
Real Analysis

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Abstract

Notes of Real Analysis mainly from Abbott's *Understanding Analysis* and Rudin's *Principles of Mathematical Analysis*.

1 Real Numbers

Rational numbers have *holes*.

Theorem 1. *There is no rational number whose square is 2.*

Proof. Proof is by contradiction. Let there be a rational number p/q where p and q have no common factors and whose square is 2. Find the common factor 2 and reach the contradiction. \square

Further, let A be the set of all positive rational numbers q such that $q^2 < 2$ and B be the set of all rational numbers p such that $p^2 > 2$. Then, A contains no largest number and B has no smallest number. Consider

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2}.$$
$$\implies q^2 - 2 = 2 \frac{p^2 - 2}{(p + 2)^2}.$$

If $p \in A$ then $q > p$ and $q \in A$. If $p \in B$ then $q < p$ and $q \in B$. Clearly, rational number system has certain holes. Interestingly between every two rational numbers r and s there is a rational $\frac{r+s}{2}$. Still the rational number system has gaps and the real system will try to fill these gaps by defining a new *irrational number* wherever there are holes.

1.1 Ordered Set

A set is a *collection* of objects called *elements* of the set. An order is a *relation* defined on a set, say S and is denoted by $<$. Order has 2 properties:

1. For $x \in S$ and $y \in S$, either $x < y$ or $x = y$ or $x > y$.
2. If $x, y, z \in S$, if $x < y$ and $y < z$ then $x < z$.

An *Ordered Set* is a set on which an order is defined.

Bound. Let S be an ordered set and $E \subset S$. If there is a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$ then E is *bounded above* and β is the *upper bound* of E . Note that β might not belong to E . Similar definition for *lower bound*.

Least Upper bound. Let S be an ordered set and $E \subset S$ which is bounded above. Then if there is an $\alpha \in S$ such that

1. α is an upper bound of E and
2. if $\gamma < \alpha$, then γ is not an upper bound of E

then α is the *least upper bound* of E or *supremum* of E . There is at most one such number.

$$\alpha = \sup E.$$

Greatest Lower Bound. For the same S and E defined above, if there is a $\beta \in S$ such that (1) β is a lower bound of E and (2) if $\gamma > \beta$ then γ is not a lower bound of E then β is the *greatest lower bound* of E or *Infimum* of E .

$$\beta = \inf E.$$

Least Upper Bound Property. S has least upper bound property if for any $E \subset S$, if E is not empty and E is bounded above then $\sup E$ exists in S .

Theorem 2. Suppose S has least upper bound property, then for every $B \subset S$, B is not empty and B is bounded below, $\inf B$ exists in S .

Proof. Let L be the set of all lower bounds of B i.e. L consists of all $y \in S$ such that $y < x$ for every $x \in B$. Then L is not empty. L is bounded above by every element of B . So L must have a supremum in S , say α . α might not be in L .

Let $\gamma < \alpha$, then by definition, γ is not an upper bound of L but every element of B is an upper bound of L , so $\gamma \notin B$. This means that $\alpha \leq x$ for every $x \in B$. So α is a lower bound of $B \implies \alpha \in L$. Also, if $\beta > \alpha$ then $\beta \notin L$ since α is an upper bound of L . So, α is a lower bound of B but β is not if $\beta > \alpha$. $\implies \alpha = \inf B$. \square

1.2 Field

A Field is a set with two operations - *addition* and *multiplication* which satisfy *Field Axioms* - A (addition), M (multiplication), D (Distributive Law).

Addition Axioms:

1. *Closure.* If $x \in F$ and $y \in F$, then $x + y \in F$.
2. *Commutative.* $x + y = y + x$ for all $x, y \in F$.
3. *Associative.* $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.
4. *Identity.* F contains an element 0 such that, $0 + x = x$ for every $x \in F$.
5. *Inverse.* For every $x \in F$ there is an element $-x \in F$ such that, $x + (-x) = 0$.

Multiplication Axioms:

1. *Closure.* If $x \in F$ and $y \in F$, then $xy \in F$.
2. *Commutative.* $xy = yx$ for all $x, y \in F$.
3. *Associative.* $(xy)z = x(yz)$ for all $x, y, z \in F$.
4. *Identity.* F contains an element 1 ($1 \neq 0$) such that, $1x = x$ for every $x \in F$.
5. *Inverse.* For every $x \in F$ (except 0) there is an element $1/x \in F$ such that, $x(1/x) = 1$.

Distributive Law: For all $x, y, z \in F$, $x(y + z) = xy + xz$.
Clearly, \mathbb{Q} , the set of all rational numbers is a Field.

1.2.1 Ordered Field

Ordered Field F is a Field that is also an ordered set such that

1. $x + y < x + z$ if $y < z$ for $x, y, z \in F$.
2. $xy > 0$ if $x > 0$ and $y > 0$ for $x, y \in F$.

1.3 The Real Field

There exists an ordered field R which has *least upper bound property* and contains Q as a sub-field. Elements of R are *real numbers*. R can be constructed from Q .

Archimedean Property For $x, y \in R$, if $x > 0$ you can always find a positive integer n such that $nx > y$.

Q is dense in R For every $x, y \in R$ you can find a $p \in Q$ such that $x < p < y$.

Existence of n^{th} roots For every positive real $x > 0$ and positive integer $n > 0$ there is one and only one positive real $y > 0$ such that $y^n = x$. **Need to study proof of this.**

Extended Real System . For every $x \in R$, define the symbols $-\infty$ and ∞ such that $-\infty < x < \infty$. So even if a subset E of R was not bounded above, in the extended real system it has a least upper bound $Sup E = \infty$. But this is **not** a field!

1.4 Complex Field

A complex number is an ordered pair (a, b) such that if $x = (a, b)$ and $y = (c, d)$:

1. $x + y = (a + c, b + d)$
2. $xy = (ac - bd, ad + bc)$

This (definition of addition and multiplication) makes the set of complex numbers a field. We can define $i = (0, 1)$. If $z = a + bi$ then its *conjugate* $\bar{z} = a - bi$. Also $a = Re(z)$ and $b = Im(z)$. *Absolute Value* of z is positive square root of $z\bar{z}$.

Schwarz Inequality If a_i and b_j are complex numbers then

$$\left| \sum_{j=1}^{j=n} a_j \bar{b}_j \right|^2 \leq \left| \sum_{j=1}^{j=n} a_j \right|^2 \left| \sum_{j=1}^{j=n} b_j \right|^2$$

1.5 Euclidean Space

Let R^k be the set of ordered k -tuples. Define

1. **Addition of vectors** $x + y = x_1 + y_1 + \dots + x_k + y_k$
2. **Multiplication by scalar** $\alpha x = \alpha x_1 + \dots + \alpha x_k$

Doubt: What's the conjugate of a vector?

2 Topology

Function Let there be 2 sets A and B . For **each** element of A associate an element of B called $f(x)$. Then f is a *function*, A is its *domain* and the set $f(A)$ is its *range*. (f is also called a mapping of A into B)

Image Let $f : A \rightarrow B$. If $E \subset A$ define $f(E)$ to be set of $f(x)$ for $x \in E$. Then $f(E)$ is the *image* of f under E . If $f(E) = B$ then f maps A *onto* B .

Inverse Image If $E \subset B$, define $f^{-1}(E)$ as set of all $x \in A$ where $f(x) \in E$. **Note:** All elements of E need not have a corresponding x in A . If for every $y \in B$, $f^{-1}(y)$ is **at most one** element of A then f is a *one-to-one* mapping.

Equivalence If f is a *one-one* and maps A onto B then A and B are *equivalent* written as $A \sim B$. \sim is a relation with following properties

1. **reflexive** $A \sim A$
2. **symmetric** If $A \sim B$ then $B \sim A$
3. **transitive** If $A \sim B$ and $B \sim C$ then $A \sim C$

Another way to say: A and B have 1-1 *correspondance*, A and B have same *cardinal number*.

Countable Let A be a set, J_n be integers $1, 2..n$ and J be set of all positive integers. A is **finite** if $J_n \sim A$ else A is **infinite**. A is **countable** if $J \sim A$ else A is **uncountable**. A is *at most countable* if A is finite or countable. Countable sets are a.k.a. *enumerable* or *denumerable*.

Sequence A sequence is a function f defined on J . Let $f(n)$ be x_n then sometimes f is denoted by $x_1, x_2...x_n$ or $\{x_n\}$.

Theorem 3. *Every infinite subset of a countable set is countable.*

Theorem 4. *Countable union of countable sets is countable. i.e. if $\{E_n\}$ is a sequence of countable sets then $S = \cup_{n=1}^{\infty} E_n$ is countable.*

Use it to prove: Set of all n -tuples of elements of a countable set is countable.

Set of all binary sequences is uncountable. (Cantor's diagonal process). Since every real number can be represented as a binary sequence, set of all real numbers is uncountable.