
Theory of Deep Learning

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Abstract

Notes of Theory of Deep Learning mainly from lectures at IISC.

1 Introduction

Define error of a classifier (aka *true error*) as probability of it making a mistake given a random data point.

$$L_{D,f}(h) = \mathbb{P}_{x \sim D}[h(x) \neq f(x)] = D(\{x : h(x) \neq f(x)\})$$

where D is the distribution from where a data point x is drawn, f is a known *correct* function which always gives the correct labels to a data point. By this definition $D(A)$ is the probability of observing a random point x from A .

Define training error or *empirical risk* as

$$L_S(h) = \frac{|\{i \in [m] : h(x_i) \neq y_i\}|}{m}$$

where S is the training *set* (it is actually a sequence since points can repeat and classifiers often take into account their order) of the form $\{(x_i, y_i)\}$. If you naively minimize this empirical risk then you are likely to overfit. To avoid it, you use some prior knowledge about the *kind of classifier* that can possibly fit to the data and restrict your hypothesis search space to those types of classifiers.

This kind of restriction induces a *bias* in the model (aka *inductive bias*). In this setting, define

$$h_S = \text{ERM}_h(S) \in \underset{h \in \mathcal{H}}{\text{argmin}} L_S(h).$$

This is a tradeoff – choosing a restricted \mathcal{H} can add too much bias but choosing a large \mathcal{H} may lead to overfitting.

Finite hypothesis class If we restrict \mathcal{H} to have an upper bound on its size then ERM_h will not overfit if we have *large* training data (how large will depend on size of \mathcal{H}).

Realizability Assumption There exists $h^* \in \mathcal{H}$ such that $L_{D,f}(h^*) = 0$ i.e. it never makes a mistake which means that $L_S(h^*) = 0$. Since this is the least possible error, this means that for every ERM hypothesis $L_S(h_S) = 0$. We are however interested in true error of h_S i.e. $L_{D,f}(h_S)$.

iid assumption Assume that elements of S are identically and independently distributed according to D denoted by $S \sim D^m$.

Now, we would like to have an h_S such that $L_{D,f}(h_S)$ is not too large. Let's say h_S *fails* if $L_{D,f}(h_S) > \epsilon$.

We want to upper bound the probability of sampling a training set that leads to a failure i.e. $D^m(S : L_{D,f}(h_S) > \epsilon)$. Define bad hypothesis as $\mathcal{H}_B = \{h \in \mathcal{H} : L_{D,f}(h_S) > \epsilon\}$ and misleading training sets as $M = \{S : \exists h \in \mathcal{H}_B, L_S(h) = 0\}$. So, all the training sets for which h_S fails must be misleading (there can be other misleading sets also). So

$$\{S : L_{D,f}(h_S) > \epsilon\} \subseteq M = \bigcup_{h \in \mathcal{H}_B} \{S : L_S(h) = 0\}.$$

This means that

$$D^m(\{S : L_{D,f}(h_S) > \epsilon\}) \leq D^m(M) = D^m\left(\bigcup_{h \in \mathcal{H}_B} \{S : L_S(h) = 0\}\right).$$

Take union bound of RHS to get

$$D^m(\{S : L_{D,f}(h_S) > \epsilon\}) \leq \sum_{h \in \mathcal{H}_B} D^m(\{S : L_S(h) = 0\}) = \sum_{h \in \mathcal{H}_B} \left(\prod_{i=1}^m D(\{x_i : h(x_i) = f(x_i)\}) \right).$$

and since $h \in \mathcal{H}_B$, $D(\{x_i : h(x_i) = f(x_i)\}) \leq 1 - \epsilon$. But each x_i is iid over D so $D^m(\{S : L_S(h) = 0\}) \leq (1 - \epsilon)^m \leq e^{-\epsilon m}$. As m goes large, the probability of finding a misleading set reduces. Therefore

$$D^m(\{S : L_{D,f}(h_S) > \epsilon\}) \leq |\mathcal{H}_B| e^{-\epsilon m} \leq |\mathcal{H}| e^{-\epsilon m}.$$

Take log both sides to get

$$\log D^m(\{S : L_{D,f}(h_S) > \epsilon\}) \leq \log |\mathcal{H}| - \epsilon m \implies m \leq \frac{\log(|\mathcal{H}|/\delta)}{\epsilon}$$

where $\delta = D^m(\{S : L_{D,f}(h_S) > \epsilon\})$. This also implies that if m is large enough i.e. $m \geq \frac{\log(|\mathcal{H}|/\delta)}{\epsilon}$ then $L_{D,f}(h_S) \leq \epsilon$ with probability $1 - \delta$ of choosing the iid samples S .

So with the ERM_h rule, your hypothesis will be *probably* $(1 - \delta)$ *approximately* (ϵ) *correct* (PAC). Note that the size m does not depend upon the underlying distribution or labeling function.

PAC Learnability A hypothesis class \mathcal{H} is PAC learnable if $\exists m_{\mathcal{H}} : (0, 1)^2 \rightarrow \mathbb{N}$ and a learning algorithm such that

For every $(\epsilon, \delta) \in (0, 1)$, for every distribution \mathcal{D} over \mathcal{X} and for every labeling function $f : \mathcal{X} \rightarrow (0, 1)$

If the realizability assumption holds over $\mathcal{H}, \mathcal{D}, f$

then running the algorithm on $m > m_{\mathcal{H}}(\epsilon, \delta)$ samples generated iid from \mathcal{D} and labeled by f gives a hypothesis h such that

with probability at least $1 - \delta$ over the choice of examples, $L_{\mathcal{D},f}(h) \leq \epsilon$.

Sample complexity $m_{\mathcal{H}} : (0, 1)^2 \rightarrow \mathbb{N}$ defines the *sample complexity* of learning \mathcal{H} i.e. how many samples are required to get a PAC solution. Let it be the *minimum function* that satisfies the criteria of PAC learnability.

Sample complexity of finite hypothesis class Every finite hypothesis class is PAC learnable with sample complexity $m \leq \lceil \frac{\log(|\mathcal{H}|/\delta)}{\epsilon} \rceil$

Removing realizability assumption Assuming that such an h^* exists such that

$\mathbb{P}_{x \sim \mathcal{D}}[h^*(x) = f(x)] = 1$ is too strong. Not only might such an h^* not exist, your features might not be discriminative enough. Instead assume that \mathcal{D} is a joint distribution over domain points \mathcal{X} and labels \mathcal{Y} . Now, true error

$$L_D(h) = \mathbb{P}_{(x,y) \sim \mathcal{D}}[h(x) \neq y] = \mathcal{D}(\{(x,y) : h(x) \neq y\})$$

Bayes optimal predictor is the best labeling function defined as

$$f_{\mathcal{D}}(x) = \begin{cases} 1 & \text{if } \mathbb{P}[y = 1|x] \geq 0.5 \\ 0 & \text{otherwise} \end{cases}$$

i.e. there is no other labeling function with a lower true error rate.

Agnostic PAC Learnability A hypothesis class \mathcal{H} is agnostic PAC learnable w.r.t. a set \mathcal{Z} and a loss function $l : \mathcal{Z} \rightarrow \mathbb{R}_+$ if there exists a function $m_{\mathcal{H}} : (0, 1)^2 \rightarrow \mathbb{N}$ and a learning algorithm such that

for every $\epsilon, \delta \in (0, 1)$ and for every \mathcal{D} over \mathcal{Z} when the algorithm is run $m \geq m_{\mathcal{H}}(\epsilon, \delta)$ samples iid from \mathcal{D} , the algo returns a hypothesis $h \in \mathcal{H}$ such that with probability $1 - \delta$ over the training samples

$$L_{\mathcal{D}}(h) = \min_{h' \in \mathcal{H}} L_{\mathcal{D}}(h') + \epsilon$$

where $L_{\mathcal{D}}(h) = \mathbb{E}_{z \sim \mathcal{D}}[l(h, z)]$.