complete network.

   (b) If $c < \delta - \delta^2$, the unique pairwise stable network is the complete network.

Assignment 4

Mixed Strategy Nash Equilibrium

4.1 Warm-up

1. Using first principles, find all MSNE of the matching pennies game.

2. using first principles, find all MSNE of the IISc-MG Road game.

3. Using first principles, find all MSNE of the prisoner's dilemma game.

4. Using first principles, find all MSNE of the following game:
   $N = \{1, 2\}$; $S_1 = S_2 = \{A, B\}$;
   U_1 is $2, 1, 2, 1$ for $(A, A), (A, B), (B, A)$, and (B, B) resp.
   U_2 is $2, 2, 1, 1$ for $(A, A), (A, B), (B, A)$, and (B, B) resp.

4.2 Work horse

1. Find all mixed strategy Nash equilibria, applying the NASC, of the following game:
   $N = \{1, 2\}$; $S_1 = S_2 = \{A, B\}$;
   U_1 is $4, 0, 1, 1$ for $(A, A), (A, B), (B, A)$, and (B, B) resp.
   U_2 is $1, 4, 5, 1$ for $(A, A), (A, B), (B, A)$, and (B, B) resp.

2. Find all mixed strategy Nash equilibria, applying the NASC, of the following game:
   $N = \{1, 2\}$; $S_1 = S_2 = \{A, B\}$;
   U_1 is $2, 1, 2, 1$ for $(A, A), (A, B), (B, A)$, and (B, B) resp.
   U_2 is $2, 2, 1, 1$ for $(A, A), (A, B), (B, A)$, and (B, B) resp.

3. Find all mixed strategy Nash equilibria, applying the NASC, of the following game:
   $N = \{1, 2\}$; $S_1 = S_2 = \{A, B\}$;
   U_1 is $1, 2, 3, 4$ for $(A, A), (A, B), (B, A)$, and (B, B) resp.
   U_2 is $4, 3, 2, 1$ for $(A, A), (A, B), (B, A)$, and (B, B) resp.

4. Show that any strictly dominant (mixed) strategy in a strategic form game must be a pure strategy.

4.3 Thought Provoking

1. I claim that in the theorem that provides NASC for a strategy profile to be MSNE, I can replace the ">=" sign in the second condition by a strict ">". Is my claim right? Prove or disprove.

2. In all definitions involving mixed strategies, note that we have implicitly assumed finite strategy sets. Think about what will happen when we have countable or even uncountably infinite sets. For example, can you write down the proposition that expresses the utility in a mixed strategy profile as a convex combination with infinite strategy sets.

3. Consider a zerosum, strategic form game with $N = \{1, 2\}$; $S_1 = S_2 = [a, b]x[a, b]$ where a and b are positive real numbers such that a is strictly less than b. Essentially, each player picks simultaneously a point in the square $[a, b]x[a, b]$. Define the utility function $u_1(s_1, s_2) = -u_2(s_1, s_2) = d(s_1, s_2)$ where $d(s_1, s_2)$ is the Euclidean distance between the two points. Compute all MSNE.

4. Consider the following strategic form game where the numbers a, b, c, d are real numbers:
   $N = \{1, 2\}$; $S_1 = S_2 = \{A, B\}$;
   U_1 is a, b, c, d for $(A, A), (A, B), (B, A)$, and (B, B) resp.
   U_2 is a, c, b, d for $(A, A), (A, B), (B, A)$, and (B, B) resp.
   It is known that the game has a strongly dominant strategy equilibrium. Prove or disprove: The above SDSE is the only possible MSNE for this game.

Assignment 5

Maxmin and Minmax Values

5.1 Warm-up

1. Find the maxmin values, minmax values, maxmin strategies, minmax strategies of the following game:
   $N = \{1, 2\}$; $S_1 = S_2 = \{A, B\}$
   u_1 is $0, 1, 1, 1$ for $(A, A), (A, B), (B, A)$, and (B, B) resp.
   u_2 is $1, 1, 1, 0$ for $(A, A), (A, B), (B, A)$, and (B, B) resp.

2. Find the maxmin values, minmax values, maxmin strategies, minmax strategies of the following game:
   $N = \{1, 2\}$; $S_1 = S_2 = \{A, B\}$
   u_1 is $4, 0, 1, 1$ for $(A, A), (A, B), (B, A)$, and (B, B) resp.
   u_2 is $1, 4, 5, 1$ for $(A, A), (A, B), (B, A)$, and (B, B) resp.

3. Find the maxmin values, minmax values, maxmin strategies, minmax strategies of the Prisoner's Dilemma Problem.

4. Find the maxmin values, minmax values, maxmin strategies of the Rock-Paper-Scissors game.

5.2 Work horse

1. Consider the following two player zero-sum game where a, b, c, d are real numbers with $a > b, d > c, d > b$, and $a > c$. Compute all mixed strategy Nash equilibria for this game. Also compute the maxmin value and minmax value in mixed strategies. Determine the maxmin mixed strategies of each player and the minmax mixed strategies against each player.

	A	B
A	a,-a	b,-b
B	c,-c	d,-d

2. Prove the following propositions.

   (a) Suppose a strategic form game $\Gamma =< N, (S_i), (u_i) >$ has a mixed strategy Nash equilibrium $(\sigma_1^*, ..., \sigma_n^*)$. Then
   $$u_i(\sigma_1^*, ..., \sigma_n^*) \geq \underline{v_i} \;\forall\; i \in N$$
   where $\underline{v_i}$ is the maxmin value in mixed strategies of player i.

(b) Consider a strategic form game $\Gamma =< N, (S_i), (u_i) >$. Then

$$\overline{v_i} \geq \underline{v_i} \ \forall \ i \in N$$

where $\underline{v_i}$ is the maxmin value in mixed strategies of player i and $\overline{v_i}$ is the minmax value in mixed strategies of player i.

(c) Suppose a strategic form game $\Gamma =< N, (S_i), (u_i) >$ has a mixed strategy Nash equilibrium $(\sigma_1^*, ..., \sigma_n^*)$. Then

$$u_i(\sigma_1^*, ..., \sigma_n^*) \geq \overline{v_i} \ \forall i \in N$$

where $\overline{v_i}$ is the minmax value in mixed strategies of player i.

(d) Given a two player strategic form game, the maxmin value in mixed strategies is equal to the minmax value in mixed strategies.

3. Compute all mixed strategy Nash equilibria for the Pigou network routing game (Chapters 4 and 6) for the cases $n = 2$ and $n = 3$. Can you generalize this result?

4. This game is called the guess the average game. There are n players. Each player announces a number in the set $\{1, ..., K\}$. A monetary reward of 1 is split equally between all the players whose number is closest to $\frac{2}{3}$ of the average number. Formulate this as a strategic form game. Show that the game has a unique mixed strategy Nash equilibrium, in which each player plays a pure strategy.

5.3 Thought Provoking

1. Consider a single player game with $N = \{1\}$ and $S_1 = [0, 1]$. Note that the set $[0, 1]$ is compact. Define the utility function as a discontinuous map:

$$u_i(s_i) = \begin{cases} s_i, \text{ if } 0 \leq s_i < 1 \\ 0, \text{ if } s_i = 1 \end{cases}$$

Show that the above game does not have a mixed strategy equilibrium.

2. Consider a single player game with $N = \{1\}$ but with $S_1 = [0, 1)$ (not compact). Define the utility function as a continuous map:

$$u_1(s_1) = s_1 \ \forall s_1 \in [0, 1]$$

Show that this game also does not have a mixed strategy equilibrium.

3. Given a strategic form game $\Gamma =< N, (S_i), (u_i) >$, show for any two mixed strategies, σ_i^*, σ_i that

$$u_i(\sigma_i^*, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}) \ \forall \sigma_{-i} \in \Delta(S_{-i})$$

if and only if

$$u_i(\sigma_i^*, s_{-i}) > u_i(\sigma_i, s_{-i}) \ \forall s_{-i} \in S_{-i}$$

4. There are two sellers 1 and 2 and there are three buyers A, B and C.

(a) A can only buy from seller 1.

(b) C can only buy from seller 2.

(c) B can buy from either seller 1 or seller 2.

(d) Each buyer has a budget (maximum willingness to pay) of 1 and wishes to buy one item.

(e) The sellers have enough items to sell.

(f) Each seller announces a price as a real number in the range $[0, 1]$. Let s_1 and s_2 be the prices announced by sellers 1 and 2, respectively.

(g) Naturally, buyer A will buy an item from seller 1 at price s_1 and buyer C will buy an item from seller 2 at price s_2.

(h) In the case of buyer B, if $s_1 \leq s_2$, then he will buy an item from seller 1, otherwise he will buy from seller 2.

We have shown in Chapter 6 that the above game does not have pure strategy Nash equilibrium. Does this game have a mixed strategy Nash equilibrium?

Assignment 6

Matrix Games

6.1 Warm-up

1. Show that a matrix A will have a saddle point if and only if the maxmin value is equal to the minmax value
2. Given a matrix $A = [a_{ij}]$, show that if a_{ij} and a_{hk} are saddle points, then a_{ik} and a_{hj} are also saddle points.
3. Establish, in matrix games, an equality or inequality relationship between maxmin value in mixed strategies and maxmin value in pure strategies
4. Establish, in matrix games, an equality or inequality relationship between minmax value in mixed strategies and minmax value in pure strategies

6.2 Work horse

1. For the following matrix game with $(2x2)$ matrix A with $a_{11} = 1$, $a_{12} = 3$, $a_{21} = 4$ and $a_{22} = 1$, write down the primal and dual LPs and compute all mixed strategy Nash equilibria.
2. Compute maxmin and minmax values in mixed strategies for the game with:
   $N = \{1, 2\}$; $S_1 = S_2 = \{A, B\}$;
   $u_1 = 2, 3, 3, 4$ for $(A, A), (A, B), (B, A)$, and (B, B) respectively
   $u_2 = 1, 2, 4, 3$ for $(A, A), (A, B), (B, A)$, and (B, B) respectively
3. Complete the proof of the theorem (Theorem 9.3 on page 145) that provides NASC for a mixed strategy profile to be an MSNE of a matrix game.
4. Give an example of a matrix game for each of the following cases.
   (a) It has only pure strategy Nash equilibria
   (b) It has exactly one Nash equilibrium
   (c) It has exactly two Nash equilibria
   (d) It has infinite number of nash equilibria
   (e) It has a strongly dominant strategy equilibrium

6.3 Thought Provoking

1. An (m x m) square matrix is called a "Latin Square" if each row and each column is a permutation of (1,2, ..., m). Compute all saddle points of a Latin Square Matrix game.

2. Consider a square matrix game where the matrix is symmetric. What can you say about the value in mixed strategies of such a game. Repeat your analysis for a skew-symmetric matrix.

3. Consider a matrix game A that is a 2 x 2 matrix with first row elements a,b and second row elements c,d, where a,b,c,d are real numbers. Derive the conditions on a,b,c,d for which the game is guaranteed to have an MSNE. Compute all MSNEs.

4. Suppose you are given a matrix game with 3 pure strategies for each player. Which numbers among 0,1, ..., 9 cannot be the total number of saddle points for the game. Justify your answer.

Assignment 7

Correlated Strategies and Correlated Equilibrium

7.1 Warm-up

1. Show that the inequalities in (25.1) on page 374 are equivalent to the inequalities (25.2) on Page 375.
2. Problem 5 on Page Page 378
3. Show that the payoff of any player under any correlated equilibrium is at least the maxmin value of that player.
4. Show in a matrix game that the row player's utility in any correlated equilibrium is equal to the value of the game in mixed strategies.

7.2 Work horse

1. Show given any mixed strategy profile that we can always find a correlated strategy that produces the same expected payoff to each player as the mixed strategy profile.
2. Consider the following two player game:

	A	B
A	4,1	0,0
B	3,3	1,4

   For the above game, compute:

   (a) the space of all payoff allocations under mixed strategy profiles
   (b) the space of all payoff allocations under mixed strategy Nash equilibria
   (c) the space of all payoff allocations under correlated strategies
   (d) the space of all payoff allocations under individually rational correlated strategies
   (e) the space of all payoff allocations under correlated equilibria

3. Compute all correlated equilibria of the modified prisioner's dilemma example discussed in the class, with payoffs (2,2), (0,6), (6,0), and (1,1).
4. Compute all correlated equilibria of the second example discussed in the class, with payoffs (5,1), (0,0), (4,4), and (1,5).

7.3 Thought Provoking

1. Is it possible that a finite strategic form game may not have correlated equilibria?
2. Given a finite strategic form game, show that the following sets are closed and convex.
   (a) The space of all payoff allocations achievable under correlated strategies
   (b) The space of all payoff allocations achievable under individually rational correlated strategies
   (c) The space of all payoff allocations achievable under correlated equilibria
3. Given a correlated strategy, can we find a mixed strategy profile that produces the same expected payoff to each player as the correlated strategy. Justify your answer.
4. Given a mixed strategy Nash equilibrium of a strategic form game, define the following correlated strategy that assigns to each pure strategy profile the product of the probabilities of these pure strategies under the given MSNE. Show that this correlated strategy is a correlated equilibrium.

Assignment 8

Nash Bargaining Problem

8.1 Warm-up

1. Investigate whether it is necessary that the default point should belong to the feasible set.
2. Why should the feasible set be convex?
3. Why should the feasible set be closed?
4. If the space of all correlated equilibria is chosen as the feasible set, which of the axioms will not be required?

8.2 Work horse

1. Problem 1 on page 397
2. Problem 2 page 397 and problem 3 on page 398
3. Problem 4 on page 398
4. Problem 6 on Page 398

8.3 Thought Provoking

1. Do you think the Nash bargaining approach will generalize to three or more players?
2. Investigate what will happen if axiom 3 (scale covariance) is not satisfied.
3. Investigate what will happen if axiom 4 (independence of irrelevant alternatives) is not satisfied.
4. Investigate what will happen if axiom 5 (symmetry) is not satisfied.

Assignment 9

Coalitional Games with Transferable Utility

9.1 Warm-up

1. Question 3 on page 414
2. Question 4 on page 414
3. Is the majority voting game superadditive? Convex?
4. Is the minimal spanning tree game superadditive? Convex?

9.2 Work horse

1. Question 1 on page 413
2. Question 2 on page 413
3. Question 8 on page 414
4. Question 9 on page 414

9.3 Thought Provoking

1. Why does a natural extension of the Nash bargaining result not work for the divide the dollar game?
2. How does a TU game differ from an NTU game? Write down a situation where an NTU game is needed to capture the underlying dynamics.
3. Describe all pure strategy Nash equilibria of each of the four versions of the divide-the-dollar game
4. Page 409: Observe the definition of strategic equivalence of any two given TU games. Prove the important result that any superadditive, essential YU game is strategically equivalent to a unique game (page 409).

Assignment 10

The Core

10.1 Warm-up

1. Problem 5 on page 427
   Find the core of the communication satellites game defined as follows:
   $v(1) = v(2) = v(3) = 0$
   $v(12) = 5.2$; $v(13) = 2.5$; $v(23) = 3$; $v(123) = 5.2$

2. Problem 6 on Page 427

3. Problem 7 on Page 427
   Compute the core of the logistics game discussed in Chapter 27. Recall that $N = \{1, 2, 3, 4\}$ and the characteristic function is

$$v(1) = v(2) = v(3) = v(4) = 0$$

$$v(12) = v(13) = v(14) = v(23) = v(24) = v(34) = v(234) = v(123) = 0$$

$$v(134) = 40;$$

$$v(124) = 45;$$

$$v(1234) = 65$$

10.2 Work horse

1. Problem 3 on Page 427
   Consider the following variant of the real estate example. Player 1 has a value of Rs.1 million; player 2 has value of Rs.2 million; and player 3 has a value of Rs.3 million for the house. Player 2 has Rs.3 million cash, so also player 3. Formulate an appropriate TU game and compute the core.

2. Problem 9 on Page 428
   Let us consider a version of divide the dollar problem with 4 players and total worth equal to 400. Suppose that any coalition with three or more players will be able to achieve the total worth. Also, a coalition with two players will be able to achieve the total worth only if player 1 is a part of the two player coalition. Set up a characteristic function for this TU game and compute the core.

3. Problem 10 on page 428
   Consider another version of divide the dollar problem with 4 players and total worth equal to 400. Any coalition containing at least two players and having player 1 would be able to achieve the total wealth of 400. Similarly, any coalition containing at least three players and containing player 2 also would be able to achieve the total wealth of 400. Set up a characteristic form game for this situation and compute the core.

4. Problem 13 on Page 428

5. Which of the four versions of the divide the dollar game are (a) monotone (b) superadditive (c) convex.

10.3 Thought Provoking

(a) It has been stated that the core of a TU game is convex and compact. Prove this result.

(b) A market game is a TU game that consists of a set B of buyers and a set S of sellers such that $N = B \cup S$ and $B \cap S = \varnothing$, and $v(C) = min(\mid C \cap B \mid, \mid C \cap S \mid);\ \forall C \subseteq N$. Compute the core of a market game.

(c) Consider the glove market example. What will be the core of this game if there are $1,000,000$ left glove suppliers and $1,000,000$ right glove suppliers?

(d) Give an example of a non-convex game for which the core is non-empty and the Shapley value belongs to the core. Now give an example of another non-convex game for which the core is non-empty and the Shapley value does not belong to the core.

Assignment 11

Shapley Value

11.1 Warm-up

(a) Problem 1 on page 444
Show using the expression for Shapley value that the sum of Shapley values of all players will be equal to the value of the grand coalition.

(b) Problem 6 on Page 445
Let us consider a version of divide the dollar problem with 4 players and total worth equal to 400. Suppose that any coalition with three or more players will be able to achieve the total worth. Also, a coalition with two players will be able to achieve the total worth only if player 1 is a part of the two player coalition. Set up a characteristic function for this TU game and compute the Shapley value.

(c) Problem 7 on Page 445
There are four players $\{1, 2, 3, 4\}$ who are interested in a wealth of 400 (real number). Any coalition containing at least two players and having player 1 would be able to achieve the total wealth of 400. Similarly, any coalition containing at least three players and containing player 2 also would be able to achieve the total wealth of 400. Set up a characteristic form game for this situation and compute the Shapley value.

(d) Problem 9 on Page 445

11.2 Work horse

(a) Problem 4 on page 444
Consider a three person superadditive game with

$$v(1) = v(2) = v(3) = 0;$$

$$v(12) = a; v(13) = b; v(23) = c;$$

$$v(123) = d$$

where $0 \leq a, b, c \leq d$. Compute the Shapley value for this game.

(b) Problem 5 on page 445
Consider the following characteristic form game with three players.

$$v(1) = v(2) = v(3) = 0;$$

$$v(12) = a; v(13) = b; v(23) = c;$$

$$v(123) = 1$$

Assume that $0 \leq a, b, c \leq 1$.

i. Find the conditions under which the core is non-empty.
ii. Compute the Shapley value.
iii. Assuming the core is non-empty, does the Shapley value belong to the core? Under what conditions will the Shapley value belong to the core of this game.

(c) Assuming the core is non-empty, does the Shapley value belong to the core:

6. Problem 10 on page 445

7. Problem 13 on page 446
Consider a TU game with four players where

$$v(12) = v(13) = v(123) = v(134) = v(124) = v(234) = v(1234) = 1$$

The characteristic function takes zero value for the rest of the coalitions. Is this game monotonic? If yes, compute the Shapley - Shubik power index for this game.

11.3 Thought Provoking

1. Problem 2 on page 444
Suppose (N, v) is a TU game and we define a unique imputation as follows.

$$\xi_i(N, v) = v(\{i\}) \ \forall i \in N$$

Which of the Shapley axioms does the above satisfy and which of the Shapley axioms does it violate?

2. Problem 3 on page 444
Suppose (N, v) is a TU game and we define a unique imputation as follows.

$$\xi_i(N, v) = v(\{1, ..., i-1, i\} - v(\{1, ..., i-1\}) \ \forall i \in N$$

Which of the Shapley axioms does the above satisfy and which of the Shapley axioms does it violate.

3. Problem 8 on Page 445

4. Problem 11 on page 445
Given a TU game (N, v), define the dual game (N, w) by

$$w(C) = v(N) - v(N \setminus C) \ \forall C \subseteq N$$

Show that the dual of the dual game is the original game (primal game) itself. Also show that the Shapley values of the primal game and the dual game are identical.

Assignment 12

Nucleolus

12.1 Reference Games

1. Village land game
   $N = \{1, 2, 3, 4\}$
   $v(1) = v(2) = v(3) = v(4) = 0$
   $v(23) = v(24) = v(34) = v(234) = 0$
   $v(12) = v(13) = v(14) = 10$
   $v(123) = v(124) = v(134) = 16$
   $v(1234) = 18$

2. Divide the Dollar
   $N = \{1, 2, 3\}$
   $v(1) = v(2) = v(3) = v(23) = 0$
   $v(12) = v(13) = v(123) = 300$

3. Voting Game
   $N = \{1, 2, 3, 4\}$
   $v(1) = v(2) = v(3) = v(4) = v(23) = v(24) = v(34) = 0$
   Rest of the v values are equal to 1.

4. MST Game
   $N = \{1, 2, 3\}$ $v(1) = 5; v(2) = 9; v(3) = 7$
   $v(12) = 15; v(13) = 12; v(23) = 17; v(123) = 23$

12.2 Warm-up

1. Define the Nucleolus

2. Intuitively, how is Shapley value different from Nucleolus

3. Why do you think nucleolus always belongs to core if the core is non-empty

4. Why do you think Shapley value may not belong to the core?

12.3 Work horse

1. Compute the nucleolus and Shapley value of Game 1

2. Compute the nucleolus and Shapley value of Game 2

3. Compute the nucleolus and Shapley value of Game 3

4. Compute the nucleolus and Shapley value of Game 4

12.4 Thought Provoking

1. Give an example of a game other than the above four games and other than divide-the-dollar where the nucleolus is the same as Shapley value
2. Which of the above games 1,2,3,4 are convex?
3. What is the computational complexity of Shapley value?
4. What is the computational complexity of nucleolus?

Assignment 13

Introduction to Mechanism Design

13.1 Warm-up

1. Consider the following set of preference lists:

Number of Voters	3	1	1	1	1	1	1
First Choice	A	A	B	B	C	C	D
Second Choice	D	B	C	C	B	D	C
Third Choice	B	C	D	A	D	B	B
Fourth Choice	C	D	A	D	A	A	A

   (a) Is there a Condorcet winner?
   (b) Who is the winner using Plurality Voting?
   (c) Who is the winner using Borda count?

2. Consider the following set of preference lists:

Number of Voters	2	2	1	1	1	1
First Choice	A	E	A	B	C	D
Second Choice	B	B	D	E	E	E
Third Choice	C	D	D	C	D	A
Fourth Choice	D	C	A	D	A	B
Fifth Choice	E	A	A	A	B	C

   (a) Is there a Condorcet winner?
   (b) Who is the winner using Plurality Voting?
   (c) Who is the winner using Borda count?

3. Consider the following set of preference lists:

Number of Voters	2	2	1	1	1
First Choice	A	B	A	C	D
Second Choice	D	D	B	B	B
Third Choice	C	A	D	D	A
Fourth Choice	B	C	C	A	C

   (a) Is there a Condorcet winner?
   (b) Who is the winner using Plurality Voting?
   (c) Who is the winner using Borda count?

4. Consider the following set of preference lists:

Number of Voters	1	1	1	1	1	1	1
First Choice	C	D	C	B	E	D	C
Second Choice	A	A	E	D	D	E	A
Third Choice	E	E	D	A	A	A	E
Fourth Choice	B	C	A	E	C	B	B
Fifth Choice	D	B	B	C	B	C	D

   (a) Is there a Condorcet winner?

   (b) Who is the winner using Plurality Voting?

   (c) Who is the winner using Borda count?

13.2 Workhorse

1. (a) Does Borda count satisfy pareto efficiency ? Explain.

   (b) Does Borda count satisfy monotonicity? Explain.

2. (a) Does Plurality voting satisfy pareto efficiency ? Explain.

   (b) Does Plurality voting satisfy monotonicity? Explain.

3. (a) Does Condorcet's rule satisfy pareto efficiency ? Explain.

   (b) Does Condorcet's rule satisfy monotonicity? Explain.

4. Explain: why it is impossible with odd number of voters, to have two distinct candidates win the same election using Condorcet's method.

13.3 Thought Provoking

1. Suppose that an election has candidates A, B,C and D. There are 4 voters, who submit the following ranked ballots:

Number of Voters	1	1	1	1
First Choice	B	C	C	A
Second Choice	A	A	D	D
Third Choice	D	B	B	B
Fourth Choice	C	D	A	C

   SHOW YOUR WORK FOR EACH OF THE FOLLOWING QUESTIONS.

   (a) Find the winner using using Borda Count Voting.

   (b) Find the winner using using a modified version of Borda Count Voting, where instead of assigning 3,2,1 and 0 points to each of first place, second place, third place and fourth place ranking, respective, we assign 4,2,-2 and -4 points, respectively.

   (c) Find the winner using using a modified version of Borda Count Voting, where instead of assigning 3,2,1 and 0 points to each of first place, second place, third place and fourth place ranking, respective, we assign -1,-3,-5 and -7 points, respectively.

   (d) Find the winner using using a modified version of Borda Count Voting, where instead of assigning 3,2,1 and 0 points to each of first place, second place, third place and fourth place ranking, respective, we assign 9,4,1 and 0 points, respectively.

(e) Does the choice of points in the modified versions of Borda Count Voting make a difference? If it does, find a rule for identifying those ways of assigning points that will give the same results as Standard Borda count voting.

2. Give examples of preference profiles in each of the following cases:

   (a) there is a Condorcet winner, but the Condorcet winner is not the same as the plurality winner.

   (b) where the winner of the plurality vote is not the same as the Borda winner

   (c) With three voters and four alternatives for which there is a Condorcet winner, but the Condorcet winner is not the Borda winner.

3. If, for a given set of voters and alternatives, there exists a Condorcet winner, then the Condorcet winner will get the highest score using the Borda count." True or false? Explain briefly.

4. Assume that there is an odd number of voters. For a set of votes (i.e., ranked lists) over a set A of alternatives, we say that alternative a beats b if more than half of the voters rank a somewhere above b in their lists. A Condorcet winner is an alternative that beats every other alternative. Show by example that there is not always a Condorcet winner. *Throughout this exercise set, assume that ties are broken in some consistent way, such as lexicographically.*

Assignment 14

Arrows Impossibility Theorem

14.1 Warm-up

1. What is the difference between social choice function and social welfare function. Give an example for each.
2. Prove that with only two alternatives the majority rule satisfies all three axioms in Arrows theorem.
3. In class we considered 4 alternatives to prove pairwise neutrality condition. Prove pairwise neutrality with just 3 alternatives.
4. A voting rule satisfies the *Condorcet condition* if it elects a Condorcet winner whenever one exists.
   (a) Does the plurality rule satisfy the Condorcet condition? Provide either a proof that it does or a counterexample (i.e., a set of votes where there is a Condorcet winner a and the rule chooses an alternative different from a).
   (b) Does the Borda count satisfy the Condorcet condition? Provide either a proof that it does or a counterexample.

14.2 Workhorse

1. Which of the axioms of Arrow's Impossibility Theorem does the following social choice rule violate? The rule : if person p_1 and person p_2 have exactly the same rankings of the alternatives, then the social choice rule will exactly coincide with their ranking ; if their rankings are different in any way, then the social choice rule will coincide with person p_3's ranking.
2. Show why the Borda count procedure is not strategy proof.
3. An election uses the following social choice method: Condorcet Voting is tried first, and if that method produces a winner, then that is the winner, and if that method does not produce a winner, Borda Count Voting is used.
   (a) Does this social choice method satisfy Pareto Criterion?
   (b) Does this social choice method satisfy Monotonicity Criterion?
4. For this question, use the following preference schedule:

Number of Voters	5	4	2
First Choice	A	B	D
Second Choice	B	D	A
Third Choice	C	C	C
Fourth Choice	D	A	B

(a) Find the winner of this election using the plurality method.

(b) Suppose that candidate B drops out of the race. Find the new winner under the plurality method.

14.3 Thought-Provoking

1. Consider that there are 3 alternatives and n voters. Also consider the following restriction on the preference orderings. At least $\lceil \frac{2n}{3} \rceil$ agents have exactly the same preference ordering. Prove that with this preference profile and using pairwise majority rule as social welfare function,

   (a) Gives the strict total ordering

   (b) Is Arrow-consistent (i.e. satisfies PE, IIA and Non-Dictatorial)

2. Consider the following restriction on the domain. Every agent places a fixed alternative e either at the top or at the bottom in their preference order. Prove that, with this restriction, a social welfare function satisfying PE and IIA conditions has to place an alternative e either at the top or at bottom.

3. Let F(.) be a social welfare function satisfying pareto efficiency and IIA conditions. Further, consider the sequence of preference profiles $(\Pi_i)_{i=1}^n$ where Π_i denote that exactly first $i-1$ agents place an alternative e at the bottom and rest of the agents place an alternative e at top in their preference,

   (a) Prove that $F(\Pi_0)$ places an alternative e at top.

   (b) Show that there exists an unique agent i^* such that $F(\Pi_{i^*})$ places an alternative e at the bottom whereas $F(\Pi_{i^*+1})$ places an alternative e at top.

4. Instead of strict ordering if the agents are allowed to be indifferent between two (or multiple) alternatives. Does the Arrows impossibility result still hold? If yes prove it, if no give an example.

Assignment 15

Gibbard- Satterthwaite Theorem

15.1 Warmup

1. A social welfare function F is said to satisfy *unanimity* if for every $\lhd \in L, F(\lhd, \lhd, \ldots, \lhd) = \lhd$. Prove that pareto efficiency is same as unanimity.

2. Find the social choice function which is incentive compatible and non-dictatorial with $|A| = 2$.

3. In the proof for Gibbard-Satterthwaite Theorem, we constructed a SWF F from SCF f. Prove that F satisfies Pareto Efficiency.

4. Which of Gibbard-Satterthwaite conditions commonly used voting methods such as method of majority rule, plurality rule and using points/rankings(borda count) violate.

15.2 Workhorse

1. Suppose that alternative A is selected given some preference profile. Modify the profile by raising some alternative X in individual i's ranking (holding everything else fixed). Then prove that either A or X is now selected.

2. In the proof for Gibbard-Satterthwaite Theorem, we constructed a SWF F from SCF f. Prove that F satisfies the axiom of Independence of Irrelevant Alternatives.

3. In the proof for Gibbard-Satterthwaite Theorem, we constructed a SWF F from SCF f. Prove that F does not satisfy dictatorship.

4. What happens when you remove the assumption regarding the onto nature of the Social Choice Function from the statement of GS Theorem?
   Moreover, can mechanisms that violate this assumption get some unusual and fascinating properties?

15.3 Thought-Provoking

1. Prove that any SCF which is not a dictatorship (i.e., the choice is not made according to the preferences of a single voter), and has at least three alternatives in its range, can be strategically manipulated.

2. Consider a preference profile $(\lhd_1, \ldots, \lhd_m)$ and an unanimous and incentive compatible social choice function f with $f(\lhd_1, \ldots, \lhd_m) = a,\ a \in A$. Now, consider that the agent i starts to move his $X \neq A$ upwards in his preference order one by one,

(a) Prove that there exists an agent i^* such that when he/she moves X upwards the outcome changes from a

(b) Prove that the new outcome should be X.
Hint: show that if an outcome is $C \neq a \neq X$ then an agent i^* could strategically manipulate the function.

3. In class, we proved the claim that for any $S \subset A, \triangleleft \in L^n$, a SCF f is incentive compatible and onto on A with $|A| \geq 3$, then $f(\triangleleft_1^S, \ldots, \triangleleft_n^S) \in S$. What happens when we remove the assumption that f is incentive compatible?

4. For a SCF f, incentive compatiblity $\Leftrightarrow$ monotonicity. One direction is proved in the class, prove the other side.

Assignment 16

Mechanisms with Money

16.1 Warm-up

1. The seller will run a sealed-bid, second-price auction. Your firm will bid in the auction, but it does not know for sure how many other bidders will participate in the auction. There will be either two or three other bidders in addition to your firm. All bidders have independent, private values for the good. Your firm's value for the good is c. What bid should your firm submit, and how does it depend on the number of other bidders who show up? Give a brief (1-3 sentence) explanation for your answer.

2. Assume an ascending auction (English Auction) with 3 bidders with the following private valuations: $v_1 = 500, v_2 = 450, v_3 = 440$. All the bidders decide to continue bidding till the price is lower than their valuations. The starting price(reserve price) is 430. The bid increment is 10.

   (a) Who is the winner?

   (b) What is the price to be paid by the winning bidder?

3. Assume a descending auction (Dutch Auction) with 3 bidders with the following private valuations: $v_1 = 500, v_2 = 450, v_3 = 440$. All the bidders decide to enter the bidding process if the price is strictly lower than their valuations. The bid decrement is 10.

   (a) Who is the winner?

   (b) What is the price to be paid by the winning bidder?

4. Are auctions with a random winner and price to be paid $p = 0$ DSIC? Prove or disprove.

16.2 Workhorse

1. Consider a second-price, sealed-bid auction with one seller who has one unit of the object which he values at s and two buyers 1,2 who have values of v_1 and v_2 for the object. The values s, v_1, v_2 are all independent, private values. Suppose that both buyers know that the seller will submit his own sealed bid of s, but they do not know the value of s. Is it optimal for the buyers to bid truthfully; that is should they each bid their true value? Give an explanation for your answer.

2. Prove that for every false bid $b_i \neq v_i$ by a bidder in a second-price auction, there exist bids b_{-i} by the other bidders such that i's utility when bidding b_i is strictly less than when bidding v_i.

3. There are two types of auctions A and B explained as follows:
   Auction A: Each bidder tries to outbid the other and bidders drop out one by one until only one bidder is left. Assume that the ongoing price increases by an infinitesimal increment. The item is awarded to the lone bidder left in the fray. This winner pays whatever is her latest bid.

Auction B: The auctioneer announces a high price first and waits to see if anyone is interested. If none is interested, lowers the price by a small amount(assume infinitesimal amount) and waits to see if anyone is interested. This is repeated until someone expresses interest in buying it at that announced price. The winner will pay this current price.

(a) Comment whether Auction A is equivalent to first price sealed bid auction.

(b) Comment whether Auction A is equivalent to second price sealed bid auction.

(c) Comment whether Auction B is equivalent to first price sealed bid auction.

(d) Comment whether Auction B is equivalent to second price sealed bid auction.

4. Prove that the game associated with the first-price auction with the players' valuations v has no Nash equilibrium iff v_n is the unique highest valuation.

16.3 Thought-Provoking

1. Consider a single-item auction with at least three bidders. Prove that awarding the item to the highest bidder, at a price equal to the third-highest bid, yields an auction that is not truthful.

2. Consider the auction for a single good between three bidders. For simplicity, each bidder has value $v_1 = v_2 = v_3 = v$ for the object. Consider a third price sealed-bid auction. That is, the winner of the auction is the one with the highest bid (ties broken randomly) and the winner must pay the third highest bid for the object (with only three bidders, the third highest bid is also the lowest submitted bid). Is it a dominant strategy for a bidder to bid its value v? Why or why not?

3. In second-price sealed-bid auction with n bidders, assume that the players' valuations of the object are all different and all positive; number the players 1 through n in such a way that $v_1 > v_2 > ... > v_n > 0$. In case of tie, assumption is that the winner is the player among those submitting the highest bid whose number is smallest (i.e. whose valuation of the object is highest).

   (a) Show that bids $(b_1, b_2, ..., b_n) = (v_1, v_2, ..., v_n)$ form a Nash Equilibrium of the game. Who gets the object?

   (b) Is $(b_1, b_2, ..., b_n) = (v_1, 0, 0, ..., 0)$ a Equilibrium of the game. Who gets the object?

   (c) $(b_1, b_2, ..., b_n) = (v_2, v_1, 0, ..., 0)$ a Nash Equilibrium of the game. Who gets the object?

   (d) Find a Nash-Equilibrium in which player n obtains the object.

4. A seller will run a second-price, sealed-bid auction for an object. There are two bidders, a and b, who have independent, private values v_i which are either 0 or 1. For both bidders the probabilities of $v_i = 0$ and $v_i = 1$ are each $\frac{1}{2}$. Both bidders understand the auction, but bidder b sometimes makes a mistake about his value for the object. Half of the time his value is 1 and he is aware that it is 1; the other half of the time his value is 0 but occasionally he mistakenly believes that his value is 1. Let's suppose that when b's value is 0 he acts as if it is 1 with probability $\frac{1}{2}$ and as if it is 0 with probability $\frac{1}{2}$. So in effect bidder b sees value 0 with probability $\frac{1}{4}$ and value 1 with probability $\frac{3}{4}$. Bidder a never makes mistakes about his value for the object, but he is aware of the mistakes that bidder b makes. Both bidders bid optimally given their perceptions of the value of the object. Assume that if there is a tie at a bid of x for the highest bid the winner is selected at random from among the highest bidders and the price is x. Is bidding his true value still a dominant strategy for bidder a? Explain briefly.

16.4 Other Questions

1. A ticket to a newly staged opera is on sale through a sealed-bid auction, which means none of the bidders can see the other players' bids. There are three bidders: Alice, Bert and Carl. Alice values the ticket at \$10, Bert at \$20, and Carl at \$30. The bidders are free to submit a bid of any positive amount.

   (a) Assume this is a second-price auction, that is, the highest bidder wins the ticket and pays the second-highest bid. If everyone bids his or her own valuation, what is the payoff of each bidder?

   (b) Show that the strategy "everyone bids his or her own valuation" is a Nash Equilibrium.

2. Are auctions with dictatorship DSIC? Prove or disprove.

Assignment 17

Myerson's Lemma

17.1 Warm-up

1. Consider a single-item auction with two bidders with valuations drawn i.i.d. from the uniform distribution on $[0, 1]$. Prove that the expected revenue obtained by the Vickrey auction (with no reserve) is $\frac{1}{3}$.

2. Consider a single-item auction with two bidders with valuations drawn i.i.d. from the uniform distribution on $[0, 1]$. Prove that the expected revenue obtained by the Vickrey auction with reserve $\frac{1}{2}$ is $\frac{5}{12}$.

3. Compute the virtual valuation function of the uniform distribution on $[0, a]$ with $a > 0$. Is this distribution regular (meaning the virtual valuation function is strictly increasing)?

4. Compute the virtual valuation function of the exponential distribution with rate $\lambda > 0$ (on $[0, \infty)$. Is this distribution regular (meaning the virtual valuation function is strictly increasing)?

17.2 WorkHorse

1. Compute the virtual valuation function of the distribution given by $F(v) = 1 - \frac{1}{(v+1)^c}$, where $c > 0$ is some constant. Is this distribution regular (meaning the virtual valuation function is strictly increasing)?

2. Use the "payment difference sandwich" in the proof of Myerson's Lemma to prove that if an allocation rule is not monotone, then it is not implementable.

3. Consider an arbitrary single-parameter environment, with feasible set X. The surplus-maximizing allocation rule is $x(b) = argmax_{(x_1,...,x_n)\in X} \sum_{i=1}^{n} b_i x_i$. Prove that this allocation rule is monotone.
   You should assume that ties are broken in a deterministic and consistent way, such as lexicographically.

4. Consider the following extension of the sponsored search setting discussed in lecture. Each bidder i now has a publicly known quality β_i (in addition to a private valuation v_i per click). As usual, each slot j has a CTR α_j, and $\alpha_1 \geq \alpha_2 \ldots \geq \alpha_k$. We assume that if bidder i is placed in slot j, its probability of a click is $\beta_i \alpha_j$, thus, bidder i derives value $v_i \beta_i \alpha_j$ from this outcome.
   Describe the surplus-maximizing allocation rule in this generalized sponsored search setting. Argue that this rule is monotone. Give an explicit formula for the per-click payment of each bidder that extends this allocation rule to a DSIC mechanism.

17.3 Thought Provoking

1. Use Myerson's Lemma to prove that the Vickrey auction is the unique single-item auction that is DSIC, always awards the good to the highest bidder, and charges losers 0.

2. We concluded the proof of Myerson's Lemma by giving a "proof by picture" that coupling a monotone and piecewise constant allocation rule x with the payment formula $p_i(b_i, b_{-i}) = \sum_{j=1}^{l} z_j.[\textit{jump in } x_i(., b_{-i}) \textit{ at } z_j]$ where $z_1, ..., z_l$ are the breakpoints of the allocation function $x_i(., b_{-i})$ in the range $[0, b_i]$, yields a DSIC mechanism. Where does the proof-by-picture break down if the piecewise constant allocation rule x is not monotone?

3. Compare and contrast an eBay auction with the sealed-bid second-price auction described in class. (Read up on eBay auction bidding rules if you don't already know how they work.) Should you bid differently in the two auctions?

4. Consider the distribution given by $F(v) = 1 - \frac{1}{(v+1)^c}$, with $c = 1$. Argue that when bidder valuations are drawn from this distribution, it is not necessarily the case that the expected revenue of an auction equals its expected virtual surplus.

Assignment 18

VCG Mechanisms

1. Consider a single-item auction with n bidders with valuations drawn i.i.d. from a regular distribution F. Prove that the expected revenue of the Vickrey auction (with no reserve price) is at least $\frac{n-1}{n}$ times that of the optimal auction (with the same number n of bidders). [Hint: deduce this statement from the Bulow-Klemperer theorem. When one new bidder is added, how much can the maximum-possible expected revenue increase?]

2. Consider an arbitrary single-parameter environment, with feasible set X. Suppose bidder i's valuation is drawn from a regular distribution F_i, with strictly increasing virtual valuation function φ_i. The virtual surplus-maximizing allocation rule is $x(b) = argmax_{(x_1,\ldots,x_n)\in X} \sum_{i=1}^{n} \varphi_i(b_i)x_i$. Prove that this allocation rule is monotone. *You should assume that ties are broken in a deterministic and consistent way, such as lexicographically.*

3. Prove that the payment $p_i(b)$ made by a bidder i in the VCG mechanism is at least 0 and at most $b_i(\omega^*)$,where ω^* is the outcome chosen by the mechanism.