Game Theory

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Abstract

Notes of Game Theory mainly from lectures at IISC.

1 Introduction and Examples

The course can be divided into four parts. We start with Non cooperative Game Theory.

Game Theory vs Mechanism Design In Game Theory, you are given a game and you must analyze it. *Analyzing* a game typically means finding and analyzing the *Equilibrium*. An informal definition of equilibrium is when all the players of the game are *happy* i.e. if they were to perform some other action, they would not be happy.

In Mechanism Design, you are given a *desirable behavior* of the players i.e. a *Social Choice Function*. Your task, is to design a game that implements the SCF in the *optimal case*. Optimal case means equilibrium?

Student Coordination Problem Consider two students, call them P_1 and P_2. They wish to meet and chat. They can either do that in IISC or on MG Road. These two actions are called *strategies*. Let "going to IISC" be strategy A and "going to MG Road" be strategy B. Then the *strategy sets* of the two players are $S_1 = S_2 = \{A, B\}$.

Assume that *strategies are chosen simultaneously*. This is called a *Simultaneous Moves Game*. A game like chess is not a simultaneous moves game. The fact that player must choose strategies means that this is a *Strategic form Game*. What other form of game can exist?.

The *utility* of a player is a function $u_i : S_1 \times S_2 \to \mathbb{R}$ (subscript i for player). It maps a *vector or profile* of strategies to a real number called *utility or payoff?*.

Assumption of Common Knowledge This means that

1. Each player knows the payoff matrix.
2. Each player knows 1.
3. Each player knows 2.
4. .
5. .

ad infinitum.

1/2	A	B
A	100,100	0, 0
B	0, 0	10, 10

Table 1: Payoff matrix.

Players are Rational and Intelligent A *rational player* maximizes the *Expected value of his own payoff*. This is called *Expected Utility Maximization*. An *Intelligent* player is capable of computing the optimal strategy.

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Equilibrium In this example, there are two equilibria – (A, A) and (B, B). The intuition is that if a player would choose some other action, he won't be as happy. This is not clear. This is an example of *Pure Strategy Equilibrium* but another possibility is to have a *probabilistic strategy* leading to *Mixed Strategy Equilibrium*. A strategy will then be a vector of probabilities of choosing each action.

In this case the mixed strategy equilibrium is $(1/11, 10/11), (1/11, 10/11)$ i.e. players can choose to be in MG road most of the time and still be pretty happy.

Cake cutting problem This is an example of Mechanism Design. Suppose you and your brother want to eat a cake and your mom (The mechanism designer) is to decided how to divide the cake. The SCF is not just fair division, it is to make everyone involved *happy*.

If she cuts the cake, you might feel that your brother will pick the bigger part leading to dissatisfaction. Consider this strategy: you cut the cake but your brother picks the first piece. Since you cut the cake and there is *common knowledge*, you are absolutely sure that it is exactly in half so you're satisfied. Since he picks the first, he might pick what he thinks is the bigger part and he'll be happy. Possible extensions for this could be to n cakes and m players.

Braess Paradox Adding more links should help but doesn't. Why?

Vickrey Auction Auctions are essential. Google has a few slots to display for ads and many companies would like to place their ad in the slot. If the user clicks on an ad, the company pays Google for the click. How to decide the price for obtaining the slot? Best way is to bid for it.

Winner Determination: The winner is decided by whoever bids the highest.

Pricing: In *First price auctions* the winning bid would be the price of the slot. But Vickrey introduced *Second price auctions* where the second highest bid or the highest non-winning bid is the price of the slot.

This extracts the exact *validation* of each player i.e. the maximum willingness to buy the item. This also satisfies *allocative efficiency* i.e. the item goes to the person who values it the most. It also has the *DSIC property (Dominant Strategy Incentive Compatibility)* which means that you can safely ignore other players and always bid your true validation. (There also exists others like BIC – Bayesian Incentive Compatibility) There is also no complexity of computing the strategy.

This has led to VCG (Vickrey Clarke Groves) mechanism

Cooperative Game Theory This is the form of GT when people form a coalition or a group for their benefit.

Divide the dollar problem Suppose you have 300$ and you want to divide it amongst three people such that *at least two should agree* on the division. Let's say P_1 proposes $(150, 150, 0)$. This is also a simultaneous moves game. We say that coalition $\{1, 2\}$ is formed. But P_3 will not sit quietly. He proposes $(200, 0, 100)$. Coalition $\{1, 3\}$ is formed since both P_1 and P_3 have increased their utility. But now P_2 will not sit quietly, he proposes $(0, 100, 200)$ and thus Coalition $\{2, 3\}$ is formed. This is a highly unstable situation where players will keep cycling.

Variations of this could be similar game but "a leader and one other should agree" or "all three should agree". The dynamics will completely change.

2 Utility Theory

Every game of chess ends in either a win for white, win for black, or draw. Simple proof idea: think of the game of chess as a tree with children being the reachable board positions from parent. Start from the leaf and build up.

preference relation A player wants his *best possible outcome*. To measure this we want a relation between all possible outcomes. An order?. This relation is *preference relation*. It is a binary relation over the set of possible outcomes O i.e. $x \geq_i y$ implies that player i prefers outcome x over y or is indifferent between the two.

Assumptions about the preference relation

1. It is complete i.e. for any pair of outcomes x, y in O either $x \geq_i y$ or $x \leq_i y$ or both.
2. It is reflexive i.e. $x \geq_i x$ for all $x \in O$.
3. It is transitive i.e. for any $x, y, z \in O$ if $x \geq_i y$ and $y \geq_i z$ then $x \geq_i z$.

Utility function is a mapping from $O \rightarrow \mathbb{R}$ such that if $x \geq_i y$ then $u(x) \geq_i u(y)$. Essentially it is an association of preference with a real number. Note that for any monotonically increasing function $v : \mathbb{R} \rightarrow \mathbb{R}$, $v \cdot u = v(u(x))$ is also a utility function for the same preference relation. So a utility function u is only *ordinal* i.e the real number is not a measure of intensity of a player's preference.

3 Strategic form games a.k.a. Normal form

Definition A Strategic Form Game is an ordered triple $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$. N is the *finite* set of players $N = 1, 2, .., n$, S_i is the set of strategies of player i. Denote the set of all possible strategy vectors as $S = S_1 \times S_2 ... \times S_n$. And $u_i : S \rightarrow \mathbb{R}$ maps a vector of strategies $s = (s_i)_i$ to the payoff of the i^{th} player.

Note the strategy set of a player need not be finite. If it is finite, we call it a *finite game*.

Notation Denote X as the cross product $\times_i X_i$. Then $X_{-i} = \times_{j \neq i} X_j$ i.e. an element of X_{-i} is an n-1 dimensional vector $x_{i-1} = (x_i, ...x_{i-1}, x_{i+1}...x_n)$.

Domination The strategy s_i of a player is dominated by t_i if for *each* strategy of other players $s_{-i} \in S_{-i}$

$$u_i(s_i, s_{-i}) < u_i(t_i, s_{-i})$$

So t_i strictly dominates s_i or s_i is strictly dominated by t_i.

Assumptions A rational player will never choose a strictly dominated strategy and all players in the game are rational. Also assume rationality is common knowledge.

Strictly dominant strategy of a player is one which strictly dominates all other strategies of the player.