Inversion on the Fly

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When I tried to learn inversion, I found myself hopelessly stuck trying to muck through difficult definitions and not finding concrete examples until much later. Although the concept may be considered to be nontrivial by some, I find the idea quite natural. Instead of throwing a flurry of definitions at the reader and expecting him or her to imbibe all of this knowledge, we instead present a series of problems while introducing the ideas behind inversion and the tools it provides "on the fly".

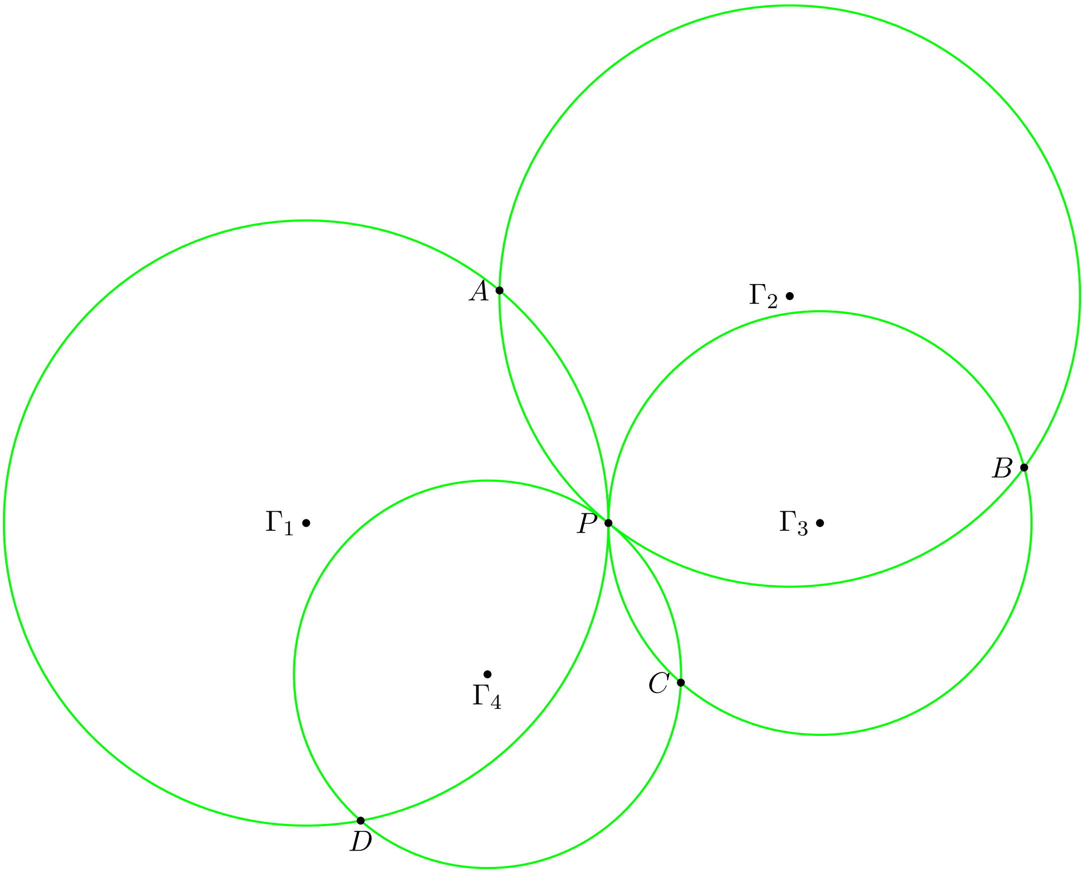
Contents

1 Power of Inversion

Let's begin!

Example 1 (IMO Shortlist 2003/G4)**.** Let Γ_1, Γ_2, Γ_3, Γ_4 be distinct circles such that Γ_1, Γ_3 are externally tangent at P, and Γ_2, Γ_4 are externally tangent at the same point P. Suppose that Γ_1 and Γ_2, Γ_2 and Γ_3, Γ_3 and Γ_4, Γ_4 and Γ_1 meet at A, B, C, D, respectively, and that all these points are different from P. Prove that

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$



First of all, there are four circles in this problem, which is not a good sign, as circles are generally hard to deal with. However, they all pass through the same point, which is suspicious. This motivates an inversion, as we will soon see.

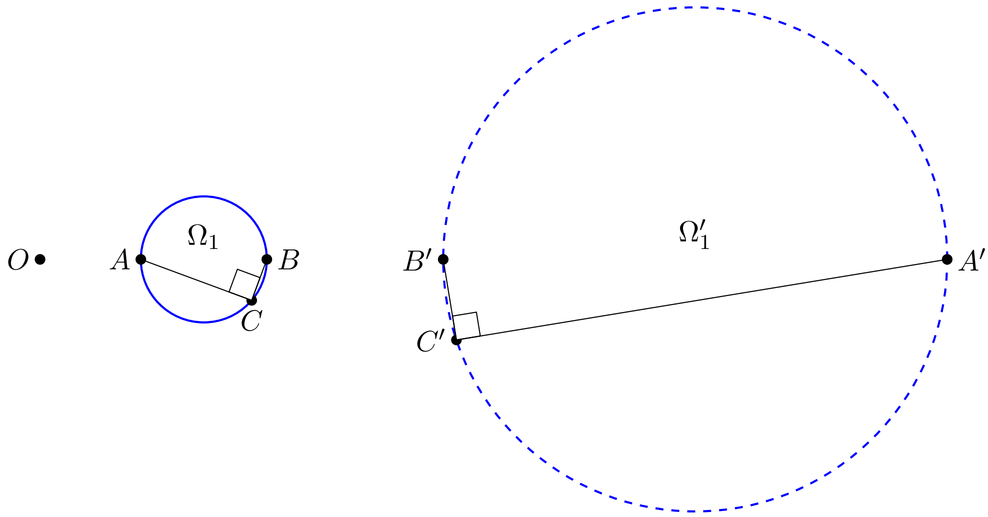
Definition 1.1. Let Ω be a circle centered at O with radius r. Then for any point P in the plane of Ω, we define its **inverse** P' as the point on ray OP such that $OP \cdot OP' = r^2$. Moreover, the inverse of P' is P.

Wait, what the heck? This feels unnatural. Why should we care about such a random transformation? Let's take a quick "local and global" perspective on the matter.

Before we continue, we do have to deal with a perhaps obvious question: what happens to O under inversion about it? This is problematic, so we extend the Euclidean plane to have a *point at infinity* (really, really far away from everything else) which O will map to. If we look at the metric relation which defines inversion, we get that $\frac{r^2}{\infty} = 0$ and $\frac{r^2}{0} = \infty$, which intuitively makes sense. Although this is somewhat nonsensical, the intuition holds.

Proposition 1.1. Circles in the plane of Ω invert to other circles.

Sketch. Let Ω_1 be a circle in the plane of Ω, we claim that the inverses of the two points in the diameter of Ω_1 that passes through O (let's call them A and B) map to points A' and B' which are a diameter of Ω_1'.



Now we angle chase. Take any point C on Ω_1; note that $\angle ACB = \angle OCB - \angle OCA$. But since $OA \cdot OA' = OC \cdot OC'$, triangles OAC and $OC'A'$ are similar. Similarly (no pun intended), triangles OBC and $OC'B'$ are similar. That means that

$$\angle ACB = \angle OCB - \angle OCA = \angle OB'C' - \angle OA'C'$$

$$= 180 - \angle A'B'C' - \angle B'A'C' = \angle B'C'A' = 90^\circ$$

so the conclusion is clear! □

Remark. Note that the circle that we inverted about is invisible in the above diagram, but it doesn't really matter since we can pick any radius and the resulting figures are just off by some scale (in the code for the above diagram, I picked $r = 1$).

Exercise 1.1. Check that the centers of Ω and Ω_1 are not, in fact, inverses.

What happens when we take a circle through O? Surprisingly, it's a line! This is intuitive because the circle gets inverted to a circle with "infinite radius" (because the point at infinity lies on the "circle", but it's really, really far away) which is clearly a line. If you don't believe me, try drawing a really big circle and fixing a reference frame at the bottom; the more and more you stretch it and your view remain fixed on that part of the circle, the more it starts to look like a line.

Now we see the beauty of inversion; it lets us get rid of difficult circles and turn them into lines, which are usually easier to deal with.

Exercise 1.2. Let Ω be a circle through O with center P. Prove that the image of Ω is perpendicular to OP.

Okay, now we might be ready to invert.

Solution. Seeing the circles all passing through P, we invert about P with radius 1 (I picked the radius 1 arbitrarily; again, it doesn't really affect anything). So what happens?

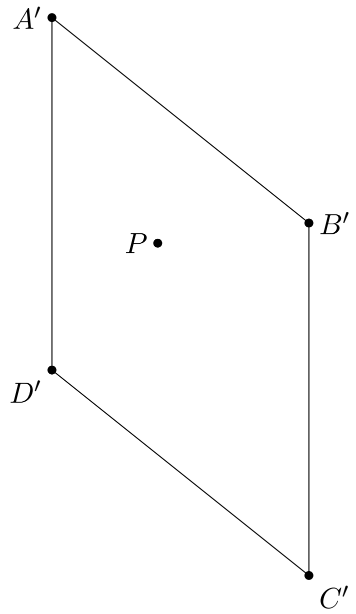
1. Γ_1 and Γ_3 were circles passing through P, so they become lines passing through the point at infinity, which makes them parallel.

2. Similarly, Γ_2 and Γ_4 invert to parallel lines.

3. A was the intersection of Γ_1 and Γ_2, so A' is the intersection of lines Γ_1' and Γ_2'.

4. Define B', C', and D' similarly.

Wait a minute. If A', B', C' and D' form intersections of two sets of parallel lines, they must form a parallelogram!



Okay, now what? Since the problem gives us a metric relation, we seek some values. Off the bat, it's obvious that $A'B' = C'D'$ and that $A'D' = B'C'$. But how do we actually calculate these values? Recall our observation of similar triangles earlier. Here, we have $\triangle PAB \sim \triangle PB'A'$, so that

$$\frac{A'B'}{AB} = \frac{PB'}{PA}$$

$$\implies A'B' = \frac{PB' \cdot AB}{PA} = \frac{PB \cdot PB' \cdot AB}{PA \cdot PB} = \frac{r^2 \cdot AB}{PA \cdot PB}$$

but here $r = 1$, so we will ignore the factor of r^2. Great! Now we repeatedly apply this derivation to obtain

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PA \cdot PB \cdot A'B' \cdot PB \cdot PC \cdot B'C'}{PA \cdot PD \cdot C'D' \cdot PD \cdot PC \cdot A'D'}$$

But since $A'B' = C'D'$ and $A'D' = B'C'$, everything cancels and we are left with $\dfrac{PB^2}{PD^2}$, so we're done. $\square$

That's a lot of information. To summarize:

Definition 1.2 (Inversion)**.** An *inversion* about a circle Ω centered at O with radius r is a transformation of the plane which:

1. Sends any point P to the unique point P' along ray OP such that $OP \cdot OP' = r^2$ (note that this fixes all points on Ω).
2. Sends P' to P.
3. Sends O to the point at infinity.

Circles in the plane of Ω invert to other circles, in particular:

1. Circles through O invert to lines not through O and vice versa.
2. Circles not through O invert to different circles not through O.

Finally, for any two inverted points A', B',

$$A'B' = \frac{r^2 \cdot AB}{OA \cdot OB}$$

Exercise 1.3. Show that lines through the center of inversion invert to themselves.

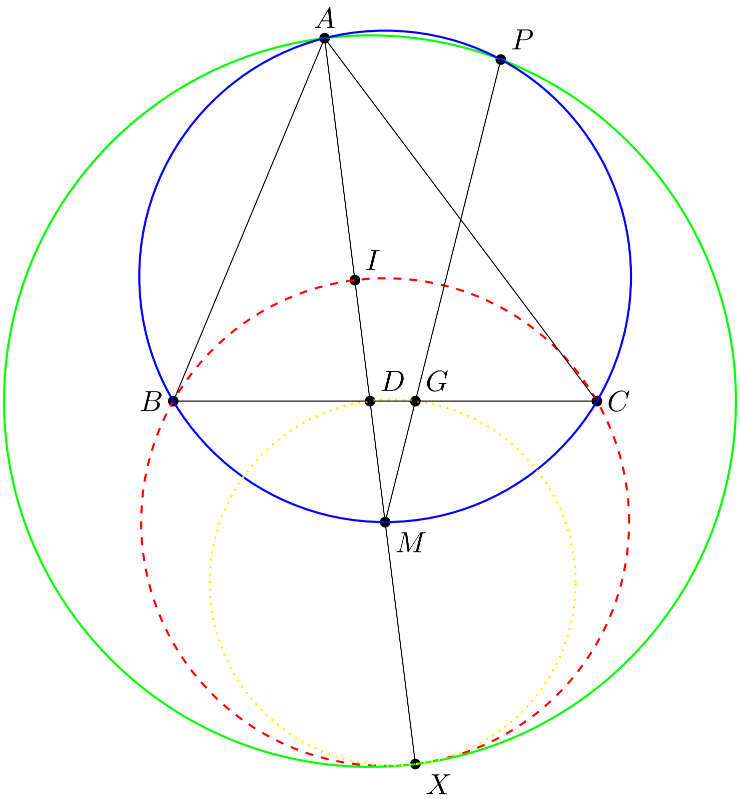
Question 1.1. What's the relationship between A, B, B', and A'?

2 Inversive Overlay

In the last example, we inverted and completely ignored the original diagram. Often, problems will invert to themselves or to similar configurations, with an appropriate choice of radius.

2.1 $\sqrt{bc}$ Inversion

Example 2 (Not-USAMO Mock Olympiad 2015-2016/5, Shortened)**.** Let ABC be a triangle where X is the A-excenter and I is the incenter. Let M be the circumcenter of $\triangle BIC$, and let G be the point on BC such that $XG \perp BC$. Construct a circle Ω with diameter AX. If Ω and the circumcircle of $\triangle ABC$ intersect at a second point P, prove that M, G, and P are collinear.



Solution. We want to try to invert this problem; it's "easy" in the sense that the desired conclusion is a collinearity, which doesn't have much to do with circles at first glance. From what I've seen, inverting about A or P isn't really that helpful, because we lose properties of the problem.

Well, what about inverting about (BIC)? It's well-known that $BICX$ is cyclic, and that M is the midpoint of minor arc BC in the circumcircle of $\triangle ABC$. Furthermore,

A, I, M and X are collinear, and line GP should pass through this point, so these lines stay put.

Let's take this step by step.

1. The circumcircle of ABC passes through M, so it inverts to a line. Since B and C are fixed by this inversion, the line BC is precisely our desired image.

2. Since A lies on the circumcircle of $\triangle ABC$, A' lies on BC. Since it also lies on line AM, A' is the intersection of AM with BC, which is the foot of the angle bisector, D.

3. The circle with diameter $\overline{AX}$ inverts to another circle. Since A inverts to D, and X is fixed by this inversion, this circle must be the circle with diameter $\overline{DX}$.

4. P' is the intersection of the circle with diameter $\overline{DX}$ and BC.

We want to show that P' lies on MG. Since $\overline{DX}$ is a diameter, we have that $\angle DP'X = 90°$. But $\angle DGX = 90°$, so in fact, $P' \equiv G$, and the conclusion is obvious!

□

This problem demonstrates a specific version of a more general trick.

Definition 2.1 ($\sqrt{bc}$ Inversion)**.** We define a $\sqrt{bc}$ inversion by inverting about A with radius $\sqrt{AB \cdot AC}$ in $\triangle ABC$ and reflecting the resulting image over the A-angle bisector. In particular, this transformation swaps B and C, so that $\triangle ABC$ is fixed.

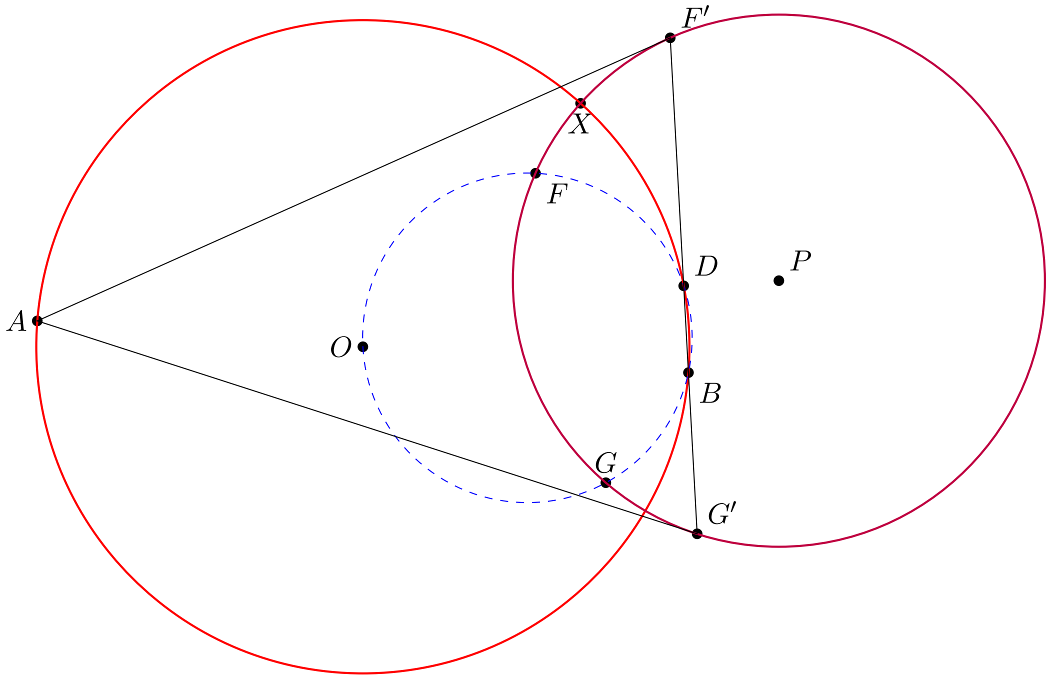
In the problem above, we did a $\sqrt{bc}$ inversion about M, but the reflection over the angle bisector was unnecessary, since $MB = MC$.

Question 2.1. What happens to the circumcircle of $\triangle ABC$? What about the incenter and A-excenter?

2.2 Orthogonal Circles

Example 3 (ELMO Shortlist 2013/G5)**.** Let ω_1 and ω_2 be two circles centered at O and P, and suppose they intersect at two points X and Y such that $\angle OXP = \angle OYP = 90°$. Diameter AB of ω_1 is selected so that B lies strictly inside ω_2. The two circles tangent to ω_2, passing through O and A, touch ω_2 at F and G. Prove that $FGOB$ is cyclic.

Solution. Pick a point on ω_2 (call it R). What happens when we invert about ω_1? R gets sent to the point on ray OR such that $OR \cdot OR' = r_1^2 = OX^2$, so R' lies on ω_2 by Power of a Point. Since we picked R arbitrarily, that means ω_2 inverts to itself with respect to ω_1. Now let's apply this inversion to the whole diagram.



1. The circle (AFO) passes through O, so it inverts to a line. Since this circle only intersects ω_2 at one point, its image only intersects ω_2 at one point as well; i.e., the image is one of the tangents of ω_2 through A (this is important: inversion preserves tangency!). F' is the resulting point of tangency.

2. Define G' in the same manner. The circle (FGO) inverts to a line; since F' and G' lie on this circle, the desired image is the line $F'G'$.

B inverts to itself, so we want to show that F', G', and B are collinear. This is easy using projective geometry, but outside the scope of this handout. Trying to angle chase, we let D be the midpoint of $\overline{F'G'}$, so that $\angle ADG' = 90°$, since AD is the perpendicular bisector of $\overline{F'G'}$. So if D, B, and G' were collinear, then $\angle ADB = 90°$ as well, which would mean that D lies on ω_1. Let's invert again, but this time around ω_2; we want to show that D' lies on ω_1. Since $F'G'$ inverts to (FGP), and A lies on this circle (because $\angle AFP = \angle AGP = 90°$), we derive that $D' \equiv A$, from which the conclusion readily follows. □

Remark. ω_1 and ω_2 are called **orthogonal circles**; that is to say, their intersections form right angles with the centers. It is important to know that both circles remain fixed when inverted about the other.

Exercise 2.1. Check that the individual points on ω_2 do not remain fixed, even though the entire circle does.

This problem demonstrates the construction of an inverse with respect to a circle.

Question 2.2. Let Ω be a circle with chord AB. Let the tangents to Ω from A and B intersect at P. What is P'?

Question 2.3. Let a and b be two complex numbers on the unit circle. If the tangents from a and b intersect at p, what is the inverse of p with respect to the unit circle? For a general complex number z, what is its inverse with respect to the unit circle?

3 Problems

Inversion may not be the main idea in all of these problems, but it can help to prove smaller results which as a whole comprise a problem.

3.1 Easier Problems

Problem 1. Prove that two circles cannot intersect at more than two points.

Problem 2 (Appolonius)**.** Let P' be the inverse of P with respect to a circle Ω. Show that for any point A on Ω, $\dfrac{AP'}{AP}$ holds constant. What is the locus of points K in $\triangle ABC$ such that $\dfrac{AB}{AC} = \dfrac{KB}{KC}$?

Problem 3 (USAMO 1993/2)**.** Let $ABCD$ be a convex quadrilateral such that diagonals AC and BD intersect at right angles, and let E be their intersection. Prove that the reflections of E across AB, BC, CD, DA are concyclic.

Problem 4. Prove Ptolemy's Theorem, which states that for a cyclic quadrilateral $ABCD$,

$$AB \cdot CD + AD \cdot BC = AC \cdot BD$$

Generalize this statement to all quadrilaterals.

Problem 5. Let H be the orthocenter of $\triangle ABC$ and D be the foot of the altitude from A onto BC. Identify the images of all of the points when an inversion about A with radius $\sqrt{AH \cdot AD}$ is performed.

1. (NUSAMO 2015-2016/1). In acute triangle ABC, M is the midpoint of $\overline{AB}$, O is the circumcenter, and H is the orthocenter. Let O_c be the reflection of O over line AB, and the circle centered at O_c through B and the circle with diameter $\overline{HM}$ intersect at two points. Prove that one of these points lies on the C-median.

2. (AOPS). Let ABC be a triangle with orthocenter H and let D, E, F be the feet of the altitudes lying on the sides BC, CA, AB respectively. Let $T = EF \cap BC$. Prove that TH is perpendicular to the A-median of triangle ABC.

Problem 6 (USAMO 1995)**.** Given a nonisosceles, nonright triangle ABC, let O denote the center of its circumscribed circle, and let A_1, B_1, and C_1 be the midpoints of sides BC, CA, and AB, respectively. Point A_2 is located on the ray OA_1 so that OAA_1 is similar to OA_2A. Points B_2 and C_2 on rays OB_1 and OC_1, respectively, are defined similarly. Prove that lines AA_2, BB_2, and CC_2 are concurrent, i.e. these three lines intersect at a point.

3.2 Harder Problems

Problem 7 (IMO 1996/2)**.** Let P be a point inside a triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incenters of triangles APB, APC, respectively. Show that the lines AP, BD, CE meet at a point.

Problem 8. Let A, B and C be three collinear points and suppose P is any point in the plane. Prove that the circumcenters of triangles PAB, PAC, PBC, and P are concyclic.

Problem 9 (IMO 2010/2)**.** Given a triangle ABC, with I as its incenter and Γ as its circumcircle, AI intersects Γ again at D. Let E be a point on the arc BDC, and F a point on the segment BC, such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. If G is the midpoint of IF, prove that the meeting point of the lines EI and DG lies on Γ.

Problem 10 (ISL 2011/G4)**.** Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G and X are collinear.

Problem 11 (EGMO 2013/2)**.** Let Ω be the circumcircle of the triangle ABC. The circle ω is tangent to the sides AC and BC, and it is internally tangent to the circle Ω at the point P. A line parallel to AB intersecting the interior of triangle ABC is tangent to ω at Q. Prove that $\angle ACP = \angle QCB$.

Problem 12 (ELMO 2010/6)**.** Let ABC be a triangle with circumcircle ω, incenter I, and A-excenter I_A. Let the incircle and the A-excircle hit BC at D and E, respectively, and let M be the midpoint of arc BC without A. Consider the circle tangent to BC at D and arc BAC at T. If TI intersects ω again at S, prove that SI_A and ME meet on ω.

Problem 13 (ISL 2002/G7)**.** The incircle Ω of the acute-angled triangle ABC is tangent to its side BC at a point K. Let AD be an altitude of triangle ABC, and let M be the midpoint of the segment AD. If N is the common point of the circle Ω and the line KM (distinct from K), then prove that the incircle Ω and the circumcircle of triangle BCN are tangent to each other at the point N.

Problem 14 (IMO 2015/3)**.** Let ABC be an acute triangle with $AB > AC$. Let Γ be its cirumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA = 90^\circ$ and let K be the point on Γ such that $\angle HKQ = 90^\circ$. Assume that the points A, B, C, K and Q are all different and lie on Γ in this order.

Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

Problem 15 (USAMO 2015/2)**.** Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$. As X varies on segment $\overline{PQ}$, show that M moves along a circle.

3.3 Hints

Hint 1. Assume there exists two circles which intersect at three points, then invert.

Hint 2. Use the distance formula we derived in **Example 1**.

Hint 3. Same as previous hint.

Hint 4. The quadrilateral formed by the reflections inverts to a rectangle.

Hint 5. B and C invert to the feet of the altitudes; H and D swap.

Hint 6. What is the relationship between this and **Question 2.2**?

Hint 7. Angle bisector theorem on $\triangle ABP$ and $\triangle ACP$. Invert about A.

Hint 8. Invert about P with radius 1, then look at the P-Simson line of triangle $A'B'C'$.

Hint 9. Show that $AE \cdot AF = AB \cdot AC$. Angle chase.

Hint 10. The second intersection of line DG with the circumcircle is the reflection of A about the perpendicular bisector of BC; $\sqrt{bc}$ inversion.

Hint 11. Let the tangency point on AC be X; do a $\sqrt{bc}$ inversion about C with radius $\overline{CX}$.

Hint 12. S is the antipode of A in ω; invert about M with radius MB.

Hint 13. Let T be the midpoint of minor arc BC; prove that T has equal power with respect to B and ω. Invert about T.

Hint 14. Perform an inversion about H with radius $-\sqrt{HA \cdot HF}$ (wait, what?).

Hint 15. Let the projection from A onto ST be F; show that $MPQF$ is cyclic by inverting about A with radius $\overline{AP}$ and then inverting about M with radius $\overline{MT}$.