# INTEGRALS

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## Contents

# 1 PREFACE

Greetings!
The goal of this document is to provide exposure to some unconventional ideas which are used in solving integration problems but are generally not taught in Indian high schools. Also, this document would be specifically geared towards JEE (Joint Entrance Exam) Advance preparation.

The motivation for writing this document comes from the fact that there exist quite a few theorems and techniques that can trivialize otherwise difficult problems (it is specifically useful for an exam like JEE where time is a deciding factor) but still, most students are unaware of them.

The handout has been divided into 4 Parts. First, we start with some unconventional theorems which can help to solve integrals much easier. We have included a few examples which would help the reader better understand the applications of these theorems. Moving forward, Section 2 and 3 comprises the synopsis and certain standard results of indefinite and definite integrals. Last but not least in Section 4, we have included roughly 100 fun integrals to practice. Note that these problems are not arranged in any order of difficulty.

Although, I tried my best to shed light on some of these techniques still there might be ambiguities present at various places due to lack of experience, I apologize for that. Hopefully, you would have fun and learn something new while going through this.
Happy learning! Thanks,

Note:

1. This document is meant for students who are already familiar with or have exposure to calculus at the high school level. Complete novice to the topic might find it hard going.

2. For suggestion, solutions, and reporting errors mail me at akiiiceoak1310@gmail.com

## 2 ACKNOWLEDGEMENT

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Last but not least, I'm eternally grateful to [Dylan Yu] for allowing me to use his STY without which this document wouldn't look this beautiful.

## 3 About the Author

[AKIII] is a rising High-School Senior from India who has a keen interest in mathematics and physics. He also qualified with several Regional and National levels exams such as RMO and NSEs, He aspires to pursue his career in the research area in the future after graduating from an IIT.

# 4 Some unfamiliar Theorems for JEE

## 4.1 Generalized integration by parts

**Theorem 4.1**
As an example of something that might be obscure, the formula for "general integration by parts " for n functions

$$\int f_1'(x)\prod_{j=2}^{n} f_j(x)dx = \prod_{i=1}^{n} f_i(x) - \sum_{i=2}^{n}\int f_i'(x) \prod_{j=1, j\neq i}^{n} f_j(x)$$

which is not necessarily useful nor difficult to derive, but is interesting nonetheless.So I'm keeping proof part for readers.

### 4.1.1 Gamma function

**Example 4.2**
The Gamma function is defined by the integral $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt$.
If we choose $x=e^{-t}$ and $u=t^z/z$ (so that $du=t^{z-1}dt$). Hence we have

$$\begin{aligned}\Gamma(z) &= e^{-t}\frac{t^z}{z}\Big|_0^\infty - \int_0^\infty \frac{t^z}{z}\left(-e^{-t}\right)dt\\ &= \frac{1}{z}\int_0^\infty t^z\left(-e^{-t}\right)dt\\ &= \frac{1}{z}\Gamma(z+1)\end{aligned}$$

or $\Gamma(z+1)=z\Gamma(z)$.
Since we can easily determine that $\Gamma(1)=1$, we conclude that, when $n$ is a positive integer, $\Gamma(n+1)=n\cdot(n-1)\cdot\dots\cdot 2\cdot 1=n!$ Hence, the Gamma function is the generalization of the factorial function.
Notes
The integration by parts formula may be re-applied.
For example, if $f_n$ stands for the $n$ -th derivative of $f$ and $g_n$ stands for the $n$ -th integral of $g$ then

$$\int_a^b fg dx = fg_1|_a^b - f_1g_2|_a^b + f_2g_3|_a^b - f_3g_4|_a^b + \cdots$$

## 4.2 Beta Function

**Theorem 4.3**

$$B(u,v)=\int_0^1 t^{u-1}(1-t)^{v-1}dt, \quad \operatorname{Re} u>0, \quad \operatorname{Re} v>0$$

This integral is called the beta integral. we easily obtain the symmetry

$$B(u,v)=B(v,u),$$

since we have by using the substitution $t=1-s$

$$\begin{aligned} B(u,v)=\int_0^1 t^{u-1}(1-t)^{v-1}dt=-\int_1^0(1-s)^{u-1}s^{v-1}ds=\int_0^1 s^{v-1}(1-s)^{u-1}ds \\ =B(v,u)\end{aligned}$$

The connection between the beta function and the gamma function is given by the following theorem:

### 4.2.1 Theorems

**Theorem 4.4**

$$B(u,v)=\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \quad \operatorname{Re} u>0, \quad \operatorname{Re} v>0$$

In order to prove this theorem we use the definition to obtain

$$\Gamma(u)\Gamma(v)=\int_0^\infty e^{-t}t^{u-1}dt\int_0^\infty e^{-s}s^{v-1}ds=\int_0^\infty\int_0^\infty e^{-(t+s)}t^{u-1}s^{v-1}dtds$$

Now we apply the change of variables $t=xy$ and $s=x(1-y)$ to this double integral. Note that $t+s=x$ and that $0<t<\infty$ and $0<s<\infty$ imply that $0<x<\infty$ and $0<y<1$. The Jacobian of this transformation is

$$\frac{\partial(t,s)}{\partial(x,y)}=\begin{vmatrix} y & x \\ 1-y & -x \end{vmatrix}=-xy-x+xy=-x$$

Since $x>0$ we conclude that $dtds=\left|\frac{\partial(t,s)}{\partial(x,y)}\right|dxdy=xdxdy$. Hence we have

$$\begin{aligned}\Gamma(u)\Gamma(v)&=\int_0^1\int_0^\infty e^{-x}x^{u-1}y^{u-1}x^{v-1}(1-y)^{v-1}xdxdy \\ &=\int_0^\infty e^{-x}x^{u+v-1}dx\int_0^1 y^{u-1}(1-y)^{v-1}dy=\Gamma(u+v)B(u,v).\end{aligned}$$

This proves it. There exist many useful forms of the beta integral which can be obtained by an appropriate change of variables. For instance, if we set $t=s/(s+1)$ and obtain

$$\begin{aligned}B(u,v)&=\int_0^1 t^{u-1}(1-t)^{v-1}dt=\int_0^\infty s^{u-1}(s+1)^{-u+1}(s+1)^{-v+1}(s+1)^{-2}ds \\ &=\int_0^\infty \frac{s^{u-1}}{(s+1)^{u+v}}ds, \quad \operatorname{Re} u>0, \quad \operatorname{Re} v>0\end{aligned}$$

This proves it.

### 4.2.2 Theorem

**Theorem 4.5**

$$B(u,v)=\int_0^\infty \frac{s^{u-1}}{(s+1)^{u+v}}ds, \quad \mathrm{Re}\, u>0, \quad \mathrm{Re}\, v>0$$

If we set $t=\cos^2\varphi$ , we find that

$$\begin{aligned} B(u,v) &= \int_0^1 t^{u-1}(1-t)^{v-1}dt = -2\int_{\pi/2}^0 (\cos\varphi)^{2u-2}(\sin\varphi)^{2v-2}\cos\varphi\sin\varphi d\varphi \\ &= 2\int_0^{\pi/2}(\cos\varphi)^{2u-1}(\sin\varphi)^{2v-1}d\varphi, \quad \mathrm{Re}\, u>0, \quad \mathrm{Re}\, v>0 \end{aligned}$$

**Theorem 4.6**

$$B(u,v)=2\int_0^{\pi/2}(\cos\varphi)^{2u-1}(\sin\varphi)^{2v-1}d\varphi, \quad \mathrm{Re}\, u>0, \quad \mathrm{Re}\, v>0$$

Finally, the substitution $t=(s-a)/(b-a)$ in first form leads to

$$\begin{aligned} B(u,v) &= \int_0^1 t^{u-1}(1-t)^{v-1}dt \\ &= \int_a^b (s-a)^{u-1}(b-a)^{-u+1}(b-s)^{v-1}(b-a)^{-v+1}(b-a)^{-1}ds \\ &= (b-a)^{-u-v+1}\int_a^b (s-a)^{u-1}(b-s)^{v-1}ds, \quad \mathrm{Re}\, u>0, \quad \mathrm{Re}\, v>0 \end{aligned}$$

**Example 4.7**

1. $\int_0^{\pi/2}(\cos\varphi)^5(\sin\varphi)^7 d\varphi = \frac{1}{2}\cdot B(3,4) = \frac{1}{2}\cdot\frac{\Gamma(3)\Gamma(4)}{\Gamma(7)} = \frac{1}{2}\cdot\frac{2!3!}{6!} = \frac{1}{2}\cdot\frac{2}{4\cdot5\cdot6} = \frac{1}{120}$,
2. $\int_0^{\pi/2}(\cos\varphi)^7(\sin\varphi)^4 d\varphi = \frac{1}{2}\cdot B(4,5/2) = \frac{1}{2}\cdot\frac{\Gamma(4)\Gamma(5/2)}{\Gamma(13/2)} = \frac{1}{2}\cdot\frac{3!}{\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}\cdot\frac{11}{2}} = \frac{1}{2}\cdot\frac{6\cdot2^4}{5\cdot7\cdot9\cdot11} = \frac{16}{1155}$
3. $$\begin{aligned} \int_0^{\pi/2}(\cos\varphi)^4(\sin\varphi)^6 d\varphi &= \frac{1}{2}\cdot B(5/2,7/2) = \frac{1}{2}\cdot\frac{\Gamma(5/2)\Gamma(7/2)}{\Gamma(6)} \\ &= \frac{1}{2}\cdot\frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma(1/2)\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma(1/2)}{5!} = \frac{5\cdot3^2\cdot\pi}{2^6\cdot2\cdot3\cdot4\cdot5} = \frac{3\pi}{2^9} = \frac{3\pi}{512} \end{aligned}$$

## 4.3 Integration through Matrix representation

**Theorem 4.8**

Integration can be done by inverting the matrix representation of the differentiation operator with respect to a clever choice of a basis and then apply the inverse of the operator to function we wish to integrate. For example, consider the basis $B = (e^{ax}\cos bx, e^{ax}\sin bx)$.

$$\frac{d}{dx}e^{ax}\cos bx = ae^{ax}\cos bx - be^{ax}\sin bx$$
$$\frac{d}{dx}e^{ax}\sin bx = ae^{ax}\sin bx + be^{ax}\cos bx$$

and the matrix representation of the linear operator is

$$T = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

To then solve something like $\int e^{ax}\cos bx dx$, this is equivalent to calculating

$$T^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}_B = \frac{1}{a^2+b^2}\begin{bmatrix} a \\ b \end{bmatrix}_B$$

So,

$$\int e^{ax}\cos bx dx = \frac{a}{a^2+b^2}e^{ax}\cos bx + \frac{b}{a^2+b^2}e^{ax}\sin bx$$

## 4.4 The Feyman Technique

**Theorem 4.9**
The technique of "Feynman Integration" is a simple application of a theorem attributed to Leibniz. In this section we state the theorem in its most basic form, and end by stating a more general version that allows for even weaker hypotheses. In both cases, we address situations where the following equation (which we would love to be true) holds:

$$\frac{d}{dx}\int_Y f(x,y)dy = \int_Y \frac{\partial}{\partial x} f(x,y)dy$$

Before stating these theorems, recall that differentiation is simply a particular example of a limit insofar as we define

$$\frac{d}{dx}f(x) =: f'(x) := \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

with a true definition on the far right. Thus, we see that (2) will hold whenever we may make the following statement,

$$\lim_{x\to a}\int_Y f(x,y)dy = \int_Y \lim_{x\to a} f(x,y)dy$$

### 4.4.1 Elementary Calculus Version

- Let $f : [a,b]\times Y \to \mathbb{R}$ be a function, with $[a,b]$ being a closed interval, and $Y$ being a compact subset of $\mathbb{R}^n$. Suppose that both $f(x,y)$ and $\partial f(x,y)/\partial x$ are continuous in the variables $x$ and $y$ jointly. Then $\int_Y f(x,y)dy$ exists as a continuously differentiable function of $x$ on $[a,b]$, with derivative

$$\frac{d}{dx}\int_Y f(x,y)dy = \int_Y \frac{\partial}{\partial x} f(x,y)dy$$

  As mentioned above, the veracity of it is completely dependent upon if we can exchange the operations of limiting and integration. If we were to prove the above theorem, our argument would make full use of the compactness of $Y$, which of course implies uniform continuity. From this fact, we could show that it is justified to switch change the order of limits and integration, thus proving it.

  However, in many cases the restriction of compactness can be too severe. Often times we would like $Y$ to be $(-\infty,a),(a,\infty),(-\infty,\infty)$,etc... In these situations, the following measure theoretic version of the above comes to our rescue:

### 4.4.2 Measure Theory Version

- Let $X$ be an open subset of $\mathbb{R}$, and $\Omega$ be a measure space. Suppose $f : X\times\Omega\to\mathbb{R}$ satisfies the following conditions: (1) $f(x,\omega)$ is a Lebesgue-integrable function of $\omega$ for each $x\in X$. (2) For almost all $\omega\in\Omega$, the derivative $\partial f(x,\omega)/\partial x$ exists for all $x\in X$. (3) There is an integrable function $\Theta : \Omega\to\mathbb{R}$ such that $|\partial f(x,\omega)/\partial x| \le \Theta(\omega)$ for all $x\in X$. Then for all $x\in X$,

$$\frac{d}{dx}\int_\Omega f(x,\omega)d\omega = \int_\Omega \frac{\partial}{\partial x} f(x,\omega)d\omega$$

### 4.4.3 Illustrations

**Example 4.10**
Compute the definite integral,

$$\int_0^1 \frac{x^2-1}{\log x}dx$$

In order to apply our theorems, we obviously need to be dealing with an integrand in two variables. In this example, we "generalize" by introducing a parameter $b$ in the exponent of our $x$ term. In particular, we could choose to define the following function:

$$I(b) = \int_0^1 \frac{x^b-1}{\log x}dx$$

As long as $b > -1$, all conditions of 'Theorem 1.3.1 are satisfied and we may differentiate under the integral sign:

$$\begin{aligned} I'(b) &= \frac{d}{db}\int_0^1 \frac{x^b-1}{\log x}dx = \int_0^1 \frac{\partial}{\partial b}\left[\frac{x^b-1}{\log x}\right]dx \\ &= \int_0^1 x^b = \left.\frac{x^{b+1}}{b+1}\right|_0^1 \\ &= \frac{1}{b+1} \end{aligned}$$

whereupon integration yields

$$I(b) = \log(b+1) + C$$

In order to find out constant of integration, we let $b = 0$ so that our integrand is 0 , implying that $C = 0$. Letting $b = 2$ will of course solve our original problem:

$$\int_0^1 \frac{x^2-1}{\log x}dx = I(2) = \log(3)$$

**Example 4.11**
Compute the improper definite integral,

$$\int_0^{\pi/2} x\cot(x)dx$$

This particular example is tricky because it is not immediately obvious where to introduce the extra parameter. However, it turns out that the following is an appropriate choice:

$$I(b) = \int_0^{\pi/2} \frac{\tan^{-1}(b\tan(x))}{\tan(x)}dx$$

so that we will have the answer to our original integral upon setting $b = 1$. After briefly verifying that the conditions of Theorem 1.3.1 are satisfied, we proceed as follows

$$\begin{aligned} I'(b) &= \frac{d}{dx}\int_0^{\pi/2} \frac{\tan^{-1}(b\tan(x))}{\tan(x)}dx = \int_0^{\pi/2} \frac{\partial}{\partial b}\left[\frac{\tan^{-1}(b\tan(x))}{\tan(x)}\right]dx \\ &= \int_0^{\pi/2} \frac{dx}{(b\tan(x))^2+1} \\ &= \frac{\pi}{2(b+1)} \end{aligned}$$

Integrating w.r.t. $b$ (and noting that our constant of integration will vanish) gives us

$$I(b) = \frac{\pi}{2}\log(b+1)$$

So that our original integral is obtained via

$$\int_0^{\pi/2} x\cot(x)dx = I(1) = \frac{\pi}{2}\log(2)$$

We conclude this example by performing integration by parts on our original integral. This yields the integral of another relatively famous integral often dealt with in introductory complex analysis courses:

$$\int_0^{\pi/2} \log(\sin(x))dx = -\int_0^{\pi/2} x\cot(x)dx = -\frac{\pi}{2}\log(2)$$

**Example 4.12**

As our final example, we compute the following definite integral,

$$\int_0^{\pi} e^{\cos(x)}\cos(\sin(x))dx.$$

We introduce the parameter $b$ as follows:

$$I(b) = \int_0^{\pi} e^{b\cos(x)}\cos(b\sin(x))dx$$

and note that all of our conditions in Theorem 1.3.1 are satisfied. However, before we compute as we did in the previous problems, we transform our integrand slightly so that we are working with complex exponentials:

$$\begin{aligned} I(b) &= \int_0^{\pi} e^{b\cos(x)}\cos(b\sin(x))dx = \frac{1}{2}\int_{-\pi}^{\pi} e^{b\cos(x)}\cos(b\sin(x))dx \\ &= \frac{1}{2}\int_0^{2\pi} e^{b\cos(x)}\cos(b\sin(x))dx \\ &= \Re\left[\frac{1}{2}\int_0^{2\pi} e^{be^{ix}}dx\right] \end{aligned}$$

With the problem posed in this fashion, now we proceed as before:

$$\begin{aligned} I'(b) &= \frac{1}{2}\frac{d}{dx}\int_0^{2\pi} e^{be^{ix\pi}}dx = \frac{1}{2}\int_0^{2\pi}\frac{\partial}{\partial b}\left[e^{be^{ix}}\right]dx \\ &= \frac{1}{2}\int_0^{2\pi} ibe^{be^{ix}}e^{ix}dx = \frac{1}{2}e^{be^{ix}}\Big|_0^{2\pi} \\ &= 0 \end{aligned}$$

since our derivative is 0,we know that I(b) is constant wrt b. so we conclude:

$$I(a) = I(0) = \pi$$

## 4.5 Frullani's Theorem

**Theorem 4.13**

Let $f : [0, +\infty) \to \mathbf{R}$ be a function such that $(f(ax) - f(bx))/x$ is integrable on $[0, +\infty)$ for every pair of positive real numbers $a, b$. Then there exists a constant $A \in \mathbf{R}$ such that

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}dx = A\log\frac{a}{b}$$

for every $a, b > 0$

$$\begin{aligned} \int_0^\infty \frac{f(ax) - f(bx)}{x}dx &= \int_0^\infty \left[\frac{f(xt)}{x}\right]_{t=b}^{t=a} dx \\ &= \int_0^\infty \int_b^a f'(xt)dtdx \\ &= \int_b^a \int_0^\infty f'(xt)dxdt \\ &= \int_b^a \left[\frac{f(xt)}{t}\right]_{x=0}^{x\to\infty} dt \\ &= \int_b^a \frac{f(\infty) - f(0)}{t}dt \\ &= (f(\infty) - f(0))(\ln(a) - \ln(b)) \\ &= (f(\infty) - f(0))\ln\left(\frac{a}{b}\right) \end{aligned}$$

# 5 SYNOPSIS OF INDEFINITE INTEGRAL

**Example 5.1**

Some standard formulae:

1. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, n \neq -1$

2. $\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C$

3. $\int e^{ax+b} dx = \frac{1}{a}e^{ax+b} + C$

4. $\int a^{px+q}\ dx = \frac{a^{px+q}}{p\ell na} + C; a > 0$

5. $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + C$

6. $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C$

7. $\int \tan(ax+b)dx = \frac{1}{a}\ln|\sec(ax+b)| + C$

8. $\int \cot(ax+b)dx = \frac{1}{a}\ln|\sin(ax+b)| + C$

9. $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + C$

10. $\int \operatorname{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b) + C$

11. $\int \sec(ax+b)\cdot\tan(ax+b)dx = \frac{1}{a}\sec(ax+b) + C$

12. $\int \operatorname{cosec}(ax+b)\cdot\cot(ax+b)dx = -\frac{1}{a}\operatorname{cosec}(ax+b) + C$

13. $\int \sec x dx = (\ln|\sec x + \tan x| + C)$ . (or) $\ln\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + C$

14. $\int \operatorname{cosec} x dx = \ell n|\operatorname{cosec} x - \cot x| + C$ (or) $\ln\left|\tan\frac{x}{2}\right| + C$ (or) $-\ell n|\operatorname{cosecx} + \cot x| + C$

15. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

16. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

17. $\int \frac{\mathrm{dx}}{\mathrm{x}\sqrt{\mathrm{x}^2-\mathrm{a}^2}} = \frac{1}{\mathrm{a}}\sec^{-1}\frac{\mathrm{x}}{\mathrm{a}} + \mathrm{C}$

18. $\int \frac{\mathrm{dx}}{\sqrt{\mathrm{x}^2+\mathrm{a}^2}} = \ln\left|\mathrm{x}+\sqrt{\mathrm{x}^2+\mathrm{a}^2}\right| + \mathrm{C} \quad (\mathrm{or})\ \sinh^{-1}\frac{\mathrm{x}}{\mathrm{a}} + \mathrm{C}$

19. $\int \frac{\mathrm{dx}}{\sqrt{\mathrm{x}^2-\mathrm{a}^2}} = \ln\left|\mathrm{x}+\sqrt{\mathrm{x}^2-\mathrm{a}^2}\right| + \mathrm{C} \quad (\mathrm{or}) \quad \cosh^{-1}\frac{\mathrm{x}}{\mathrm{a}} + \mathrm{C}$

20. $\int \frac{\mathrm{dx}}{\mathrm{a}^2-\mathrm{x}^2} = \frac{1}{2\mathrm{a}}\ln\left|\frac{\mathrm{a}+\mathrm{x}}{\mathrm{a}-\mathrm{x}}\right| + \mathrm{C}$

21. $\int \frac{\mathrm{dx}}{\mathrm{x}^2-\mathrm{a}^2} = \frac{1}{2\mathrm{a}}\ell\mathrm{n}\left|\frac{\mathrm{x}-\mathrm{a}}{\mathrm{x}+\mathrm{a}}\right| + \mathrm{C}$

**Example 5.2**

1. $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

2. $\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + C$

3. $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C$

4. $\int \mathrm{e}^{\mathrm{ax}}\cdot\sin\mathrm{bxdx} = \frac{\mathrm{e}^{\mathrm{ax}}}{\mathrm{a}^2+\mathrm{b}^2}(\mathrm{a}\sin\mathrm{bx}-\mathrm{b}\cos\mathrm{bx}) + \mathrm{C}$

5. $\int e^{ax}\cdot\cos bxdx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$

6. $\int \mathrm{C}f(x)dx = C\int f(x)\cdot dx$

7. $\int(\mathrm{f(x)} \pm \mathrm{g(x)})\mathrm{dx} = \int \mathrm{f(x)dx} \pm \int \mathrm{g(x)dX}$

8. $\int \mathrm{f(x)dx} = \mathrm{g(x)} + \mathrm{C}_1 \Rightarrow \int \mathrm{f(ax+b)dx} = \frac{\mathrm{g(ax+b)}}{\mathrm{a}} + \mathrm{C}_2$

9. $\int(f(x))^n f'(x)dx = \frac{(f(x))^{n+1}}{n+1} + c, n \neq 1$

10. $\int \frac{f'(x)}{(f(x))^n}dx = \frac{(f(x))^{-n+1}}{-n+1} + c$ where $n \neq 1$

11. $\int \frac{f'(x)}{f(x)}dx = \log_e(|f(x)|) + c$

12. $\int \frac{f(x)g'(x)-g(x)f'(x)}{f(x)g(x)}dx = \ln\left(\frac{g(x)}{f(x)}\right) + c$

    for ex: i) $\int \frac{x^2}{(x\cos x-\sin x)(x\sin x+\cos x)}dx$

    ii) $\int \frac{e^{2x}-e^{t}+1}{(e^x\sin x+\cos x)(e'\cos x-\sin x)}$

13. Examples:

1) $\int \frac{\sqrt{5+x^{10}}}{x^{16}} dx$ Is equal to $\frac{-1}{75}\left(1+\frac{5}{x^{10}}\right)^{\frac{3}{2}}+c$

2) $\int \frac{5x^4+4x^5}{(x+1+x^5)^2} dx =, \ldots \cdot \frac{x^5}{x+1+x^5}+c$

3) $\int \frac{(x^2-1)}{x^3\sqrt{2x^4-2x^2+1}} dx = \frac{1}{2}\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}+c$

4) $\int \frac{2x^{12}+5x^9}{(x^5+x^3+1)^3} dx = \ldots \cdot \frac{x^{10}}{2(x^5+x^3+1)^2}+c$

5) $\int \left(x^{7m}+x^{2m}+x^m\right)\left(2x^{6m}+7x^m+14\right)^{\frac{1}{m}} dx$
$= \frac{1}{14(m+1)}\left(2x^{7m}+7x^{2m}+14x^m\right)^{\frac{m+1}{m}}$

6) $\int \left(x^2+x\right)\left(x^{-8}+2x^{-9}\right)^{\frac{1}{10}} dx = \frac{5}{11}\left(x^2+2x\right)^{\frac{11}{10}}+C$

**Example 5.3**

1. Integration of Type: $\int \frac{L_1(x)}{L_2(x)} dx$ where $L_1(x)$ and $L_2(x)$ are linear functions in $x$ To evaluate such integrals write $\mathrm{L_1(x)}$ in terms of $\mathrm{L_2(x)}$ (i.e. $\mathrm{L_1(x) = A \cdot L_2(x) + B}$) then $\int \frac{L_1(x)}{L_2(x)} dx = \int \frac{AL_2(x)+B}{L_2(x)} dx = Ax + B\int \frac{1}{L_2(x)} dx$ OR divide and proceed

2. Integration of Type: $\int \frac{L_1(x)}{\sqrt{L_2(x)}} dx$ where $\mathrm{L_1(x)}$ and $\mathrm{L_2(x)}$ are linear functions in x To evaluate such integrals write $\mathrm{L_1(x)}$ in terms of $\mathrm{L_2(x)}$ (i.e. $\mathrm{L_1(x) = A \cdot L_2(x) + B}$ ) then $\int \frac{L_\downarrow(x)}{\sqrt{L_2(x)}} dx = \int \frac{A \cdot L_2(x)+B}{\sqrt{L_2(x)}} dx = A \cdot \int \sqrt{L_2(x)} dx + B\int \frac{1}{\sqrt{L_2(x)}} dx$ OR take $t^2 = L_2(x)$ and proceed

3. Integration of Type: To evaluate such integrals write $\mathrm{L_1(x)}$ in terms of $\mathrm{L_2(x)}$ (i.e. $\mathrm{L_1(x) = A \cdot L_2(x) + B}$ ) then $\int L_1(x)\sqrt{L_2(x)} dx = \int A.\left(L_2(x)\right)^{\frac{3}{2}} + B\sqrt{L_2(x)} dx$ OR take $t^2 = L_2(x)$ and proceed

4. Integration of Type: $\int \sqrt{\frac{L_1(x)}{L_2(x)}} dx$ where $L_1(x)$ and $L_2(x)$ are linear functions in $x$ take $t^2 = L_2(x)$ and proceed

5. Integration of Type: $\int \frac{1}{\text{Quadratic}} dx$ or $\int \frac{1}{\sqrt{\text{Quadratic}}} dx$ or $\int \sqrt{Q \text{ uadratic}}\, dx$ Examples: $\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c} dx$ Express $ax^2+bx+c$ in the form of perfect square & then apply the suitable formula In case of $\int \frac{1}{\text{Quadratic}}$, Quadratic equation can be factorized, then partial fraction will integrate.

6. Integration of type: $\int \frac{\text{linear}}{\text{quadratic}} dx$ or $\int \frac{\text{linear}}{\sqrt{\text{quadratic}}} dx$ or $\int$ linear $\sqrt{\text{quadratic}}\, dx$
$\int \frac{px+q}{ax^2+bx+c} dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx, \int (px+q)\sqrt{ax^2+bx+c} dx$
Express $\mathrm{px + q = A}$ (differential co - efficient of denominator) $+\mathrm{B}$ and find the

values of $A$ and $B$ and proceed. In case of $\int \frac{\text{linear}}{\text{quadratic}} dx$, Quadratic equation can be factorized, then partial fraction that will to integrate.

7. $\int \frac{dx}{(x-\alpha)\sqrt{(x-\alpha)(\beta-x)}}$ ( Let : $x=\alpha\cos^2\theta+\beta\sin^2\theta$)

8. $\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$ (Let : $x=\alpha\sec^2\theta-\beta\tan^2\theta$)

**Example 5.4** 1. Integration of the type

$\int \frac{1}{x(x''+1)} dx$ or $\int \frac{1}{x^n(x^n+1)^{\frac{1}{n}}} dx$ or $\int \frac{1}{x^2(x^n+1)^{\frac{n-1}{n}}} dx$

Take $x$ common and let $t=\left(1+\frac{1}{x^n}\right)$ & proceed.

Examples:

1. $\int \frac{dx}{x^{20}(1+x^{20})^{\frac{1}{20}}} = -\frac{1}{19}\left(1+\frac{1}{x^{20}}\right)^{\frac{19}{20}}+C$
2. $\int \frac{dx}{x^{22}(x^7-6)} = A\left\{\log u^6+9u^2-2u^3-18u\right\}+\mathrm{c}\ A=\frac{1}{54432}, u=\left(\frac{x^7-6}{x^7}\right)$

2. Integration of the type $\mathrm{I}=\int \frac{x^{\mathrm{m}}}{(ax+b)^n} dx$ where $\mathrm{m,n}$ are natural numbers Put $\mathrm{t}=\mathrm{ax}+\mathrm{b}$ then $\mathrm{I}=\frac{1}{a^{m-1}}\int \frac{(t-b)^{\mathrm{m}}}{t^n} dt$ [example: $\int \frac{x^2}{(x+2)^3} dx$]

3. Integration of the type $\int \frac{dx}{x^m(ax+b)^n}$ where $m, n$ are natural numbers Put $t=\frac{ax+b}{x}$ [example: $\int \frac{dx}{x^3(ax+b)^2}$]

4. Integration of the type:
   $\int \frac{\mathrm{dx}}{\mathrm{L}_1\sqrt{\mathrm{L}_2}}$ OR $\int \frac{\mathrm{dx}}{\text{Quadratic }\sqrt{\text{Linear}}}$ take Linear $=\mathrm{t}^2$ and proceed
   Example:
   $\int \frac{dx}{(ax+b)\sqrt{px+q}}$ OR $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$

   Put $\mathrm{px}+\mathrm{q}=\mathrm{t}^2$. And proceed

5. Integration of the type:
   $\int \frac{\mathrm{dx}}{\text{Linear }\sqrt{\text{ Quadratic}}}$, take $\mathrm{L}=1/\mathrm{t}$ and proceed Example: $\int \frac{\mathrm{dx}}{(\mathrm{ax+b})\sqrt{\mathrm{px}^2+\mathrm{qx+r}}}$, put $\mathrm{ax}+\mathrm{b}=\frac{1}{\mathrm{t}}$;

   Integration of the type: $\int \frac{dx}{Q_1\sqrt{Q_2}}, \int \frac{dx}{(ax^2+bx+c)\sqrt{px^2+qx+r}}$

6. Case $-1$
   Breaking in linear factors of Quadratic
   So we would prefer to proceed in this manner which helps us to evaluate it
   $\int \frac{1}{L_1 L_2 \sqrt{Q}} dx = \int \left(\frac{A}{L_1} + \frac{B}{L_2}\right) \frac{1}{\sqrt{Q}} = \int \frac{1}{L_1\sqrt{Q}} dx + \int \frac{1}{L_2\sqrt{Q}} dx$

7. Case - II
   If $ax^2 + bx + c$ is a perfect square say $(lx+m)^2$ the put $lx + m = 1/t$

**Example 5.5**

Case $-III$
If $\mathrm{b}=0; \mathrm{q}=0 \mathrm{eg} \cdot \int \frac{\mathrm{dx}}{\left(\mathrm{ax}^2+\mathrm{b}\right)\sqrt{\mathrm{px}^2+\mathrm{r}}}$ then put $\mathrm{x}=\frac{1}{\mathrm{t}}$ or the trigonometric substitution are also helpful.
Case IV If factors of $ax^2+bx+c$ is not possible. Then put $\frac{px^2+qx+r}{ax^2+bx+c} = t^2$ 11)Integration of the type:
(i) $\int \frac{dx}{a \pm b\sin^2 x}$ OR $\int \frac{dx}{a \pm b\cos^2 x}$ OR $\int \frac{dx}{a\sin^2 x \pm b\sin x\cos x \pm c\cos^2 x}$
$OR \int \frac{dx}{a \pm b\sin 2x} OR \int \frac{dx}{a \pm b\cos 2x} \int \frac{1}{a\cos^2 x \pm b\sin^2 x} dx * OR \int \frac{1}{(a\cos x \pm b\sin x)^2} dx$
OR $\int \frac{1}{a\sin^2 x \pm b\cos^2 x \pm c} dx$

(Denominator is the expression in terms of $\sin 2x$ or $\cos 2x$ or $\sin^2 x$ or $\cos^2 x$ or $\sin^4 x$ or $\cos^4 x$) .
Multiply Numerator and Denominator by $\sec^2 x$ and hence convert the question in the form of $f(\tan x), \sec^2 x \&$ put $\tan x = t$ or Multiply Numerator & Denominator by $\operatorname{cosec}^2 x$ and hence convert the question in the form of $f(\cot x), \cos ec^2 x \&$ put $\cot x = t$
12) Integration of the type: $\int \frac{dx}{a \pm b\sin x}$ OR $\int \frac{dx}{a \pm b\cos x}$ OR $\int \frac{dx}{a \pm b\sin x \pm c\cos x}$ Convert sines & cosines into their respective tangents of half the angles and then, put $\tan\frac{x}{2} = t, \frac{1}{2}\sec^2 x dx = dt$ and proceed

$$\int \frac{\mathrm{dx}}{a+b\sin x} = \begin{cases} \frac{2}{\sqrt{a^2-b^2}} \tan^{-1}\left(\frac{a\tan\left(\frac{x}{2}\right)+b}{\sqrt{a^2-b^2}}\right) + c, \text{ if, } a^2 > b^2 \\ \frac{1}{\sqrt{b^2-a^2}} \log_e \left(\frac{b-\sqrt{b^2-a^2}+a\tan\left(\frac{x}{2}\right)}{b+\sqrt{b^2-a^2}+a\tan\left(\frac{x}{2}\right)}\right) \text{ if } a^2 < b^2 \\ \frac{1}{a}(\tan x - \sec x) + c, \text{ if } b = a \\ \frac{1}{a}(\tan x + \sec x) + c, \text{ if } b = -a \end{cases}$$

13)Integration of the type:
$\int \frac{a\cdot\cos x + b\cdot\sin x}{\ell\cdot\cos x + m\cdot\sin x} dx. \int \frac{1}{a+b\tan x} dx, \int \frac{1}{a+b\cot x} dx, \int \frac{a+b\cot x}{c+d\cot x} dx, \int \frac{\tan x}{a+b\tan x} dx \int \frac{1}{a\sin x + b\cos x} dx$
Express Numerator $=$ A(Denominator) $+\mathrm{B}\frac{\mathrm{d}}{\mathrm{dx}}$ (Denominator), And find the value of the constants A and B by comparing the coefficients of $\cos \mathrm{x}$ and $\sin \mathrm{x}$ and proceed i.e.

$$\int \frac{N}{D} dx = \int \frac{A\cdot(D) + Bd(D)}{D} dx = Ax + B\ln|D| + c$$

Examples:
1) $\int \frac{4e^4 + 6e^{-1}}{9e^x - 4e^{-x}} dx$ 2) $\int \frac{3\sin x + 2\cos x}{2\sin x + 3\cos x} dx$ 3) $\int \frac{1}{2+3\tan x} dx$

**Example 5.6** 1. Integration of the type: $\int \frac{a \cdot \cos x+b \cdot \sin x+c}{\ell \cdot \cos x+m \cdot \sin x+n} dx$.

2. Express Numerator = A(Denominator) +B$\frac{\mathrm{d}}{\mathrm{dx}}$ (Denominator) +K and find the values of A, Band $K$ by comparing the coefficients of $\cos x$ and $\sin x$ and proceed. (i.e. $\int \frac{N}{D} dx=\int \frac{A.(D)+B d(D)+K}{D} dx=A x+B \ln |D|+K \int \frac{1}{D}\Big)$

3. Integration of the type $\int \frac{1}{a \sin x+b \cos x} dx$ or $\int \frac{1}{(a \cos x \pm b \sin x)^{2}} dx$

4. convert $a \sin x+b \cos x$ as a single term $=\sqrt{a^{2}+b^{2}}(\sin (A+x))$ OR $\sqrt{a^{2}+b^{2}}(\cos (x-A))$ and proceed.

5. Integration of the type $\int \cos m x \cdot \cos n x d x, \int \sin m x \cdot \sin n x \cdot d x, \int \cos m x \cdot \sin n x d x$ and Write the integrant as a sum of two terms and proceed

6. $\int \tan (a+b) x \cdot \tan a x \cdot \tan b x \cdot d x=\int(\tan (a+b) x-\tan \alpha x-\tan b x) d x$

7. $\int \frac{1}{\sin (x-a) \sin (x-b)} d x=\frac{1}{\cos (a-b)} \int \frac{\cos ((x-b)-(x-a))}{\sin (x-a) \sin (x-b)} d x$

8. $\int \frac{\sin (x+a)}{\sin (x+b)} d x=\int \frac{\sin ((x+b)+(a-b))}{\sin (x+b)} d x$

9. Integration of type $\int \sin^{\mathrm{m}} x \cdot \cos^{n} x d x$
   Case: i If atleast one of m or n is odd natural number, say m is odd put cos x = t and vice versa.
   Case: ii When m + n is a negative even integer (say **k** ), multiply and divide by $\cos^{2} x$ hence convert the question in the form of $f(\tan x), \sec^{2} x$ then put $\tan x=t$ to evaluate the integration.
   Case: iii If $m$ and $n$ are even natural number then converts higher power into higher angles.
   Examples:

10. $\int\left(\frac{\sin^{2} x}{\cos^{14} x}\right)^{\frac{1}{3}} d x$

11. $\int \frac{\sqrt{\tan x}}{\sin x \cos x} d x$

12. $\int \frac{\cos^{4} x}{\sin^{3} x\left(\sin^{5} x+\cos^{5} x\right)^{\frac{3}{5}}} x d x$

13. Integrals of the form $\int_{\alpha}^{\beta} \frac{d x}{\sqrt{(x-\alpha)(\beta-x)}}, \int_{\alpha}^{\beta} \sqrt{\frac{x-\alpha}{\beta-x}} d x, \int_{\alpha}^{\beta} \sqrt{(x-\alpha)(\beta-x)} d x$ can also be solved by using the substitution $x=\alpha \cos^{2} \theta+\beta \sin^{2} \theta$ , or, $x=\alpha \sec^{2} \theta-\beta \tan^{2} \theta$

**Example 5.7** 1. Integration by parts: Product of two functions

f(x) and g(x) can be integrated, using formula:
$\int(\mathrm{f(x)g(x)})\mathrm{dx} = \mathrm{f(x)}\int(\mathrm{g(x)})\mathrm{dx} - \int\left(\frac{\mathrm{d}}{\mathrm{dx}}(\mathrm{f(x)})\int(\mathrm{g(x)})\mathrm{dx}\right)\mathrm{dx}$ or

$\int u(x).d(v(x)) = u(x).v(x) - \int v(x)d(u(x))$

(i) When you find $\int \mathrm{g(x)dx}$ then it will not contain arbitrary constant.

(ii) $\int g(x)dx$ should be taken as same at both places.

(iii) The choice of f(x) and g(x) can be decided by ILATE guideline. The function will come later is taken an integral function
(g(x)). [ I $\rightarrow$ Inverse function, L $\rightarrow$ Logarithmic function A $\rightarrow$ Algebraic function function E $\rightarrow$ Exponential function.

2. Examples:
i). $\int e^{2x}\left(2x^3 + 3x^2 - 8x + 1\right)dx$

ii). $\int\left(3x^2 + x - 2\right)\sin^2(3x+1)dx$

**Example 5.8**

Some standard Reduction formula:

1. $I_n = \int \sin^n x dx = \frac{-\sin^{n-1} x \cdot \cos x}{n} + \frac{n-1}{n} I_{n-2}$

2. $I_n = \int \cos^n x dx = \frac{\cos^{n-1} x \cdot \sin x}{n} + \frac{n-1}{n} I_{n-2}$

3. $I_n = \int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - I_{n-2}, n \geq 2$

4. $I_n = \int \cot\ ^n x dx = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}, n \geq 2$

5. $I_n = \int \sec^n x dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} I_{n-2}$

6. $I_h = \int \operatorname{cosec}^n x dx = \frac{\cot x \cos ec^{n-3} x}{-(n-1)} + \frac{n-2}{n-1} I_{n-2}$.

7. If $I_{(m,n)} = \int x^m (\log_e x)^n dx$, then $l_{(m,n)} = \frac{x^{m+1}}{m+1} (\ln x)^n - \frac{n}{m+1} I_{(m,n-1)}$

8. If $l_n = \int \frac{dx}{(a^2+x^2)^n}$, then $l_n = \frac{x}{a^2(2n-2)(a^2+x^2)^{n-1}} + \frac{2n-3}{2n-2a^2} I_{m-1}$

9. If $I_n = \int \frac{dx}{(1+x^2)^{11}}$, then $I_n = \frac{x}{(2n-2)(1+x^2)^{n-1}} + \frac{2n-3}{2n-2} I_{n-1}$

10. Let $I_{(\ m,n)} = \int \frac{\sin^m x}{\cos^n x} dx (n \neq 1)$ then $I_{(m,n)} = \frac{1}{(n-1)} \cdot \frac{\sin^{m-1} x}{\cos^{n-1} x} + \frac{m-1}{n-1} I_{(\ m-2,n-2)}$

11. If $I_n = \int \frac{x^n}{\sqrt{ax^2+2bx+c}} dx$ then $(n+1)aI_{n+1} + (2n+1)bI_n + ncI_{n-1} =$
$$x^n \sqrt{ax^2 + 2bx + c}$$

12. If $I_n = \int \cos nx \cdot \cos ecx dx$ then $I_n - I_{n-2} = \frac{2\cos(n-1)x}{n-1}$

13. If $I_{m,n} = \int x^m (1-x)^{n-1} dx$ then $I_{m,n} - \frac{n-1}{m+n} I_{m,n-1)} = \frac{x^{m+1}(1-x)^{n-1}}{m+n}$

14. If $\mathrm{I_n} = \int \mathrm{e^x} (\sin \mathrm{x})^{\mathrm{n}} \mathrm{dx}$, then $\frac{\mathrm{I_3}}{\mathrm{I_1}}$ is equal to $\ldots 3/5$

15. Let n be a non-negative integer and, Let $\mathrm{I_n} = \int \mathrm{x}^n \sqrt{\mathrm{a}^2 - \mathrm{x}^2} \mathrm{dx} (\mathrm{a} > 0)$ $I_n = -\frac{x^{n-1}(a^2-x^2)^{3/2}}{A} + a^2 B I_{n-2}$ where $A$ and $B$ are constants. $A = n+2$ and $B = \frac{n-1}{n+2}$

16. $I_n = \int (\sin x + \cos x)^n dx = (\sin x + \cos x)^{n-1} (\sin x - \cos x) + 2(n-1) I_{n-2}$

# 6 SYNOPSIS OF DEFINITE INTEGRAL

**Example 6.1**
Properties of Definite Integral:

1. $\int_a^b \mathrm{f(x)dx} = \int_a^{\mathrm{b}} \mathrm{f(t)dt}$
i.e. definite integral is independent of variable of integration.
$\int_a^b f(x)dx = -\int_b^a f(x)dx$

2. $\int_a^b f(x)dx = \int_{-b}^{-a} f(-x)dx$

3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
where $c$ may lie inside or outside the interval $[a, b]$.

4.
$$\int_{-a}^{a} f(x)dx = \int_0^a f(x) + f(-x)dx$$
$$= \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(-x) = f(x) \text{ ie.f } (x) \text{ is even} \\ 0, & \text{if } f(-x) = -f(x) \text{ ie.f } (x) \text{ is odd} \end{cases}$$

[example: The value of $\int_{-2014}^{2014} \frac{f'(x)+f^1(-x)}{(2014)^x+1} dx$ equals to $f(2014) - f(-2014)$]

5. $\int_{\frac{a}{k}}^{\frac{b}{k}} f(kx)dx = \frac{1}{k}\int_a^b f(x)dx$ ( Expansion and contraction property)

6. If $f(x) \geq 0$ for all $a \leq x \leq b$ then $\int_a^b f(x)dx \geq 0$

7. $\int_0^a \mathrm{f(x)dx} = \int_0^a \mathrm{f(a-x)dx}$, (Shift property)

8. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

9. $\int_a^b f(x)dx = (b-a)\int_0^1 f(a+(b-a)x)dx$
example:
$\int_{-4}^{-5} \sin^2\left(x^2-3\right)dx + \int_{-2}^{-1} \sin^2\left(x^2+12x+33\right)dx = 0$

**Theorem 6.2**
Let $\frac{d}{dx}(F(x)) = f(x) \forall x \in (a, b)$. Then $\int_a^b f(x)dx = \lim_{x\to a-} F(x) - \lim_{x\to b^-} F(x)$

$a$. If F(x) is continuous at a and b, then $\int_a^{\mathrm{b}} \mathrm{f(x)dx} = \mathrm{F(b)} - \mathrm{F(a)}$.

b. Leibnitz Theorem: If $F(x) = \int_{f(x)}^{h(x)} f(t)dt$, then $\frac{dF(x)}{dx} = h'(x)f(h(x)) - g'(x)f(g(x))$

c. If the functions are defined on $[a, b]$ and differentiable on $(a, b)$ and the function $f(x, t)$ is continuous then $\frac{d}{dx}\left(\int_{m(x)}^{n(x)} f(x, t)dt\right) = \int_{m(x)}^{n(x)} \frac{\partial}{\partial x} f(x, t)dt + \frac{d(m(x))}{dx} f(x, m(x))$- $\frac{d(n(x))}{dx} f(x, n(x))$

**Example 6.3**
Integral of an inverse function

1. If f(x) is invertible function and f(x) is continuous then a definite integral of can be expressed in terms of f(x) i.e. $\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y)dy$ or $\int_a^b f(x)dx + \int_c^d f^{-1}(y)dy = bd - ac$

2. Examples:
i) If the value of $\int_\pi^2 e^{x^2}dx$ is k, then the value of $\int_e^{e^4} \sqrt{\ln x}dx$ is $2e^4 - e - k$
ii) $\int_0^{\frac{1}{2}} \sin x dx + \int_0^1 \sin^{-1} x dx = \frac{\pi}{2}$
iii) $\int_0^1 \sqrt[3]{1-x^7}dx - \int_0^1 \sqrt[7]{1-x^3}dx = 0$

3. Reduction formula:
i. $I_{m,n} = \int_0^1 x^m(1-x)^n dx = \frac{m!\ n!}{(m+n+1)!}$
ii. $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx$
According as n is even or odd. $I_0 = \frac{\pi}{2}, I_1 = 1$
Hence $I_n = \begin{cases} \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\cdots\cdots\left(\frac{1}{2}\right)\cdot\frac{\pi}{2}, & \text{if } n \text{ is even} \\ \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\cdots\cdots\left(\frac{2}{3}\right)\cdot 1, & \text{if } n \text{ is odd} \end{cases}$
iii. If $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cdot \cos^n x dx = \frac{(m-1)(m-3)(m-5)....(n-1)(n-3)(n-5).....}{(m+n)(m+n-2)(m+n-4)........}\frac{\pi}{2}$ when both $m, n$ are even.
$= \frac{(m-1)(m-3)(m-5).....(n-1)(n-3)(n-5).....}{(m+n)(m+n-2)(m+n-4)..........}$, otherwise

4. Examples: $\left(\int_0^{\frac{\pi}{2}} \sin^6 x \cos^8 x dx = \frac{5.3.1.7.5.3.1}{14.12.10.8.6.4.2}\frac{\pi}{2}\right)$

5. Reduction formula - some examples:

6. $I_n = \int_0^{\frac{\pi}{2}} \frac{\sin nx}{\sin x}dx = \frac{\pi}{2}$, for all positive odd positive integers n.

7. $I_n = \int_0^\pi \frac{\sin(2n+1)x}{\sin x}$ $\mathrm{dx} = \pi$, for all natural number n.

8. $I_n = \int_0^\pi \frac{\sin^2 nx}{\sin^2 x}dx = n\pi$, for all natural numbers $n$

9. $\mathrm{In} = \int_0^{\frac{x}{2}} \frac{\sin^2 \mathrm{nx}}{\sin \mathrm{x}}\mathrm{dx} = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2\mathrm{n}-1}$ for all natural numbers n

10. If $I_{\mathrm{n}} = \int \tan^n x dx$, then $I_n + I_{n-2} = \frac{\tan^{n-1} x}{n-1}, n \geq 2$
$I_0 + I_1 + 2\left(I_2 + \ldots\ldots.. + 1_8\right) + I_9 + I_{10}$, is equal to $\sum_{n=1}^9 \frac{\tan^n x}{n}$

11. If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x dx$ then
i. $I_n + I_{n-2} = \frac{1}{n-1}$, for all $n = 2, 3, 4, \ldots\ldots.$
ii. $\frac{1}{I_2+I_4}, \frac{1}{I_3+I_5}, \frac{1}{I_4+1_6}, \ldots$ are in A.P
iii.$I_{n+1} + I_{n-1} = \frac{1}{n}$
iv.$\frac{1}{n+1} < 2I_n < \frac{1}{n-1}$ for all natural numbers greater than one.

**Example 6.4**

Maximum and Minimum Inequality:

1. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int f(x)dx \leq M(b-a)$, where $m$ and $M$ are absolute minimum and maximum values of the function f(x) in [a, b]

2. Further if f(x) is monotonically decreasing in (a, b),
then $\mathrm{f(b)(b-a)} < \int_a^b \mathrm{f(x)dx} < \mathrm{f(a)(b-a)}$
and if $f(x)$ is monotonically increasing in $(a, b)$, then
$f(a)(b-a) < \int_a^b f(x)dx < f(b)(b-a)$

Examples:

3. $1 < \int_0^{\frac{\pi}{3}} \frac{\sin x}{x} dx < \frac{\pi}{2}$ $\left\{\text{because f(x) decreases and } f\left(\frac{\pi}{2}\right) = \frac{2}{\pi} \& f\left(0^+\right) = 1\right\}$

4. $\frac{\sqrt{3}}{8} < \int_{\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sin x}{x} dx < \frac{\sqrt{2}}{6}$ $\left[\text{because f(x) decreases and } f\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2\pi} \& f\left(\frac{\pi}{4}\right) = \frac{2\sqrt{2}}{\pi}\right\}$

5. $1 < \int_0^2 \left(\frac{5-x}{9-x^2}\right) dx < \frac{6}{5}$ { absolute maximum and minimum values of $f(x)$ in $[0, 2]$ are $f(2)$ and f(1)}

6. Schwartz inequality:
For any two integrable functions f(x) and g(x) on the interval (a, b), then $\left|\int_a^b f(x)g(x)dx\right| \leq \sqrt{\int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx}$

7. Example: The maximum value of $\int_0^1 \sqrt{(1+x)\left(1+x^3\right)}dx$ is $\sqrt{\frac{15}{8}}$

8. Other inequalities:
If the function $f(x)$ increases and has a concave up graph in the interval $[a, b]$ then $(b-a)f(a) < \int_0^b f(x)dx < (b-a)\left(\frac{f(a)+f(b)}{2}\right)$
$[(b-a)f(a)$ equals to area of the rectangle whose sides are f(a) and b − a& $(b-a)\left(\frac{f(a)+f(b)}{2}\right)$ equals to area of the trapezium whose parallel sides has a length f(a) and $f(b)$, distance between the parallel sides is [b − a]

# 7 UNFAMILIAR PRACTICE QUESTIONS

It is not death that a man should fear, but he should fear never beginning to live.

⚠ Problems are not arranged according to increasing difficulty.

> "Do share your views on each problem via akiiiceoak1310@gmail.com or mail for solution of any particular problem mentioning Subject as Solution of Integral handout"
>
> *AK-AKIII*

**Problem 1.**

If

$$\int_{-0.25}^{0.25} \sum_{n=0}^{\infty} \binom{2n}{n} k^n dk = \frac{\sqrt{A}}{A},$$

find $A$.

**Problem 2.**

Let $I_1 = \int_0^1 \frac{dx}{e^x(1+x)}$ and $I_2 = \int_0^{\pi/4} \frac{e^{\tan^2\theta}}{(2-\tan^2\theta)} \frac{\sin\theta d\theta}{\cos^3\theta}$, then $\frac{eI_1}{I_2}$ is equal to

**Problem 3.**

If the integral

$$\int_0^{\infty} \frac{dx}{x^{2/3}\left(1+\sqrt{2}x+x^2\right)}$$

evaluates to $\frac{\pi^a\sqrt{b}}{c}$ for positive integers $a, b, c$ with $b$ square-free, then find $a+b+c$.

**Problem 4.**

Let f : $[0,1] \to \mathrm{R}$ be continuously differentiable function and

$$\left|\frac{f(0)+f(1)}{2} - \int_0^1 f(x)dx\right| \leq \alpha \max_{x\in[0,1]} |f'(x)|, \text{ then } \alpha = ?$$

**Problem 5.**

If $f(0) = 0, \& f'(x) \in (0,1]$, then

$$\frac{\int_0^1 f'(x)dx}{\int_0^1 f^3(x)dx}$$

is always greater than the least integer k.Find value of k.

**Problem 6.**

Evaluate

$$\lim_{n\to\infty} \int_0^{\pi/2} n(1-\sqrt[n]{\cos x})\mathrm{d}x$$

**Problem 7.**
Evaluate

$$\int_0^1 (xlnx)^{2021} dx$$

**Problem 8.**
Find the maximum value of

$$\int_0^y \sqrt{x^4 + (y(3-y))^2} dx,$$

where $0 \leq y \leq 3$.

**Problem 9.**
$\int_0^\pi \left(\frac{8\sin^2 x}{g(x)} - 1\right)(g(x))^2 dx = 6\pi$; where $g(x)$ is a continuous function, then find the maximum value of the function $g(x)$ in $[0, \pi]$ .

**Problem 10.**
If $\int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}} = K\sqrt[3]{\frac{1+x}{1-x}} + C$. Then K is equal to?

**Problem 11.**

$$I_n = \int_0^1 \frac{x^n}{x^2 + a^2} dx$$

Define the integral $I_n$ as above for positive real variable $a$ independent of $x$ and natural number $n$. Evaluate the limit below.

$$\lim_{n\to\infty} \left[\sqrt{n}\left(nI_n - \frac{1}{a^2+1}\right)\right]$$

**Problem 12.**
If $I = \int_0^\pi \ln\left(1 - 2a\cos x + a^2\right) dx$ where $I = \lambda$ ln a, then $\lambda$ is?

**Problem 13.**
$\sum_{p=4}^{N} \frac{p^2+2}{(p-2)^4} < K, \forall N \geq 4$,If the sum does not exceed the integer value k for any value of N,Find the value of k.

**Problem 14.**
Suppose $f$ is a function on the interval $[1, 3]$ such that $-1 \leq f(x) \leq 1$ for all $x$ and $\int_1^3 f(x)dx = 0$ If maximum value of $\int_1^3 \frac{\mathrm{f(x)}}{\mathrm{x}}\mathrm{dx}$ is $\ln\left(\frac{\mathrm{a}}{\mathrm{b}}\right)$ (where $\mathrm{a}, \mathrm{b}$ are coprime numbers), then $\mathrm{a} + \mathrm{b}$ is equal to??

**Problem 15.**
evaluate $\int_{-100}^{-10} \left(\frac{x^2-x}{x^3-3x+1}\right)^2 dx + \int_{\frac{1}{100}}^{\frac{1}{11}} \left(\frac{x^2-x}{x^3-3x+1}\right)^2 dx + \int_{\frac{101}{100}}^{\frac{11}{10}} \left(\frac{x^2-x}{x^3-3x+1}\right)^2 dx$

If the final is in the form $- P(x)|_{-100}^{10}$ then answer P(1)

**Problem 16.**

For large $n$, find $\left[\ \int_0^1 \frac{nx^{n-1}dx}{1+x^2}\ \right]$ where [ ] denotes greatest integer function.

**Problem 17.**

$I_1 = \int_0^1 \left(1-(1-x^3)^{\sqrt{2}}\right)^{\sqrt{3}} x^2dx$ and $I_2 = \int_0^1 \left(1-(1-x^3)^{\sqrt{2}}\right)^{\sqrt{3}+1} \cdot x^2dx$, then $\frac{\frac{I_1}{I_2}-\frac{\sqrt{3}-1}{2\sqrt{2}}+0.2}{10}$ is equal to?

**Problem 18.**

$$\int_1^{\frac{\sqrt{5}+1}{2}} \frac{x^2+1}{x^4-x^2+1} \cdot \ln\left(x-\frac{1}{x}+1\right)\mathrm{d}x$$

**Problem 19.**

If $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x+\cos x}dx = \frac{1}{2\sqrt{a}}\ln(b+c\sqrt{2})$.

Find a+b+c.

**Problem 20.**

$\frac{\int_0^{\frac{\pi}{2}} \theta^{100}\sin\theta d\theta}{\int_0^{\frac{\pi}{2}} \theta^{99}\cos\theta d\theta} = \alpha$ and $\int_0^{\frac{\pi}{2}} \theta^{99}\sin\theta d\theta + \frac{\int_0^{\frac{\pi}{2}} \theta^{100}\cos\theta d\theta}{100} = \beta$. Then find the value of $\alpha\beta$.

**Problem 21.**

Consider the integrals $I_1 = \int_0^1 e^{x^2}\sin x dx$; $I_2 = \int_0^1 e^{-x^2}\operatorname{cosec} x dx$ and $I_3 = e^{\int\left(x^2+\log\sin x\right)dx}$. The value $(\mathrm{I}_1-\mathrm{I}_3)\left(\mathrm{I}_3-\mathrm{I}_2^{-1}\right)$ belongs to $[a,\infty)$. Find the least integral value of $a$.

**Problem 22.**

If $I_1 = \int_0^{\pi/2} \cos\theta f\left(\cos^2\theta+\sin\theta\right)\mathrm{d}\theta$ and $I_2 = \int_0^{\pi/2} \sin 2\theta f\left(\sin\theta+\cos^2\theta\right)\mathrm{d}\theta$ then $\frac{\mathrm{I}_1}{\mathrm{I}_2}$ is ?

**Problem 23.**

Suppose $f$ is continuous on $[a,b]\&f(a)=f(b)=0\&\int_a^b f^2(x)dx=1$ then the minimum value of $\int_a^b \left(f'(x)\right)^2 dx \int_a^b x^2f^2(x)dx$ is ?

**Problem 24.**

If

$$\Omega = \int_{\frac{\pi}{5}}^{\frac{3\pi}{10}} \frac{xdx}{\sin 2x} = -\frac{\pi}{a}\log\left(\tan\frac{\pi}{b}\right),$$

find $a+b$.

**Problem 25.**

If $\int \cos(2016x)\cdot\left(\sin^{2014}x\right)dx = f(x)+C, f(0)=0$ then

1. $\int_\pi^{2\pi} \mathrm{f(x)dx} = a$

2. $\int_0^\pi$ f(x)dx $= b$

3. $\lim_{x\to 0} \frac{2015 f(x)}{x^{2015}} = c$

   Find a+b+c.

## Problem 26.

f: R $\to$ R is a continuous function satisfying $\int_0^1 \left(\frac{f(x)}{x}\right)^2 dx = 2\int_0^1 f(x)dx - \frac{1}{3}$, then $256 f\left(\frac{1}{8}\right)$ is equal to?

## Problem 27.

$\int_{-9}^{0} \frac{1+x^{2011}-\sqrt{1+x^{4022}}}{2x^{2011}} dx$

## Problem 28.

When $\int_0^1 f(x)dx = 0$ and $-1 \leq f(x) \leq 1$, the maximum value of

$$\int_0^1 (f(x))^3 dx$$

is $X$. Find $[100X]$.

## Problem 29.

Evaluate the following integral. $\int_8^{13} \frac{\lfloor x^2 \rfloor}{\lfloor x^2 \rfloor + \lfloor x^2 - 42x + 441 \rfloor} dx$

,Where [ ] denotes floor function.

## Problem 30.

$\frac{1}{10} \int_0^\pi x\ cosec^{\sin(\cos x)}(x) dx$

## Problem 31.

$\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} dx$

## Problem 32.

$\int_0^{\pi/2} \frac{4\cos x}{3+12\sin^2 x - 2\cos 2x + 12\sin^2 x \cos 2x + 3\cos 4x} dx = a\pi$

Find the value of a.

## Problem 33.

Let,

$$I_1 = \int_1^3 (x^2 + x + 3) \cdot f(x^3 - 2x^2 - 5x + 2020)\mathrm{d}x$$

$$I_2 = \int_1^3 (4x^2 - 3x - 2) \cdot f(x^3 - 2x^2 - 5x + 2020)\mathrm{d}x.$$

Now, if, $\frac{2I_1}{3I_2} = \frac{p}{q}$, where $p$ and $q$ are co-prime. Now, if,

$$\int \frac{x-1}{(x+x\sqrt{x}+\sqrt{x})\sqrt{(\sqrt{x}(x+1))}} \mathrm{d}x = 4\arctan\left[g(x)\right] + C,$$

where $C$ is an arbitrary constant of Integration. Find $g^2(q-p)$.

## Problem 34.

$f : (0,\infty) \to \mathbb{R}, fis$ continuous, $f(x) - \log_3 x = 4 - f\left(5^{\log_3 x}\right), \forall x > 0$. Find: $\Omega = \int_2^3 (f(x)-2)\cdot \log_x 15dx$

## Problem 35.

Evaluate 2022I,if

I=

$$\sum_{k=0}^{2021^2} (-1)^k \frac{\binom{2021^2}{k}}{\binom{2021+k}{k}}$$

## Problem 36.

Find a closed form: $\Omega = \left(\int_{\frac{\sqrt{3}}{3}}^{1} \frac{x\log x dx}{3x^4}\right)\left(\int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{x\log x dx}{x^4+1}\right)\left(\int_1^{\sqrt{3}} \frac{x\log x dx}{x^4+3}\right)$

## Problem 37.

If $\int_{-\infty}^{\infty} e^{\mathrm{tx}} f(x)dx = \sin^{-1}\left(t - \sqrt{\frac{1}{2}}\right)$, then $\frac{1}{2\sqrt{2}}\int_{-\infty}^{\infty} xf(x)dx$ is

## Problem 38.

$$\lim_{n\to\infty} n\int_1^e x^2(\ln(x))^n dx.$$

For integer $n$, let $L$ denote the value of the limit above. Find the value of $\ln(L)$

## Problem 39.

Consider an Integral

$$I = \int_0^{\pi/2} |\cos x - kx| dx$$

If for some positive Real value of $k$ this integral can be minimized. if that value of **k** is expressed as

$$\frac{\sqrt{a}}{\pi}\left(\cos\left(\frac{\pi}{\sqrt{a}}\right)\right)$$

Then Find $a^2$ ??

## Problem 40.

Find: $\int_0^{\infty} \frac{x}{(1+x^4)(1+x)} dx$

**Problem 41.** If the following limit

$$\lim_{n\to\infty}\int_0^{n\pi} e^{-x}|\sin(x)|dx$$

can be expressed as $\alpha\left(\frac{1+e^{\beta}}{1-e^{\lambda}}\right)$ where $\alpha,\beta,\lambda$ are real numbers, $\alpha\geq 0$ and $\beta,\lambda\leq 0$ and $n$ is an integer, find the value of $2(\alpha+\ \beta-\lambda)$

**Problem 42.**

Let $f(x)=\int\frac{x^7+2}{(x^2+x+1)^2}dx$ subject to $f(0)=\frac{\pi}{3\sqrt{3}}$. Find $|f(-1)|$

**Problem 43.**

$If$

$$I_1=\int_0^1\frac{x^{\frac{7}{2}}(1-x)^{\frac{9}{2}}}{30}dx$$

and

$$I_2=\int_0^1\frac{x^{\frac{7}{2}}(1-x)^{\frac{9}{2}}}{(x+5)^{10}}dx$$

and $\frac{I_1}{I_2}=5a^3\sqrt{a}$, where $a\in\mathbb{N}$, find $a$.

**Problem 44.**

There exist a Function $f(x)=ax^2+bx+c$ for $a>0$, which has positive roots. It is found that for some real values $\alpha$ and $\beta$.

$\int_0^{\alpha}f(x)\cdot dx=0$ and $\int_0^{\phi_1}f(x)\cdot dx+\int_{\phi_2}^{\beta}f(x)\cdot dx=|\int_{\phi_1}^{\phi_2}f(x)\cdot dx|$.

where $\phi_1$, $\phi_2$ are roots of $f(x)$, and $\phi_1,\phi_2$ belong to reals.Also, $\alpha+\beta=5$.
Find the value of $x$ for which $f(x)$ is minimum.

**Problem 45.**

Find: $\int_0^{\infty}\left(\frac{x^2}{(1-x^2+x^4)(1+x)}\right)dx$

**Problem 46.**

$Find$: $I=\int_0^{\infty}\frac{\ln x}{10^2+x^2}dx$

**Problem 47.**

Let $A=\int_{\pi/5}^{\pi/4}\cot^3 x\ln\sin x dx$. $B=\int_{\pi/5}^{\pi/4}\frac{\csc^2 x}{\tan x}\ln^2\sin x dx$.

Let $C=A-B$ ; Find $\lfloor -10000C\rfloor$.

**Problem 48.**

$$\int_0^{2018\pi}(\cos x)^{2^{3^{4^{\cdot^{\cdot^{\cdot^{2018}}}}}}}(\sin x)^{3^{4^{5^{\cdot^{\cdot^{\cdot^{2018}}}}}}}dx$$

## Problem 49.

$$I_n = \int_0^1 \frac{x^n}{ax+b} dx$$

Define the integral $I_n$ as above for positive real variables $a$ and $b$ independent of $x$ and natural number $n$.

$$\lim_{n\to\infty} nI_n = \frac{1}{\lambda a + \mu b}$$

If $\lambda$ and $\mu$ are constants that satisfy the limit above, evaluate $\lambda + \mu$.

## Problem 50.

$Given \frac{(f'(x)\cdot g(x) - g'(x)\cdot f(x)]dx}{[f(x)+g(x)]\sqrt{f(x)\cdot g(x) - g^2(x)}} = \sqrt{h}\tan^{-1}\sqrt{\frac{f(x)-g(x)}{kg(x)}} + C$. Calculate the value of $[h^2 + k + 1]$ where $(h, k \in N)$ and 'c' denotes integral constant and $(g(x) > 0)$.

## Problem 51.

$A$ function $f$ satisfies $f(x) = f(c/x)$ for some real number $c(> 1)$ and all positive number $x$. if $\int_1^{\sqrt{c}} f(x)/x \quad dx = 3$ then $\int_1^c f(x)/x \quad dx$ equals?

## Problem 52.

$\int_0^\pi \ln\left\{1 - 2\alpha\cos(x) + \alpha^2\right\} dx = \begin{cases} a, & \text{when } \alpha^2 < 1, and \\ b\ln\left\{\alpha^2\right\} & \text{when } \alpha^2 > 1 \end{cases}$

Find value of $a + b$.

## Problem 53.

$Evaluate$ the following:

$$\int_0^\infty \frac{x^3 - \sin^3 x}{x^5} dx$$

If the result can be expressed as $\frac{a\pi^b}{c}$ where $a$ and $c$ are coprime. Find $a+ b + c$.

## Problem 54.

$$\int_0^4 \frac{\ln(x)}{\sqrt{4x - x^2}} dx$$

## Problem 55.

$$I = \int_1^{16} \frac{\ln(x)}{\sqrt{x}(x+4)} dx$$

If $I$ can be expressed as $A\ln(A)\tan^{-1}\left(\frac{B}{C}\right)$ where $A, B, C$ are positive integers with $\gcd(B, C) = 1$ and $A$ is a prime number. Then find the value of $A + B + C$

## Problem 56.

$\int_0^\infty \frac{(x-1)}{\sqrt{2^x - 1}\ln(2^x - 1)} dx \cdot = \frac{\pi}{a\cdot(\ln b)^c}$ whene $a, b, c$ are Natural no. then find $\frac{(a^b + b^c + c^a)}{(a+b+c)}$

**Problem 57.**

Evaluate the following integrals

1. $\int_0^{\pi/2} \ln\{2\sin(x)\}dx$ .
2. $\int_0^{\pi/2} \ln\{2\cos(x)\}dx$

**Problem 58.**

$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{\sin\left(\frac{2n+1}{2}\cdot x\right)}{\sin\left(\frac{x}{2}\right)(x^2+1)}dx = \pi\frac{(e+a)}{(e+b)}$. Find $(a+b)$

**Problem 59.**

Let $I_a = \int_0^1 x^a(\ln x)^3 dx$. Calculate $\sum_{n=0}^{\infty} I_n$.

**Problem 60.**

$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{(2n+1)\binom{2n}{n}}$$

If the summation above can be represented as $A\cdot\pi^B$, where $A, B$ are positive integers, then find the value of $(A+B)^3$ ?

**Problem 61.**

Evaluate

$$2020\cdot\int_0^{\infty}\left\lfloor \log_{2021}\left\lfloor \frac{\lceil x\rceil}{x}\right\rfloor\right\rfloor dx,$$

where $\lceil\cdot\rceil$ denotes ceiling function and $\lfloor\cdot\rfloor$ denotes the floor function.

**Problem 62.**

Let $f$ satisfy $x = f(x)e^{f(x)}$. Calculate $\int_0^e f(x)dx$

**Problem 63.**

$$\lim_{n\to\infty}\int_0^{\frac{1}{n}} x^{2018x+1}dx$$

**Problem 64.**

$$\int_0^{\pi}\frac{2x\sin(x)}{3+\cos(2x)}dx$$

**Problem 65.**

$$\int\sqrt{x\sqrt[3]{x\sqrt[4]{x\sqrt[5]{x\ldots}}}}dx$$

**Problem 66.**

$$\int e^{e^{2016x}+6048x}dx$$

**Problem 67.**

$$\int_{-\infty}^{\infty} \arctan\left(\frac{1}{2x^2}\right) dx$$

**Problem 68.**

$\Omega = \int_0^1 (\{\log(1+x)\} \cdot \log\{1+x\}) dx$

$\{x\} = x - [x], [*]-$ great integer function

**Problem 69.**

$$\lim_{n\to\infty} l_n,$$

where

$$l_1 = \int_0^1 \frac{dx}{1+\sqrt{x}}, l_2 = \int_0^1 \frac{dx}{1+\frac{1}{1+\sqrt{x}}}$$

**Problem 70.**

$$\int_0^{\frac{\pi}{4}} x \prod_{k=1}^{\infty} \cos\left(\frac{x}{2^k}\right) dx$$

**Problem 71.**

$$\lim_{n\to\infty} n \int_0^1 \frac{x^n}{2014+x^n} dx = a$$

find $\left[10^7 a\right]$ , where $[.]$ - greatest integer function

**Problem 72.**

$$\int_0^{\infty} \left(xe^{1-x} - \lfloor x \rfloor e^{1-\lfloor x \rfloor}\right) dx$$

**Problem 73.**

$$\int_0^{\int_0^{\cdots} \frac{1}{\sqrt{x}}\, \mathrm{d}x} \frac{1}{\sqrt{x}}\, \mathrm{d}x$$

**Problem 74.**

$$\int_0^{\infty} \frac{x^8 - 4x^6 + 9x^4 - 5x^2 + 1}{x^{12} - 10x^{10} + 37x^8 - 42x^6 + 26x^4 - 8x^2 + 1} dx = ?$$

**Problem 75.**

$$\lim_{n\to\infty} \int_0^1 x^{2019} \{nx\} dx$$

**Problem 76.**

$$\int_{-2021\pi}^{2021\pi} \frac{\cos^5(x)+1}{e^x+1}\, \mathrm{d}x, \quad a \in \mathbb{N}$$

**Problem 77.**

$$\int_0^1 \frac{(x^2+1)\ln(1+x)}{x^4-x^2+1}dx$$

**Problem 78.**

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sqrt{\sin(2x)})^2}dx$$

**Problem 79.**

$$\int_0^1 g(x)-f(x)\mathrm{d}x$$
$$f(x)=1 \text{ and } g(x)=\sqrt{2-x^2}$$

. **Problem 80.**

$$\int_0^1 \left(1-x^3+x^5-x^8+x^{10}-x^{13}+\ldots\right)dx$$

**Problem 81.**

$$\int_0^{2\pi} \frac{x}{\phi-\cos^2(x)}dx$$
$$\phi: \text{ Golden ratio}$$

**Problem 82.**

$$\int_0^1 x\sqrt{x\sqrt[3]{x\sqrt[4]{x\sqrt[5]{x\ldots}}}}dx$$

**Problem 83.**

$$\int_0^{\pi} \frac{\sin\frac{21x}{2}}{\sin\frac{x}{2}}dx$$

**Problem 84.**

$$\lim_{n\to\infty}\int_0^1 \sqrt{\frac{1}{x}+n^2x^{2n}}dx$$

**Problem 85.**

$$\int_0^1 \sqrt{\frac{x}{1-x}}\ln\left(\frac{x}{1-x}\right)dx$$

**Problem 86.**

$$\int_0^{\infty} \frac{\ln x}{1+x^2}\,\mathrm{d}x$$

**Problem 87.**

$$\int_0^{\frac{\pi}{2}} \tan^{2021}(x)dx$$

**Problem 88.**

$$\int_0^1 \left(\frac{x^x}{(1-x)^{1-x}} - \frac{(1-x)^{1-x}}{x^x}\right) \mathrm{d}x$$

**Problem 89.**
solve for $a+b$

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{x^2+4}dx = \frac{\pi}{ae^b}$$

**Problem 90.**

$$\int_0^{\infty} \frac{\log(x)}{\sqrt{x}(x+1)^2}dx$$

**Problem 91.**

$$\int_0^{\pi/4} \sqrt{\tan x}\sqrt{1-\tan x}dx$$

**Problem 92.**

$$\int_0^{\infty} \lfloor ne^{-x} \rfloor \, dx$$

**Problem 93.**

$$\int_1^3 \int_1^3 \int_1^3 \int_1^3 \int_1^3 \frac{x_1+x_2+x_3+x_4-x_5}{x_1+x_2+x_3+x_4+x_5} dx_1 dx_2 dx_3 dx_4 dx_5$$

**Problem 94.**

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 \frac{x_1^{505}+x_2^{505}+\ldots+x_n^{505}}{x_1^{2020}+x_2^{2020}+\ldots+x_n^{2020}} dx_1 dx_2 \ldots dx_n$$

**Problem 95.**
If

$$\left(\int_0^{\frac{d}{dx}\int_0^{\lim_{x\to\infty} e^{-x}\sum_{k=0}^{x} \frac{x^k}{k!}} \pi dt} \frac{dr}{1+8\sin^2(\tan r)}\right) = \theta$$

then, Find $\frac{(\cos(\theta)\sin(\theta))^2}{2}$

**Problem 96.** Evaluate

$$\int_0^1 (x^x)^{(x^x)^{(x^x)^{(x^x)^{\cdot^{\cdot^{\cdot}}}}}} \mathrm{d}x$$

**Problem 97.** I f

$$\int_0^1 (-1)^{\lfloor \frac{1}{x} \rfloor} dx = \ln\left(\frac{e}{k}\right)$$

then find $k$ ($\lfloor \bullet \rfloor$ = floor)

**Problem 98.** C onsider the integral

$$I_1 = \int_1^e (1+x)(x+\ell n x)^{100}\ dx$$

$$I_2 = \int_{\sin^{+}(1/e)}^{\pi/2} (1+e\sin x+\ell n(\sin x))^{101} \cos x dx$$

also, $I_1 + \frac{e}{101} I_2 = \frac{e(1+e)^{101}-k}{101}$

then "$K$" is greater than or equal to?

**Problem 99.** Evaluate

$$\int_0^\infty \frac{1-\cos(x\sqrt{e-1})}{xe^x} dx$$

**Problem 100.**

$$u_1 = \int_{\int_0^{\frac{1}{2}} xdx}^{\int_{\frac{1}{2}}^{1} xdx} xdx, \quad u_2 = \int_{\int_{\int_{\int_0^{\frac{1}{4}} xdx}^{\frac{1}{4}} xdx}^{\frac{2}{4}} xdx}^{\int_{\frac{2}{4}}^{\int_{\frac{3}{4}}^{\int_{\frac{3}{4}}^{\frac{4}{4}} xdx} xdx} xdx} xdx$$

$\{u_i\}_{i=0}^{\infty}$ is a sequence of real numbers. The first few terms are as above. If

$$P = \prod_{n=0}^{2021} \frac{1}{4u_n}$$

,find $[log_2 log_2 P]$,where [ ] denotes greatest integer function.

Details and assumptions:-

- The limits in $u_0$ when read from bottom to top are $\left[\frac{0}{2^0}, \frac{1}{2^0}\right]$.
- The limits in $u_1$ when read from bottom to top are $\left[\frac{0}{2^1}, \frac{1}{2^1}, \frac{1}{2^1}, \frac{2}{2^1}\right]$.
- The limits in $u_2$ when read from bottom to top are $\left[\frac{0}{2^2}, \frac{1}{2^2}, \frac{1}{2^2}, \frac{2}{2^2}, \frac{2}{2^2}, \frac{3}{2^2}, \frac{3}{2^2}, \frac{4}{2^2}\right]$
- In general, the limits in $u_n$ when read from bottom to top are $\left[\frac{0}{2^n}, \frac{1}{2^n}, \frac{1}{2^n}, \frac{2}{2^n}, \frac{2}{2^n}, \dots, \frac{2^n-1}{2^n}\right]$.

# 8 Answer keys

1. 02
2. 02
3. 10
4. 0.25
5. 01
6. 1.0887
7. $\frac{2021!}{2022^{2022}}$
8. 09
9. 04
10. -1.5
11. 0
12. $2\pi$
13. 05
14. 07
15. 00
16. 00
17. 0.12
18. $\frac{\pi}{8}\ln 2$
19. 07
20. $\left(\frac{\pi}{2}\right)^{100}$
21. 00
22. 01
23. 0.50
24. 13
25. 01
26. 04
27. 8.5
28. 25
29. 2.5

30. 0.399
31. $\pi^2$
32. 0.25
33. 02
34. 01
35. 01
36. 00
37. 0.50
38. 03
39. 64
40. $\frac{\pi}{8}$
41. 01
42. 2.187
43. 30
44. 08
45. $\frac{\pi}{3\sqrt{3}}$
46. $\frac{\pi \ln 10}{20}$
47. 2074
48. 0
49. 02
50. 07
51. 06
52. $\pi$
53. 46
54. 0
55. 09
56. 02
57. 0
58. 0
59. $-6.493$

60. 08

61. 01

62. $(e-1)$

63. 0.5

64. $\pi^2/4$

65. $(x^e-1)/(e-1)$

66. $$\frac{1}{2016}e^{e^{2016x}}\left(e^{4032x}-2e^{2016x}+2\right)$$

67. $\pi$

68. $e$

69. $\phi=\frac{\sqrt{5}+1}{2}$, aka Golden Ratio

70. $1-\frac{\sqrt{2}}{2}$

71. 4964

72. $e-\left(\frac{e}{e-1}\right)^2$

73. 04

74. $\pi/2$

75. $1/4040$

76. $2021\pi$

77. $\frac{\pi}{6}\ln(2+\sqrt{3})$

78. $1/3$

79. $\frac{\pi}{4}-\frac{1}{2}$

80. $\frac{\pi}{2}\sqrt{1+\frac{2}{\sqrt{5}}}$

81. $2\pi^2$

82. $1/e$

83. $\pi$

84. 03

85. $\pi$

86. 0

87. $\frac{\pi}{2}\sec(2021\pi/2)$

88. 0

89. 08
90. $-\pi$
91. $\pi\left(\frac{\sqrt{1+\sqrt{2}}}{2}-1\right)$
92. $\ln\left(\frac{n^n}{n!}\right)$
93. $96/5$
94. $2021/506$
95. $0.1$
96. $\pi^2/12$
97. 04
98. 01
99. $1/2$
100. 2027

Note: For further updated pdf after corrections do check the blog AKIII CEOs .