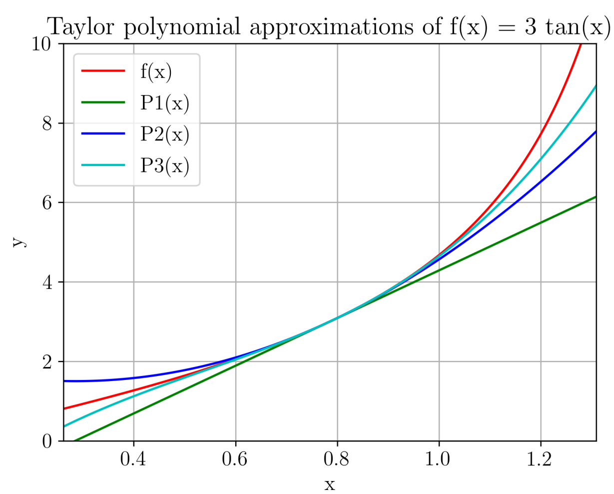
1. (a) Let $f(x) = 3\tan x$.
   It follows that:

$$f'(x) = 3\sec^2 x$$
$$f''(x) = 6\tan x \sec^2 x$$
$$f'''(x) = 6(\sec^4 x + 2\tan^2 x \sec^2 x)$$

   Then, the Taylor expansion of $f(x)$ about $\frac{\pi}{4}$ is

$$P_3(x) = 3 + 6(x - \frac{\pi}{4}) + 6(x - \frac{\pi}{4})^2 + 8(x - \frac{\pi}{4})^3$$

   (b) Source code can be found here:
   `https://github.com/codeandkey/math481-iastate-sp2020`



2. Let $f(x) = \log(1 + xe^x)$. Then,

$$f'(x) = \frac{e^x(x+1)}{xe^x + 1}$$
$$f''(x) = \frac{e^x(x - e^x + 2)}{(xe^x + 1)^2}$$

   The Taylor polynomial coefficients are then:

$$f(0) = 0$$
$$f'(0) = 1$$
$$f''(0) = 1$$

Therefore the second-degree Taylor polynomial is $P_2(x) = x + \frac{x^2}{2}$.
Substituting $f(x)$ back into $g(x)$ results in a simplifcation:

$$\begin{aligned} g(x) &= \frac{f(x)}{x} \\ &\approx \frac{P_2(x)}{x} \\ &= \frac{x + \frac{x^2}{2}}{x} \\ &= 1 + \frac{x}{2} \end{aligned}$$

Here it is clear that the limit

$$\lim_{x \to 0} \left(1 + \frac{x}{2}\right) = 1$$

.

3. (a) As all of the derivatives of e^x are identical the n-th degree Taylor polynomial is as follows:

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!}$$

The remainder $R_n(x)$ is the difference between the approximation and the exact value:

$$\begin{aligned} R_n(x) &= |e^x - P_n(x)| \\ &= e^x - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!}\right) \end{aligned}$$

(b) Trying different values for n had the following results:

$$\begin{aligned} R_1(1) &\approx 0.718 \\ R_2(1) &\approx 0.21 \\ R_3(1) &\approx 0.05 \\ R_4(1) &\approx 0.0099 \\ R_5(1) &\approx 0.00161 \\ R_6(1) &\approx 0.0002 \\ R_7(1) &\approx 0.000027 \\ R_8(1) &\approx 0.00000305 < 10^{-5} \end{aligned}$$

So, $n = 8$ was sufficient to bring $R_n(1)$ below 10^{-5}.

(c) The actual error computed at $n = 8$ was 0.00000305861.

4. (a) Let $f(x) = \dfrac{1}{1-x}$.
   The derivatives of $f(x)$ are then:

$$\begin{aligned} f'(x) &= \frac{1}{(1-x)^2} \\ f''(x) &= \frac{2}{(1-x)^3} \\ f'''(x) &= \frac{6}{(1-x)^4} \\ f''''(x) &= \frac{24}{(1-x)^5} \end{aligned}$$

   The infinite Taylor series representation about $x = 0$ is then:

$$\begin{aligned} P_n(x) &= 1 + x + \frac{2!}{2!}x^2 + \frac{3!}{3!}x^3 + ... + \frac{n!}{n!}x^n \\ P_n(x) &= 1 + x + x^2 + x^3 + ... + x^n \end{aligned}$$

   (b) Let $x = -t^2$. Then $f(x) = \dfrac{1}{1+t^2}$, and the Taylor polynomial becomes:

$$P_n(x) = 1 - t^2 + t^4 - t^6 + ...$$

   (c) Integrating the result gives the Taylor series for $tan^{-1}(x)$ about $x = 0$:

$$\int_0^x \left(1 - t^2 + t^4 - t^6 + ...\right) dt = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$$

5. (a) To solve this limit we can apply L'Hopital's rule successively:

$$\begin{aligned} \lim_{h\to 0} F(h) &= \lim_{h\to 0} \frac{\sin h - h\cos h}{h^3} \\ &= \lim_{h\to 0} \frac{h\sin h}{3h^2} \\ &= \lim_{h\to 0} \frac{\sin h + h\cos h}{6h} \\ &= \lim_{h\to 0} \frac{2\cos h - h\sin h}{6} \\ &= \frac{2-0}{6} \\ &= \frac{1}{3} \end{aligned}$$

(b) To find the rate of the convergence, the remainder can be manipulated:

$$\begin{aligned}
\lim_{h\to 0} F(h) &= \lim_{h\to 0} \frac{\sin h - h\cos h}{h^3} = \frac{1}{3} \\
&= \lim_{h\to 0} \left| \frac{1}{3} - \frac{\sin h - h\cos h}{h^3} \right| = 0 \\
&\leq \left| \frac{h^3}{3} - \sin h - h\cos h \right| \\
&\leq \left| h^3 \right|
\end{aligned}$$

So, the rate of convergence is bounded from above by $O(h^3)$.

6. (a) See the source code in the link from part 1.

   (b) The program reported $N = 4$ to be sufficiently large, with $x^N \approx 0.2575302$.