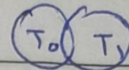
ASSIGNMENT (Week 1)



1.1 Q1

Soln: $P(\text{Today}) = .6 = P(T_0)$

$P(\text{Tomorrow}) = .5 = P(T_1)$

$P(\overline{T_0 \cup T_1}) = 1 - P(T_0 \cup T_1) = .3$

$\longrightarrow P(T_0 \cup T_1) = .7$

$P(T_0) + P(T_1) - P(T_0 \cap T_1) = .7$

$P(T_0 \cap T_1) = .4$

(a) Today or tomorrow

$= P(T_0 \cup T_1) = \underline{0.7}$ Ans

(b) $P(T_0 \cap T_1) = \underline{0.4}$ Ans

(c) $P(T_0 \cap \overline{T_1}) + P(T_0 \cap T_1) = P(T_0)$

$P(T_0 \cap \overline{T_1}) = .6 - .4 = \underline{.2}$ Ans

(d) x = today or tomorrow, but not both

$P(X) = P(T_0) + P(T_1) - 2P(T_0 \cap T_1)$

$= 1.1 - 2 \times .4$

$= \underline{0.3}$ Ans

1.2 Q2

Soln

Sample space (S) =
(1,1) (1,2) (1,3) (1,4) (1,5) (1,6)
(2,1) (2,2) (2,3) (2,4) (2,5) (2,6)
(3,1) (3,2) (3,3) (3,4) (3,5) (3,6)
(4,1) (4,2) (4,3) (4,4) (4,5) (4,6)
(5,1) (5,2) (5,3) (5,4) (5,5) (5,6)
(6,1) (6,2) (6,3) (6,4) (6,5) (6,6)

For $X_1 + X_2 = 8$
(2,6) (5,3) (6,2) (3,5) (4,4)

$$P(A) = \frac{\text{Favourable outcomes}}{\text{Total possible}}$$

$$= \frac{5}{36} \text{ Ans}$$

1.3 Q3

1.4 Q4

1.4 Que 4

1. CDF of X, $(F_X(x))$

$$F_X(x) = P(X \le x) = \underbrace{p\,F_d(x)}_{\text{if it lands heads}} + \cancel{p}\underbrace{(1-p)F_c(x)}_{\text{if it lands tails}}$$

2. PDF $[f_X(x)]$

$$f_X(x) = p\,f_d(x) + \cancel{p}(1-p)\,f_c(x)$$

3. $E[X] = p\,E[X_d] + \cancel{p}(1-p)\,E[X_c]$

$$E[X_d] = \sum x\,f_d(x)$$

$$E[X_c] = \int_{-\infty}^{\infty} x\,f_c(x)\,dx$$

$$\boxed{E[X] = p\sum x\,f_d(x) + (1-p)\int_{-\infty}^{\infty} x\,f_c(x)\,dx}$$

4. $\mathrm{Var}(X) = E[(X-\mu)^2] = E[X^2] - (E[X])^2$

$$= p\left[\sum_x x^2 f_d(x) - \left(\sum_x x f_d(x)\right)^2\right]$$

$$+ (1-p)\left[\int_{-\infty}^{\infty} x^2 f_c(x)\,dx - \left(\int_{-\infty}^{\infty} x f_c(x)\,dx\right.\right.$$

$$+ p(1-p)\left(\sum_x x f_d(x) - \int_{-\infty}^{\infty} x f_c(x)\,dx\right)$$

5. given
$X, Y \sim N(0,1)$ independent

$Z = 1 + X + XY^2 \qquad W = 1 + X$

We want
$\mathrm{Cov}(Z, W) = \mathrm{Cov}(1 + X + XY^2, 1 + X)$

→ Linearity of covariance
$\mathrm{Cov}(Z, W) = \mathrm{Cov}(X + XY^2, X)$
$= \mathrm{Cov}(X, X) + \mathrm{Cov}(XY^2, X)$

$= \mathrm{Var}(X) = 1$

$\mathrm{Cov}(XY^2, X) = E[X^2Y^2] - E[XY^2]\,E[X]$
X & Y → independent $E[X] = 0$ → 0

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$$E(X^2Y^2) = E[X^2]\,E[Y^2] = 1\cdot 1 = 1$$

$$cov(XY^2, X) = 1$$

$$\Rightarrow cov(Z, W) = 1 + 1 = \underline{2} \text{ Ans}$$

Que 6.

Soln: Let there be 4 companies A, B, C, D

Probability of receiving offer from ~~each~~ $x = P(x)$

~~$P(A) = P(B) = P$~~

~~$P(A\cap B\cap C\cap D) = 20\% = .2$~~

If $P(A\cap B\cap C\cap D) = .2$, then we don't have sufficient information.

If the chance of getting an offer from each company (not necessarily simultan-) $= .2$

$$P(A) = P(B) = P(C) = P(D) = .2$$

Probability of not getting offer from any company $= (1-.2)^4 = (.8)^4$

$1-(.8)^4$ = Probability of getting an offer from atleast one company

$= .5904$

$= \underline{59.04\ \%}$

Hence, he is wrong.

<u>Que 7</u>

Probability of more than 120 errors

$= {}^{1000}C_{121}(.1)^{121} + {}^{1000}C_{122}(.1)^{122} \cdots + {}^{1000}C_{1000}(.1)^{1000}$

Que 8.

Let T be the total no. of sandwiches required for 95%.

Expectation of one guest's demand

$E[X] = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$

$E[X^2] = 0^2 \cdot \frac{1}{4} + 1^2 \cdot \frac{1}{2} + 2^2 \cdot \frac{1}{4} = \frac{1}{2} + 1 = 1.5$

$Var(X) = E[X^2] - (E[X])^2 = 1.5 - 1 = 0.5$

$E[S] = 64 \times 1 = 64$

$Var(S) = 64 \times 0.5 = 32$

$\sigma_S = \sqrt{Var(S)} = \sqrt{32} = 5.66$

Using Central Limit Theorem

$S \approx N(64, 32)$

$P(T \leq m) = 0.95$

$Z_{0.95} \approx 1.645$ (standard Normal)

$m = \mu + Z\sigma$

$= 64 + 1.645(5.66) \approx 64 + 9.31$

≈ 73.31

74 sandwiches for 95% surity

Que 9

Sol^n

$$\mu = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} Var(X) & Cov(X,Y) \\ Cov(Y,X) & Var(Y) \end{pmatrix} = \begin{pmatrix} 1 & p \\ p & 1 \end{pmatrix}$$

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① $E[X] = \mu$ for multivariate normal

$\mu_x = 0, \mu_y = 0$

Therefore,

$E[X] = 0 \quad E[Y] = 0$

② $\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \rightarrow$ var

$Var[X] = 1 \qquad Var[Y] = 1$

③ $cov(X, Y) = E[(X - E[X])(Y - E[Y])]$

$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \rightarrow \boxed{cov(X, Y) = \rho}$

Que 10

Sol^n $\qquad \mu = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\Sigma = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}$

For a jointly gaussian vector :-

$X | Y = y \sim N(\mu_x + \Sigma_{xy} \Sigma_{yy}^{-1} (y - \mu_y),$

$\Sigma_{xx} - \Sigma_{xy} \Sigma_{yy}^{-1} \Sigma_{yx})$

$\mu_x = 1$, $\mu_y = 2$, $\Sigma_{xx} = 4$, $\Sigma_{yy} = 3$, $\Sigma_{xy} = \Sigma_{yx} = 1$

$\rightarrow E[X|Y = y] = \mu_x + \Sigma_{xx} \Sigma_{yy}^{-1} (y - \mu_y)$

$= 1 + 1 \cdot \frac{1}{3} (y - 2)$

$$\boxed{E(X|Y = y) = 1 + \frac{y-2}{3}}$$

$Var(X|Y) = \Sigma_{xx} - \Sigma_{xy} \Sigma_{yy}^{-1} \Sigma_{yx}$

$= 4 - 1 \cdot \frac{1}{3} \cdot 1$

$= \frac{11}{3}$ Ans

Hence,

$X|Y = y \sim N\left(1 + \frac{y-2}{3}, \frac{11}{3}\right)$

Soln 11: Since $Z = 3X - 2Y$ is linear,

$$\boxed{Z \sim N(\mu_Z, \sigma_Z^2)}$$

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$E[aX + bY] = aE[X] + bE[Y]$

$E[Z] = E[3X-2Y] = 3E[X] - 2E[Y] = 0$

$$\boxed{E[Z] = 0}$$

$\rightarrow \text{Var}(aX+bY) = a^2\,\text{Var}(X) + b^2\,\text{Var}(Y) + 2ab\,\text{cov}(X,Y)$

$= 9 + 16 - 12$

$$\boxed{\text{Var}(Z) = 13}$$

$$\Longrightarrow \boxed{Z \sim N(0, 13)} \text{ Ans}$$

$$\text{Corr}(Z,X) = \frac{\text{cov}(Z,X)}{\sigma_Z \sigma_X}$$

$\text{cov}(3X-2Y, X) = 3\,\text{cov}(X,X) - 2\,\text{cov}(Y,X)$

$= 3 - 2 = 1$

$\sigma_Z = \sqrt{13}$, $\sigma_X = 1$

$$\text{Final Corr}(Z,X) = \frac{1}{\sqrt{13}} \text{ Ans}$$

Que 12 Soln

Conditional gaussian formula

$$X \mid (Y,Z) \sim N\left(\mu_X + \Sigma_{X,[YZ]}\Sigma^{-1}_{[YZ]}\begin{pmatrix} y \\ z \end{pmatrix},\ \Sigma_{XX} - \Sigma_{X,[YZ]}\Sigma^{-1}_{[YZ]}\Sigma_{[YZ],X}\right)$$

Σ_{XX}

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 & 2 & 1 \\ 2 & 3 & -1 \\ 1 & -1 & 5 \end{pmatrix}\right)$$

$\Sigma_{[YZ]}$

$\Rightarrow |\Sigma_{[YZ]}| = 14$

$$\boxed{\Sigma^{-1}_{[YZ]} = \frac{1}{14}\begin{pmatrix} 5 & 1 \\ 1 & 3 \end{pmatrix}}$$

$E(X \mid Y = y, Z = z]$

$$= [2\ 1]\frac{1}{14}\begin{pmatrix} 5 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}$$

$$= \frac{1}{14}[11\ 5]\begin{pmatrix} y \\ z \end{pmatrix}$$

$$= \frac{11}{14}y + \frac{5}{14}z$$

$\mathrm{Var}(X \mid Y, Z)$

$$= 4 - [2\ 1]\frac{1}{14}\begin{pmatrix} 5 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= 4 - \frac{27}{14}$$

$$= \frac{29}{14} \text{ Ans}$$

$$\boxed{X \mid (Y = y, Z = z) \sim N\left(\frac{11}{14}y + \frac{5}{14}z, \frac{29}{14}\right)}$$