1. First, we compute the derivatives of the exact solution:

$$\begin{aligned} u(t) &= \eta e^{-\lambda t} \\ u'(t) &= -\lambda \eta e^{-\lambda t} \\ u''(t) &= \lambda^2 \eta e^{-\lambda t} \end{aligned}$$

   Then, the Taylor series method sequence is:

$$\begin{aligned} U^0 &= u(0) = \eta \\ U^{n+1} &= U^n + ku'(kn) + \frac{1}{2}k^2 u''(kn) \\ &= U^n - k\lambda\eta e^{-\lambda kn} + \frac{1}{2}k^2\lambda^2\eta e^{\lambda kn} \\ &= U^n + k\lambda\eta e^{-\lambda kn}\left(\frac{k\lambda}{2} - 1\right) \end{aligned}$$

   For $k\lambda > 0$, $-kn\lambda$ will approach $-\infty$ for large n and therefore $e^{-kn\lambda}$ approaches 0.
   As a result the change from U^n to U^{n+1} will approach 0 and the approximation must decay.

2. (a)

$$\begin{aligned} u' &= u - t^2 + 1 \\ u(0) &= 0 \\ u(t) &= c_1 e^t + t^2 + 2t + 1 \\ &= -e^t + t^2 + 2t + 1 \end{aligned}$$

   (b) The derivatives for the exact solution are as follows:

$$\begin{aligned} u'(t) &= -e^t + 2t + 2 \\ u''(t) &= -e^t + 2 \\ u'''(t) &= -e^t \\ u''''(t) &= -e^t \end{aligned}$$

   Then, the Taylor series method approximation is the following:

$$\begin{aligned} U^0 &= u(0) = 0 \\ U^{n+1} &= U^n + ku'(t^n) + \frac{1}{2!}k^2u''(t^n) + \frac{1}{3!}k^3u'''(t^n) + \frac{1}{4!}k^4u''''(t^n) \\ &= U^n - ke^{t^n} + 2kt^n + 2k - \frac{1}{2}k^2e^{t^n} + k^2 - \frac{1}{6}k^3e^{t^n} - \frac{1}{24}k^4e^{t^n} \end{aligned}$$

   (c) Results from the approximation:

k	error	ratio
0.25	1.9e-4	-
0.125	1.24e-5	15.32
0.0625	7.96e-7	15.66
0.03125	5.02e-8	15.83
0.015625	3.16e-9	15.91
0.0078125	1.98e-10	15.95
0.00390625	1.23e-11	15.97
0.001953125	7.73e-13	16.01

As N doubles and the timestep k halves, the error decreases by a consistent factor of 16.
It follows that the error can be modeled as $\frac{1}{16^{\log_2 n}}$ which can be simplified to $\frac{1}{n^4}$.
Therefore, the approximation converges at a rate of n^4.

3. (a) See the source code at `https://github.com/codeandkey/math481-iastate-sp2020`.

   (b) Results from the approximation:

k	θ error	θ error ratio	θ' error	θ' error ratio
0.1	0.055	-	0.926	-
0.05	0.013	4.19	0.220	4.191
0.025	0.003	3.50	0.05	4.06
0.0125	0.001	3.642	0.013	4.033
0.00625	0.0002	3.79	0.0033	4.014
0.003125	7.09e-5	3.869	0.0008	3.984
0.0015625	1.844e-5	3.845	0.0002	3.901