Real Analysis

Manik Bhandari[*]
Department of Computer Science
Indian Institute of Science
Bangalore, India
mbbhandarimanik2@gmail.com

Abstract

Notes of Real Analysis mainly from Abbott's *Understanding Analysis* and Rudin's *Principles of Mathematical Analysis*.

1 Real Numbers

Rational numbers have *holes*.

Theorem 1. *There is no rational number whose square is 2.*

Proof. Proof is by contradiction. Let there be a rational number p/q where p and q have no common factors and whose square is 2. Find the common factor 2 and reach the contradiction. □

Further, let A be the set of all positive rational numbers q such that $q^2 < 2$ and B be the set of all rational numbers p such that $p^2 > 2$. Then, A contains no largest number and B has no smallest number. Consider

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2}.$$

$$\implies q^2 - 2 = 2\frac{p^2 - 2}{(p + 2)^2}.$$

If $p \in A$ then $q > p$ and $q \in A$. If $p \in B$ then $q < p$ and $q \in B$. Clearly, rational number system has certain holes. Interestingly between every two rational numbers r and s there is a rational $\frac{r+s}{2}$. Still the rational number system has gaps and the real system will try to fill these gaps by defining a new *irrational number* wherever there are holes.

1.1 Ordered Set

A set is a *collection* of objects called *elements* of the set. An order is a *relation* defined on a set, say S and is denoted by $<$. Order has 2 properties:

1. For $x \in S$ and $y \in S$, either $x < y$ or $x = y$ or $x > y$.
2. If $x, y, z \in S$, if $x < y$ and $y < z$ then $x < z$.

An *Ordered Set* is a set on which an order is defined.

Bound. Let S be an ordered set and $E \subset S$. If there is a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$ then E is *bounded above* and β is the *upper bound* of E. Note that β might not belong to E. Similar definition for *lower bound*.

[*]Use footnote for providing further information about author (webpage, alternative address)—*not* for acknowledging funding agencies.

32nd Conference on Neural Information Processing Systems (NIPS 2018), Montréal, Canada.

Least Upper bound. Let S be an ordered set and $E \subset S$ which is bounded above. Then if there is an $\alpha \in S$ such that

1. α is an upper bound of E and
2. if $\gamma < \alpha$, then γ is not an upper bound of E

then α is the *least upper bound* of E or *supremum* of E. There is at most one such number.

$$\alpha = supE.$$

Greatest Lower Bound. For the same S and E defined above, if there is a $\beta \in S$ such that (1) β is a lower bound of E and (2) if $\gamma > \beta$ then γ is not a lower bound of E then β is the *greatest lower bound* of E or *Infimum* of E.

$$\beta = infE.$$

Least Upper Bound Property. S has least upper bound property if for any $E \subset S$, if E is not empty and E is bounded above then sup E esists in S.

Theorem 2. *Suppose S has least upper bound property, then for every $B \subset S$, B is not empty and B is bounded below, inf B exists in S.*

Proof. Let L be the set of all lower bounds of B i.e. L consists of all $y \in S$ such that $y < x$ for every $x \in B$. Then L is not empty. L is bounded above by every element of B. So L must have a supremum in S, say α. α might not be in L.
Let $\gamma < \alpha$, then by definition, γ is not an upper bound of L but every element of B is an upper bound of L, so $\gamma \notin B$. This means that $\alpha \leq x$ for every $x \in B$. So α is a lower bound of $B \implies \alpha \in L$. Also, if $\beta > \alpha$ then $\beta \notin L$ since α is an upper bound of L. So, α is a lower bound of B but β is not if $\beta > \alpha$. $\implies \alpha = \inf B$. $\square$

1.2 Field

A Field is a set with two operations - *addition* and *multiplication* which satisfy *Field Axioms* - A (addition), M (multiplication), D (Distributive Law).

Addition Axioms:

1. *Closure.* If $x \in F$ and $y \in F$, then $x + y \in F$.
2. *Commutative.* $x + y = y + x$ for all $x, y \in F$.
3. *Associative.* $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.
4. *Identity.* F contains an element 0 such that, $0 + x = x$ for every $x \in F$.
5. *Inverse.* For every $x \in F$ there is an element $-x \in F$ such that, $x + (-x) = 0$.

Multiplication Axioms:

1. *Closure.* If $x \in F$ and $y \in F$, then $xy \in F$.
2. *Commutative.* $xy = yx$ for all $x, y \in F$.
3. *Associative.* $(xy)z = x(yz)$ for all $x, y, z \in F$.
4. *Identity.* F contains an element 1 ($1 \neq 0$) such that, $1x = x$ for every $x \in F$.
5. *Inverse.* For every $x \in F$ (except 0) there is an element $1/x \in F$ such that, $x(1/x) = 1$.

Distributive Law: For all $x, y, z \in F$, $x(y + z) = xy + xz$.
Clearly, Q, the set of all rational numbers is a Field.

1.2.1 Ordered Field

Ordered Field F is a Field that is also an ordered set such that

1. $x + y < x + z$ if $y < z$ for $x, y, z \in F$.
2. $xy > 0$ if $x > 0$ and $y > 0$ for $x, y \in F$.

1.3 The Real Field

There exists an ordered field R which has *least upper bound property* and *contains* Q as a sub-field. Elements of R are *real numbers*. R can be constructed from Q.

Archimedean Property For $x, y \in R$, if $x > 0$ you can always find a positive integer n such that $nx > y$.

Q **is dense in R** For every $x, y \in R$ you can find a $p \in Q$ such that $x < p < y$.

Existence of n^{th} roots For every positive real $x > 0$ and positive integer $n > 0$ there is one and only one positive real $y > 0$ such that $y^n = x$. **Need to study proof of this.**

Extended Real System . For every $x \in R$, define the symbols $-\infty$ and ∞ such that $-\infty < x < \infty$. So even if a subset E of R was not bounded above, in the extended real system it has a least upper bound $SupE = \infty$. But this is **not** a field!