20F-MATH168-1 HW 1

PRATYUSHA MAJUMDER

TOTAL POINTS

39 / 45

QUESTION 1

1 Read ch. 1-6 5 / 5

✓ + **5 pts** **Completed. (I have no way of checking, so this is on the honor system.)**

QUESTION 2

2 Linear algebra theorem 10 / 10

✓ + **10 pts** **Correct**

+ **5 pts** Partial credit

QUESTION 3

3 Problem 6.1 10 / 10

✓ + **10 pts** **Full credit**

+ **9 pts** Minor mistakes

QUESTION 4

4 Problem 6.2 5 / 10

✓ + **5 pts** **Part a (maximum edges), with a correct explanation**

+ **5 pts** Part b (minimum edges), with a correct explanation

"single component" means the network is connected. If there are no edges, the network is disconnected.

QUESTION 5

5 Problem 6.4 9 / 10

✓ + **2.5 pts** **Part a**

✓ + **2.5 pts** **Part b**

✓ + **2.5 pts** **Part c**

✓ + **2.5 pts** **Part d**

+ **0 pts** Missing

- 1 Point adjustment

Missing a factor of 1/2 in part b.

Favourite theorem from linear algebra + proof:

planar - network that can be drawn on a plane without having any edges cross

tree - connected, undirected network with no loops

Thm: Dimension Theorem

Let V be a finite-dimensional space, and let $T: V \to W$ be a linear transformation. Then

$$\text{nullity}(T) + \text{rank}(T) = \dim(T).$$

Proof: · Let $n = \dim(V)$, since $\dim(V)$ finite by hypothesis.

- $N(T) \subset V \Rightarrow \dim(N(T))$ is finite
- $\therefore$ Let $k = \dim(N(T)) = \text{nullity}(T)$

- Since $k = \dim(N(T))$, let $\{v_1, \dots, v_k\}$ be the basis of $N(T)$.
- $\{v_1, \dots, v_k\}$ is a basis of $N(T)$ $\Rightarrow$ $v_1, \dots, v_k$ are linearly independent.
- Since $\{v_1, \dots, v_k\}$ is a set of k elements that is a basis of $N(T)$, it lies in $N(T)$ and $N(T) \subset V$

$\Rightarrow \{v_1, \dots, v_k\}$ is a linearly independent set of k elements lying in V

$\underset{\text{By corollary}}{\Longrightarrow} \{v_1, \dots, v_k\}$ forms part of the basis of V.

- Let $\{v_{k+1}, \dots, v_n\}$ be the extra elements added to $\{v_1, \dots, v_k\}$ to form the basis of V.

$\Rightarrow \{v_{k+1}, \dots, v_n\}$ lie in V, and $\{v_1, \dots, v_n\}$ spans V.

- Applying T to $\{v_{k+1}, \dots, v_n\}$ we get:

$$\{Tv_{k+1}, \dots, Tv_n\} \in \text{Im}(T)$$

Sub-proof A): $\{Tv_{k+1}, \dots, Tv_n\}$ spans $\text{Im}(T)$

- Let $\vec{w}$ be a vector in $\text{Im}(T)$
- By definition of $\text{Im}(T)$, $\vec{w} = T\vec{v}$ for some $\vec{v} \in V$.
- Since $\{v_1, \dots, v_n\}$ spans V,

$$\vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$$

for $a_1, \dots, a_n \in \mathbb{R}$

$$\underset{\text{applying } T}{\Longrightarrow} T\vec{v} = T(a_1\vec{v}_1 + \dots + a_n\vec{v}_n)$$

$$= T(a_1\vec{v}_1) + \dots + T(a_n\vec{v}_n)$$

$$= a_1 T\vec{v}_1 + \dots + a_n T\vec{v}_n$$

($A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2$ by linearity.

($Ak\vec{v} = kA\vec{v}$ by linearity)

Since $v_1, \dots, v_k$ lie in $N(T)$,

$$T\vec{v}_1 = T\vec{v}_2 = \dots = T\vec{v}_k = 0.$$

$\Rightarrow T\vec{v} = a_{k+1}T\vec{v}_{k+1} + \dots + a_n T\vec{v}_n = w$

Hence, for any vector $\vec{w}$ in $\text{Im}(T)$,

$$\vec{w} = a_{k+1}T\vec{v}_{k+1} + \dots + a_n T\vec{v}_n$$

1 Read ch. 1-6 **5 / 5**

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· $\{v_1, \dots, v_k\}$ is a basis of $N(T) \Rightarrow v_1, \dots, v_k$ are linearly independent.

· Since $\{v_1, \dots, v_k\}$ is a set of k elements that is a basis of $N(T)$, it lies in $N(T)$ and $N(T) \subset V$

$\Rightarrow \{v_1, \dots, v_k\}$ is a linearly independent set of k elements lying in V

$\xRightarrow[\text{By corollary}]{} \{v_1, \dots, v_k\}$ forms part of the basis of V.

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· Let $\vec{w}$ be a vector in $\text{Im}(T)$

· By definition of $\text{Im}(T)$, $\vec{w} = T\vec{v}$ for some $\vec{v} \in V$.

· Since $\{v_1, \dots, v_n\}$ spans V,

$$\vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$$

for $a_1, \dots, a_n \in \mathbb{R}$

$$\xRightarrow{\text{applying } T} T\vec{v} = T(a_1\vec{v}_1 + \dots + a_n\vec{v}_n)$$

$$= T(a_1\vec{v}_1) + \dots + T(a_n\vec{v}_n)$$

$$= a_1 T\vec{v}_1 + \dots + a_n T\vec{v}_n$$

($A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2$ by linearity.

($Ak\vec{v} = kA\vec{v}$ by linearity)

Since $v_1, \dots, v_k$ lie in $N(T)$,

$$T\vec{v}_1 = T\vec{v}_2 = \dots = T\vec{v}_k = 0.$$

$\Rightarrow T\vec{v} = a_{k+1}T\vec{v}_{k+1} + \dots + a_n T\vec{v}_n = w$

Hence, for any vector $\vec{w}$ in $\text{Im}(T)$,

$\vec{w} = a_{k+1}T\vec{v}_{k+1} + \dots + a_n T\vec{v}_n$

Thus, $\{T\vec{v}_{k+1}, \dots T\vec{v}_n\}$ spans $Im(T)$. $(*)$

Sub-proof B: $\{T\vec{v}_{k+1}, \dots, T\vec{v}_n\}$ linearly independent

Proof: Suppose $\{T\vec{v}_{k+1}, \dots T\vec{v}_n\}$ not linearly independent

$$\Rightarrow a_{k+1} T\vec{v}_{k+1} + \dots + a_n T\vec{v}_n = 0$$

for $a_{k+1}, \dots, a_n \in \mathbb{R}$ such that not all $a_{k+1}, \dots, a_n$ were 0.

- $T(a_{k+1}\vec{v}_{k+1} + \dots + a_n\vec{v}_n) = 0$ by linearity of T

$$\Rightarrow a_{k+1}\vec{v}_{k+1} + \dots + a_n\vec{v}_n \in N(T)$$

Since $\text{span}(N(T)) = \{v_1, \dots, v_k\}$, this would imply

$$a_{k+1}\vec{v}_{k+1} + \dots + a_n\vec{v}_n = a_1\vec{v}_1 + \dots + a_k\vec{v}_k$$

for $a_1, \dots, a_k \in \mathbb{R}$.

Now, $a_{k+1}\vec{v}_{k+1} + \dots + a_n\vec{v}_n = 0$

$$\Leftrightarrow a_1\vec{v}_1 + \dots + a_k\vec{v}_k = 0$$

Since $\{\vec{v}_1, \dots, \vec{v}_k\}$ linearly independent,

$$\Rightarrow a_1 = \dots = a_k = 0$$

$\lightning$ contradiction to $a_{k+1}, \dots, a_n$ not all zero

$\therefore \{T\vec{v}_{k+1}, \dots, T\vec{v}_n\}$ must be linearly independent $(**)$

By $(*)$ and $(**)$, since $\{T\vec{v}_{k+1}, \dots, T\vec{v}_n\}$ is linearly independent and spans $Im(T)$

$$\Rightarrow \{T\vec{v}_{k+1}, \dots, T\vec{v}_n\} \text{ is a basis of } T \text{ and}$$

$$\text{rank}(T) = \dim(Im(T)) = n - k$$

$$\Leftrightarrow \dim(Im(T)) = \dim(V) - \dim(N(T))$$
$$\Leftrightarrow \dim(Im(T)) + \dim(N(T)) = \dim(V)$$
$$\Leftrightarrow \text{rank}(T) + \text{nullity}(T) = \dim(V)$$

$\square$

2 Linear algebra theorem 10 / 10

✓ + **10 pts** **Correct**

+ **5 pts** Partial credit

Ex 6.1

EXERCISES

6.1 Which word or words from the following list describe each of the five networks below: *directed, undirected, cyclic, acyclic, approximately acyclic, planar, approximately planar, tree, approximate tree.*

a) The Internet, at the level of autonomous systems
b) A food web
c) The stem and branches of a plant
d) A spider web
e) A complete clique of four nodes

Give one real-life example of each of the following types of networks, not including the five examples above:

f) An acyclic (or approximately acyclic) directed network
g) A cyclic directed network
h) A tree (or approximate tree)
i) A planar (or approximately planar) network
j) A bipartite network

Describe briefly one empirical technique that could be used to measure the structure of each of the following networks (i.e., to fully determine the positions of all the edges):

k) The World Wide Web
l) A citation network of scientific papers
m) A food web
n) A network of friendships between a group of co-workers
o) A power grid

a) undirected, cyclic

b) directed, acyclic

c) undirected, acyclic, tree, planar

d) undirected, cyclic, planar

e) undirected, cyclic, planar

f) Acyclic directed network: Citation network of academic papers

g) Cyclic directed network: world wide web

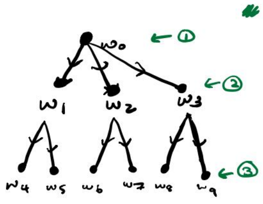
h) Tree / approximate tree: family tree with edges going from parent (node) to child (node)

i) Planar / approximately planar network: road networks

j) Bipartite networks: actors connected to the films they've appeared in

k) Measuring the structure of the World Wide Web:

1. Using a web crawler that performs a breadth-first search on the network.
2. Starting at an initial web page, w_0, the crawler can store links to all the other webpages that w_0 has links to.
3. Once it has finished scanning w_0, the crawler goes into its storage, retrieves a URL stored that was found in w_0, locates the page and proceeds to store all the URLs from that page.
4. Once the crawler has opened all the URLs found on w_0, scanned their pages and stored the links found on the pages, the crawler moves to the next level and repeats steps 3-4 for each of the stored URLs



* crawler evaluates URLs on each ~~label~~ level (i.e. 1, 2, 3...) before moving onto the next level, where w_0 is initial web page, w_i are web pages, parent web pages (nodes) are the web page that stored the links for child web pages (nodes)

l) Citation network of scientific papers

- A citation index can be built, where each scieutific paper can be submitted electronically.
- A crawler can be used to identify citations on electronic manuscript (e_0) and store them in the database, labelling the parent manuscript it came from.
- The crawler can then try to find eletronic versions of the cited manuscripts of e_0 and scan them to find citations.
- The parent manuscript's citations should be stored before visiting each of its child (cited) manuscripts and scanning those for their own citations.
- This is also a breadth-first search.

m) Food web

- Use published records and literature to assemble food-webs, taking note of predator-prey interactions that have been recorded and repeat the process for all ecosystems.

n) Friendships between co-workers

- The same experiment as Bernard et. al.

→ Ask participants from an office to read through a list of 100+ family names pulled from a phone directory and count up how many people with names that appeared they knew

→ Can also ask participants to give a 'score' to these people depending on degree of closeness, which can be represented as edge weights.

o) Power grid

- Since power grids are usually operated by a single authority, use their maps to construct a network where nodes represent transformers and each line transporting electricity is represented as an edge.

Ex 6.2

6.2 A simple network consists of n nodes in a single component. What is the maximum possible number of edges it could have? What is the minimum possible number of edges it could have? Explain briefly how you arrive at your answers.

- Maximum possible edges in a simple network (no multi-edges or self-edges) = $\frac{1}{2}n(n-1)$
- Each node can be connected to at most $(n-1)$ nodes since no self-edges.
- There are n nodes $\Rightarrow$ $n(n-1)$ possible edges
- $n(n-1)$ changes to $\frac{1}{2}n(n-1)$ to prevent edges from being double-counted.

- Minimum possible edges: 0 edges

The graph can be a set of isolated nodes because it still fulfills the definition of simple graph.

0 edges $\Rightarrow$ no self-edges

0 edges $\Rightarrow$ no multiedges.

3 Problem 6.1 **10 / 10**

- ✓ **+ 10 pts** **Full credit**
- **+ 9 pts** Minor mistakes

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4 Problem 6.2 5 / 10

✓ + **5 pts** **Part a (maximum edges), with a correct explanation**

+ **5 pts** Part b (minimum edges), with a correct explanation

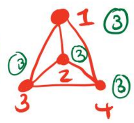
"single component" means the network is connected. If there are no edges, the network is disconnected.

Ex 6.4

6.4 Let $\mathbf{A}$ be the adjacency matrix of an undirected network and $\mathbf{1}$ be the column vector whose elements are all 1. In terms of these quantities write expressions for:

a) The vector $\mathbf{k}$ whose elements are the degrees k_i of the nodes;
b) The number m of edges in the network;
c) The matrix $\mathbf{N}$ whose element N_{ij} is equal to the number of common neighbors of nodes i and j;
d) The total number of triangles in the network, where a triangle means three nodes, each connected by edges to both of the others.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad \vec{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}$$

a) $\vec{k} = A \cdot \vec{1}$

degree of a node is the number of edges attached to each node

b) $\vec{k} \cdot \vec{1}$

c) $N = A^2$

d) $v_i \to v_j \to v_k \to v_i$ walk of length 3

A^3_{ii} = # walks of length 3 from v_i to v_i (e.g. of the form $v_i \to v_j \to v_k \to v_i$)

$\Rightarrow \dfrac{tr(A^3)}{2 \cdot 3}$ = total number of triangles

as $v_i \to v_j \to v_k \to v_i$
$= v_j \to v_k \to v_i \to v_j$
$= v_k \to v_i \to v_j \to v_k$

as $v_i \to v_j \to v_k \to v_i$
$= v_i \to v_j \to v_k \to v_i$ (other way around)

as network is undirected.

5 Problem 6.4 9 / 10

- ✓ + **2.5 pts** Part a
- ✓ + **2.5 pts** Part b
- ✓ + **2.5 pts** Part c
- ✓ + **2.5 pts** Part d
- \+ **0 pts** Missing
- **- 1** Point adjustment

Missing a factor of 1/2 in part b.