1. Let $u'(t) = 2t(1+u)$, $u(0) = 0$.

$$\begin{aligned} u_1(t) &= \int_0^t f(s, u(0))ds \\ &= \int_0^t f(s, 0)ds \\ &= \int_0^t 2sds \\ &= t^2 \end{aligned}$$

We can then derive u_2 and u_3:

$$\begin{aligned} u_2(t) &= \int_0^t f(s, u_1(s))ds \\ &= \int_0^t f(s, s^2)ds \\ &= \int_0^t 2s(1+s^2)ds \\ &= \int_0^t 2s + \int_0^t 2s^3 ds \\ &= t^2 + \frac{t^4}{2} \\ u_3(t) &= \int_0^t f(s, u_2(s))ds \\ &= \int_0^t f(s, s^2 + \frac{s^4}{2})ds \\ &= \int_0^t 2s(1 + s^2 + \frac{s^4}{2})ds \\ &= \int_0^t 2s + \int_0^t 2s^3 ds + \int_0^t s^5 ds \\ &= t^2 + \frac{t^4}{2} + \frac{t^6}{6} \end{aligned}$$

It seems that the Picard iteration for this function takes the following pattern:

$$u_k(t) = \sum_{i=1}^{k} \frac{t^{2i}}{i!}$$

Let $x = t^2$. Then the expanded series looks like a Taylor series:

$$\lim_{k\to\infty} u_k(t) = t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + ...$$
$$\lim_{k\to\infty} u_k(x) = x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

It is known that $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$, so this sum converges to $u_\infty = e^{t^2} - 1$.

2.

3.

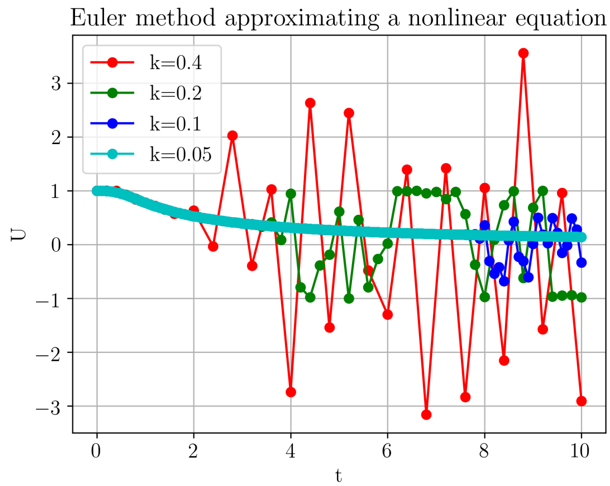
4. See the source code at `https://github.com/codeandkey/math481-iastate-sp2020`.

5. Here is the data collected from the program:

k	U^N	E^K	E^{2K}/E^K
5e-2	38.093	15.505	–
2.5e-2	44.846	8.752	1.771
1.25e-2	48.927	4.671	1.873
6.25e-3	51.182	2.417	1.933
3.125e-3	52.369	1.230	1.965
1.5625e-3	52.978	0.620	1.982
7.8125e-4	53.287	0.311	1.991
3.90625e-4	53.442	0.156	1.995
1.953125e-4	53.520	0.078	1.997
9.765625e-5	53.559	0.039	1.998

Looking at this graph it is clear that as k tends towards infinity the error E^N tends to 0. It appears that the ratio of E^{2K}/E^K tends to 2, indicating that as k increases the error will halve on every increment. However, that improvement in error will quickly tend towards 0 as well.

6. See the graph:


Euler method approximating a nonlinear equation
k=0.4
k=0.2
k=0.1
k=0.05
U
t
3
2
1
0
−1
−2
−3
0
2
4
6
8
10