The Method of Moving Points

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Introduction

Here is the list of things one should know about while using the method we will study in this paper:

- Cross ratio on lines, conics, pencils of lines.
- Inversion wrt. a circle.
- Basic projective transformations of a plane.
- Polar transformation. Duality principle.
- Algebraic and geometric definitions of a conic.

There are a lot of papers including information about every of these items. For example, read [1] and [2].

1 One side of the MMP: the coinciding maps

For the next definition, $\mathcal{F}$ is the set of objects on which the cross ratio is defined (for example, lines, conics, pencils of lines).

Definition 1.0.1. For $\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{F}$ we say a function $f : \mathcal{C}_1 \to \mathcal{C}_2$ is a *projective map* if it preserves cross-ratios. In other words, if $A, B, C, D \in \mathcal{C}_1$, then

$$(A, B; C, D) = (f(A), f(B); f(C), f(D)).$$

Excercise 1.1. Check that every projective map is bijective.

Here are some examples of projective maps:

- Given a line l and a point P, the map from l to $\mathcal{C}_P$ (the pencil of lines through P) given by $X \mapsto PX$.

- Given a conic γ and a point P on the conic, the map from ℓ to $\mathcal{C}_P$ given by $X \mapsto PX$.
- Given a conic γ and any point P, the map from γ to γ by $X \mapsto PX \cap \gamma \neq X$.
- An inversion of a circle or a line.

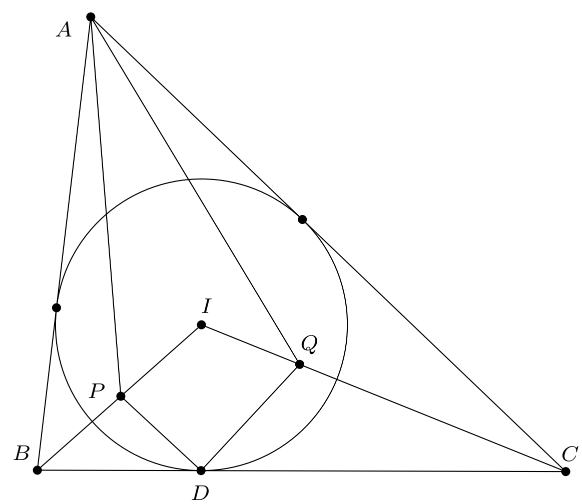
The following theorem is the main result we will use in this section.

Theorem 1.2. *Let $f : \mathcal{C}_1 \to C_2$ and $g : \mathcal{C}_1 \to C_2$ be two projective maps. Then, $f \equiv g$ if and only if $f = g$ for three distinct points on $\mathcal{C}_1$.*

Proof. The necessity is obvious. For the sufficiency, let A_1, A_2, A_3 be three distinct points on $\mathcal{C}_1$ such that $f(A_i) = g(A_i) = B_i$ for all $i = \overline{1,3}$. Then, since for any $D \in \mathcal{C}_1 \setminus \{A_1, A_2, A_3\}$ there exists a unique point X on $\mathcal{C}_2$ such that $(B_1, B_2; B_3, X) = (A_1, A_2; A_3, B)$, it follows that $X = f(B) = g(B)$, as desired. □

Now, it's time to solve some problems with the theory we've just learnt!

Example 1.1 (Serbia MO 2018). Let $\triangle ABC$ be a triangle with incenter I. Points P and Q are chosen on segments BI and CI such that $2\angle PAQ = \angle BAC$. If D is the touch point of incircle and side BC, prove that $\angle PDQ = 90°$.



Solution. Let $\angle BAC = 2\alpha$. Consider the map $f : BI \to CI$ sending a point $X \in BI$ to $Y \in CI$ such that $\angle XAY = \alpha$. Note that f is projective since $X \mapsto AX \mapsto AY \mapsto Y$ is projective since $AX \mapsto AY$ is a rotation by the fixed angle α. Similarly, the map $g : BI \to CI$ which sends a point $X \in BI$ to $Y \in CI$ such that $\angle XDY = 90°$, is also projective. Our goal is to prove that $f \equiv g$, which will imply that P is sent to Q by both maps, hence $\angle PDQ = 90°$. It follows from the theorem 1.2 that it's enough to check 3 cases:

- $P = B$. Then both f and g send P to the same point $Q = I$;
- $P = I$. Then both f and g send P to the same point $Q = C$;

- Let P and Q be the incenters of $\triangle ABD$ and $\triangle ACD$, respectively. Note that

$$\angle PAQ = \angle PAI + \angle IAQ = \frac{\angle BAI}{2} + \frac{\angle IAC}{2} = \frac{\angle BAC}{2} = \alpha,$$

  thus f maps P to Q. Similarly,

$$\angle PDQ = \angle PDI + \angle IDQ = \frac{\angle BDI}{2} + \frac{\angle IDC}{2} = 90^\circ,$$

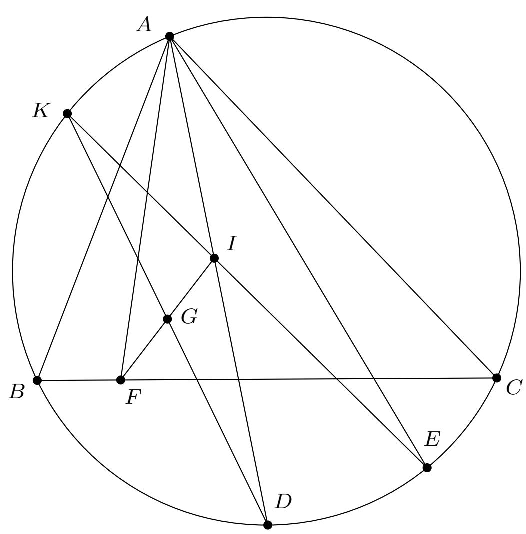
  hence g also maps P to Q, as required.

□

So, why the Method of Moving Points? Note that sketch of the previous solution can be rewritten in the following manner: "Let P be moving on the line BI. f sends it to Q_1 such that $\angle PAF = \alpha$, and g sends it to Q_2 such that $\angle PDQ = 90^\circ$. Since f and g are projective, it's enough to check three points to show that they are the same..." So, the main idea of MMP is to start moving a single point along a line/conic and check how do other parts of the diagram change. Sometimes you should try different variants of the moving point to make observations easier (the correspondence between points seems to be more difficult to investigate if moving, for example, point D).

Look again at the solution for the previous example. Note that we considered points P, Q moving on segments BI, CI, but we defined projective maps from line to line. If we add more strictness and say that the maps f and g save oriented angles $\angle(PA, AQ) = \alpha$ and $\angle(PD, DQ) = 90^\circ$, they will be projective along the whole lines BI, CI, so the problem is true for any point P on the line BI. Some problems which can be done by MMP have a generalisation by this method. Let's consider another example showing this:

Example 1.2 (IMO 2010/2). Given a triangle ABC, with I as its incenter and Γ as its circumcircle, AI intersects Γ again at D. Let E be a point on the arc BDC, and F a point on the segment BC, such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. If G is the midpoint of IF, prove that the meeting point of the lines EI and DG lies on Γ.



Solution. We will move E on Γ, which means that we have mappings generated by varying E. Let $K_1 = EI \cap \Gamma$ and $K_2 = DG \cap \Gamma$. We have that $f : E \mapsto K_1$ is projective as it's a simple projection from Γ to itself wrt. I.

We'll now show that $g : E \mapsto K_2$ is projective. Note that $E \mapsto AE \mapsto AF \mapsto F$ since $AE \mapsto AF$ is a reflection over AD which clearly preserves cross ratio. Also, the locus of G is homothetic to the locus of F wrt. I with coefficient $1/2$, thus $F \mapsto G$ is projective. Projecting through $D \in \Gamma$, we have that $G \mapsto K_2$ is projective, so we get that $g : E \mapsto AE \mapsto AF \mapsto F \mapsto G \mapsto K_2$ is projective, as desired.

Therefore, to show $K_1 = K_2$, it suffices to check three values of E.

- $E = C$. Here K_1 is the midpoint of the arc AB. By incenter lemma, $DI = DB$ and $K_1I = K_1B$, hence DK_1 passes through the midpoint G of BI, which implies $K_1 = K_2$.

- $E = B$. Similar to the previous case, but stop! The problem statement says that $\angle CAE < < \frac{1}{2}\angle BAC$, so you cannot consider $E = B$! We may try to consider other points, but let's look again at f and g. They will stay projective if E is moving along the whole circle Γ, and since we know that they must coincide on the arc DC, by theorem 1.2 they must coincide on Γ. Thus, we extend the locus of E to Γ and prove the more general version of the problem. Similarly to the previous item, we check $E = B$.

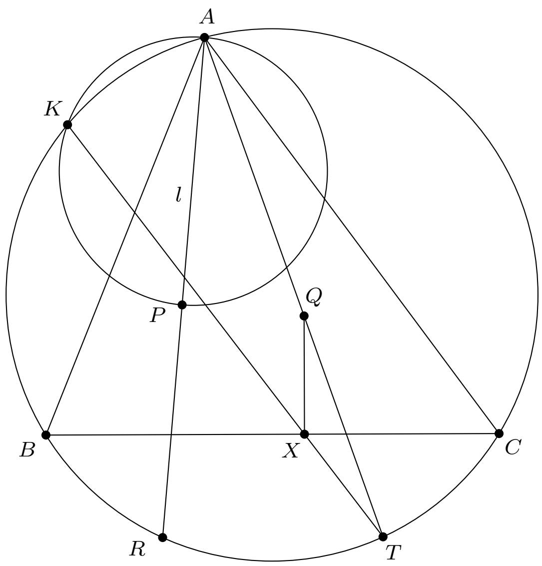
- $E = D$. Then $K_1 = A$, $F = AD \cap BC \Rightarrow G \in AD \Rightarrow K_2 = A = K_1$.

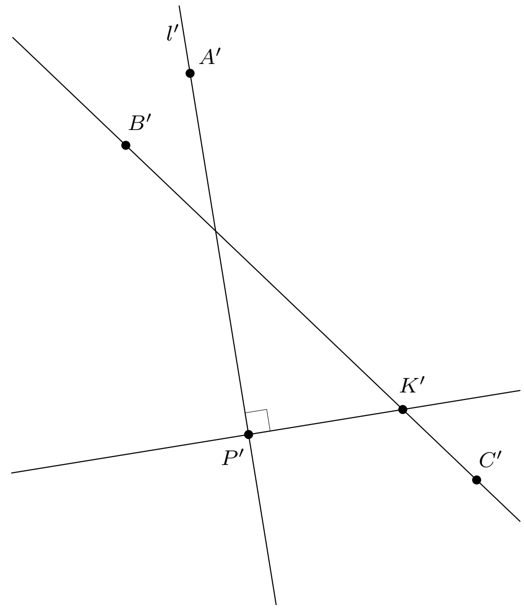
□

Note that this is *the process* of finding a solution described above. While being at the contest, for clearer solution one should write the generalised version of the problem first, and then to prove it.

The next example shows how inversion can be useful in proving that some map is projective. This transformation can easily deal with moving circles, and it's usually good to keep in mind the technique of applying it.

Example 1.3. Let P, Q be isogonal conjugate points wrt. $\triangle ABC$ (i.e. pairs of lines (AP, AQ), (BP, BQ), (CP, CQ) are symmetric to each other wrt. angle bisectors of $\angle BAC$, $\angle ABC$, $\angle ACB$, respectively). Point X is the orthogonal projection of Q on BC. The circle with diameter PA intersects $\odot(ABC)$ at $K \neq A$. AQ intersects $\odot(ABC)$ at $T \neq A$ Prove that the points K, X, T are collinear.





Solution. Let Q be moving along the line AT. Firstly, $Q \mapsto X \mapsto K_1$ is projective, where $K_1 = TX \cap \odot(ABC)$. Let l be the line AP which stays fixed as Q varies. Then, P is moving along l, and $Q \mapsto CQ \mapsto CP \mapsto P$ is projective since $CQ \mapsto CP$ is a symmetry wrt. the bisector of $\angle ACB$. We now need to show that $P \mapsto K_2$ is projective, were K is defined as the intersection of the circle with diameter PA and $\odot(ABC)$. Consider inversion at A. The circumcircle of ABC is sent to the line $B'C'$, and the circle with diameter AP is sent to the line $P'K_2'$ perpendicular to AP'. Here point P' is moving along the fixed line l, and since $\angle AP'K_2' = 90^\circ$, the map $P \mapsto$ is clearly projective, and since inversion preserves cross ratios, we've got that $P \mapsto K_2$ is projective in the initial problem. Thus, it's left to check three positions of point Q:

- $Q = T$. Then K_1 is such that $TK_1 \perp BC$. Let $R = l \cap \odot(ABC)$, and since $TR \parallel BC$,

$$\angle(PC, CB) = \angle(AC, CT) = \angle(AR, RT) = \angle(AR, BC),$$

  which means that $P = \infty_l$. Thus, $AK_2 \perp l$ as the limit case of circle with diameter AP. Finally, $90^\circ = \angle(BC, TK_1) = \angle(RT, TK_1) = \angle(RA, AK_1)$, which implies $K_1 = K_2$, as required.

- $Q = \infty_{AT}$. Similarly to the previous case, we obtain that $P = R$. Then $K_2 = R$, $X = \infty_{BC}$, $K_1 = T\infty_{BC} \cap \odot(ABC) = R = K_2$.

- $Q = BC \cap AT$. Then, $X = Q \Rightarrow K_1 = A$. Then, since CB and CA are symmetric wrt. the bisector of $\angle ACB$, $P = A \Rightarrow K_2 = A = K_1$, as desired.

□

1.1 Problems for practice

1. Let ABC be a triangle with circumcircle (O). The tangent to (O) at A intersects the line BC at P. E is an arbitrary point on the line PO, and $D \in BE$ is such that $AD \perp AB$. Prove that $\angle EAB = \angle ACD$.

2. Let ΔABC with circumcircle (O) and incircle (I). X is an arbitrary point on BC. The line through I perpendicular to IX cuts the tangent of (I) which parallel to BC at Y. AY cuts (O) again at Z. T is the tangent point of A-Mixtilinear incircle with (O). Prove that X, Z, T are collinear.

3. Let AB be a diameter of circle ω. ℓ is the tangent line to ω at B. Take two points C, D on ℓ such that B is between C and D. E, F are the intersections of ω and AC, AD, respectively, and G, H are the intersections of ω and CF, DE, respectively. Prove that $AH = AG$.

4. Let (O) be a circle and l a line. The perpendicular through O to l intersects Γ at A and B. Let P_1, P_2 be two point on Γ, and $PA \cap l = X_1$, $PB \cap l = X_2$, $QA \cap l = Y_1$, $QB \cap l = Y_2$. Prove that the circumcircles of $\triangle AX_1Y_1$ and $\triangle AX_2Y_2$ intersect on Γ.

5. In $\triangle ABC$ $\angle B$ is obtuse and $AB \neq BC$. Let O is the circumcenter and ω is the circumcircle of this triangle. N is the midpoint of arc ABC. The circumcircle of $\triangle BON$ intersects AC on points X and Y. Let $BX \cap \omega = P \neq B$ and $BY \cap \omega = Q \neq B$. Prove that P, Q and reflection of N with respect to line AC are collinear.

6. Given a $\triangle ABC$ and two isogonal points P and Q. Points D, E and F are intersections of lines AP, BP and CP with sides BC, AC, and AB, respectively. Let O be the circumcenter of $\triangle ABC$. Let X be the intersection of the line perpendicular to EF through A and the line OD. Prove that $QX \perp BC$.

2 Other side of the MMP: the Steiner Conic

In this section, we will study the projective maps between two lines or between two pencils of lines. We start from a very important theorem.

Theorem 2.1 (Steiner conic)**.** *Let $f : \mathcal{C}_A \to \mathcal{C}_B$ be a projective map between two pencils of lines $\mathcal{C}_A$ and $\mathcal{C}_B$. Then*

1. *$f(AB) = AB$ if and only if there exists a line $\mathcal{C}$ such that $l \cap f(l) \in \mathcal{C}$ for every $l \in \mathcal{C}_A \backslash \{BC\}$;*

2. *$f(AB) \neq AB$ if and only if there exists a conic $\mathcal{C}$ passing through A, B such that $l \cap f(l) \in \mathcal{C}$ for every $l \in \mathcal{C}_A$.*

Making the polar transformation, we obtain the dual of theorem 2.1:

Theorem 2.2 (Dual to Steiner conic)**.** *Let $f : a \to b$ be a projective map between two lines a and b. Then*

1. *$f(a \cap b) = a \cap b$ if and only if there exists a point C such that $Df(D)$ pass through C for every $D \in a \setminus \{a \cap b\}$;*

2. *$f(a \cap b) \neq a \cap b$ if and only if there exists a conic $\mathcal{C}$ tangent to a, b such that $Df(D)$ is tangent to $\mathcal{C}$ for every $D \in a$.*

Proof of theorem 2.1. Clearly, the maps in both cases are projective since $l \mapsto l \cap \mathcal{C} = f(l) \cap \mathcal{C} \mapsto f(l)$ is projective. Also note that the sufficiency is obvious, so let's prove the necessity.

1. Let $f(AB) = AB$, and consider three distinct lines $l_1, l_2, l_3 \in \mathcal{C}_A \setminus \{BC\}$. Let $D_i = l_i \cap f(l_i)$ for every $i = \overline{1,3}$, $\mathcal{C}$ be a line through A_1 and A_2, and $P = \mathcal{C} \cap AB$. Then,

$$(D_1, D_2; D_3, P) = (AD_1, AD_2; AD_3, AB) = (BD_1, BD_2; BD_3; AB) = (D_1, D_2; BD_3 \cap \mathcal{C}, P),$$

which implies $D_3 = BD_3 \cap \mathcal{C}$, which is equivalent to $D_4 \in \mathcal{C}$, as desired.

2. Let $f(AB) \neq AB$, and consider four distinct lines $l_1, l_2, l_3, l_4 \in \mathcal{C}_A \setminus \{BC\}$. Let $D_i = = l_i \cap f(l_i)$ for every $i = \overline{1,4}$. It's well-known that there exists a unique conic $\mathcal{C}$ passing through A, B, D_1, D_2, D_3 (no three of these points collinear, since the contrary will imply $f(AB) = AB$). Then,

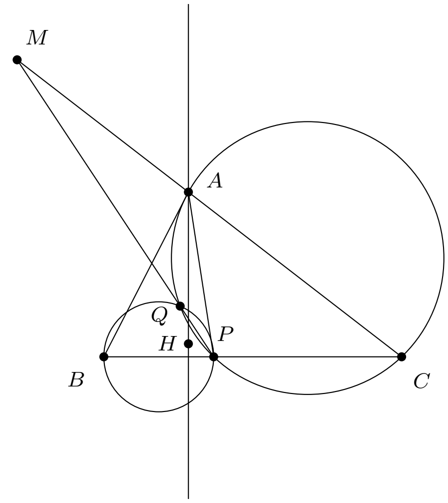
$$\begin{aligned}(D_1, D_2; D_3, D_4) = (AD_1, AD_2; AD_3, AD_4) &= (BD_1, BD_2; BD_3; BD_4) = \\ &= (D_1, D_2; BD_3, BD_4 \cap \mathcal{C}),\end{aligned}$$

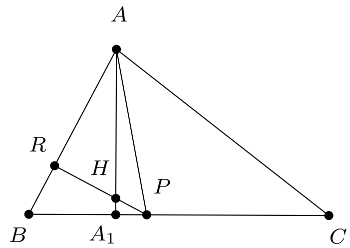
which implies $D_4 = BD_4 \cap \mathcal{C}$, which is equivalent to $D_4 \in \mathcal{C}$, as desired.

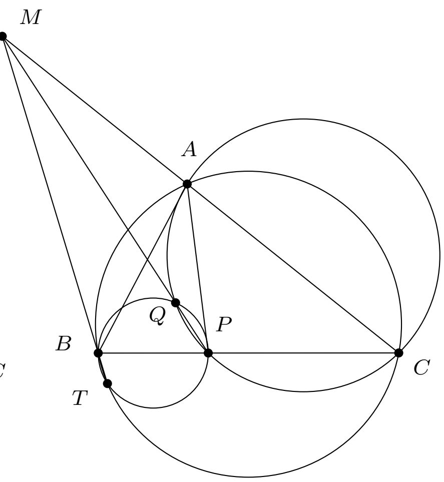
As it was said later, theorem 2.2 is the dual of theorem 2.1, thus it can be also considered to be proven. □

The two theorems proved is a strong tool in proving that the locus of some points is a line of that some set of lines pass through a single point. The next example shows the power of these results.

Example 2.1. Given a $\triangle ABC$ and a point P on the line BC. The circle with diameter BP intersects the circumcircle of APC again at point Q. Let M be the intersection of PQ and AC, and H be the orthocenter of $\triangle ABP$ Prove that when P varies on BC, MH passes through a fixed point.







Solution. Let P be moving along the line BC. First let's deal with orthocenter. Let $A_1 \in BC$ and $R \in AB$ such that $AA_1 \perp BC$ and $PR \perp AB$. Note that AA_1 is fixed and $AA_1 \cap PR = H$, hence $P \mapsto PR \mapsto H$ is projective. So, our plan is the following:

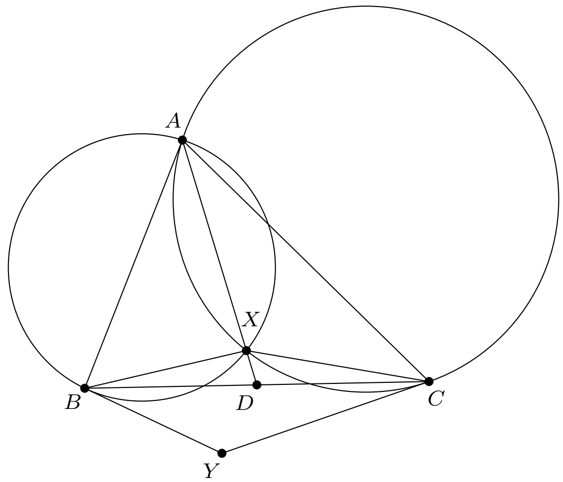
1. Show that $P \mapsto M$ is projective, which will imply $H \mapsto P \mapsto M$ is projective;

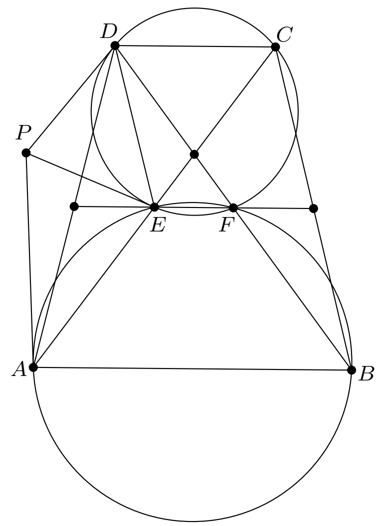
2. Show that $H \mapsto M$ sends $AA_1 \cap AC = A$ to itself, which will imply by theorem 2.2 that HM passes through a fixed point.

Why are we so sure that this plan will work, i.e. $H \mapsto M$ will be projective? Note that if the problem statement is true, then $M \mapsto H$ is simply a projection from AC to AA_1, which is clearly projective. So, our plan MUST work, or the problem is incorrect! So, let's prove that $P \mapsto M$ is projective. Let the circumcircle Γ of triangle ABC intersect the circle with diameter BP again at T. Then, by radical axes theorem BT passes through M. In the solution for Example 1.3 it was shown that $P \mapsto T$ is projective (inversion again!), and $T \mapsto M$ is clearly projective. Thus, $P \mapsto T \mapsto M$ is projective, and the conclusion follows. □

Do you see the beauty of this solution? We don't know ANYTHING about the point all the lines MH pass through, but we proved that such a point exists!

Example 2.2 (RMM 2019/2). Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. Let E be the midpoint of AC. Denote by ω and Ω the circumcircles of the triangles ABE and CDE, respectively. Let P be the crossing point of the tangent to ω at A with the tangent to Ω at D. Prove that PE is tangent to Ω.





Solution. Firstly, we provide the following lemma:

Lemma. Let ABC be a triangle, AD its median, $X \in AD$ and ω_B, ω_C the circumscribed circles of $\triangle AXB$ and $\triangle AXC$ respectively. Draw tangents l_B, l_C to ω_B, ω_C in B, C, respectively, and let $l_B \cap l_C = Y$. Then Y lies on the A-symmedian of the triangle ABC.

Proof. Move X along AD. Note that $X \mapsto BX \mapsto BY$ is projective since $\angle(BY, BX) = \angle(BA, XA) = const$, and $BX \mapsto BY$ is simply a rotation by a fixed angle. Similarly, $X \mapsto CY$ is projective, thus $BY \mapsto CY$ is projective. Since $BC \mapsto BC$ (the radical axis of circles passing through A, B and A, C respectively and both tangent to BC passes through A and D, and hence their intersection lies on AM), the locus of points Y is a line, and it's left to check that for some two positions of point X the point Y lies on the A-symmedian of $\triangle ABC$. Let ω be the circumcircle of $\triangle ABC$, $X_1 = AD \cap \omega$, $X_2 = D$. For $X = X_1$ we have $\omega_B = \omega_C = \omega$, hence Y is the intersection point of tangents through B, C to ω, and it's well-known that it lies on the symmedian. For $X = X_2$ we have:

$$\angle BYC = 180^\circ - \angle YCB - \angle YBC = 180^\circ - \angle YAB - \angle YAC = 180^\circ - \angle CAB,$$

which implies that $Y \in \omega$. Then $\angle DAC = \angle DCY = \angle BCY = \angle BAY$, and hence AY is symmedian. Our lemma is proved. $\square$

Back to the problem. Let F be the midpoint of BD, and because of symmetry ω and Ω pass through it. Applying lemma to these circles and $\triangle AED$ we get that EP is its symmedian, and hence

$$\angle(DE, EP) = \angle(EF, EC) = \angle(DC, CE),$$

which implies that PE is tangent to ω, q.e.d. $\square$

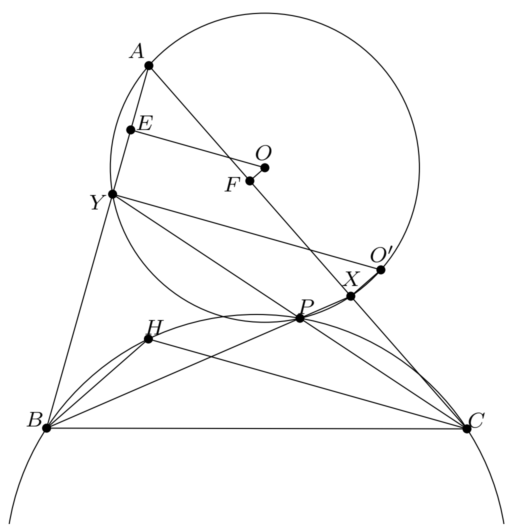
Again, we proved in the lemma that there exists a line all points Y lie on, and then we found this line checking two points. Note that since we know this line should be a symmedian l, we can consider maps $BY \mapsto BY \cap l$, $CY \mapsto CY \cap l$, and then check three points to show that they are the same.

Now, let's provide a useful lemma which will help sometimes in solving problems with maps saving angles.

Lemma 2.3. Given two lines a and b. Suppose that $f : a \to b$ is a projective map such that $f(\infty_a) = \infty_b$. Then, for any fixed α and β, the locus of points X such that $\angle(XA, a) = \alpha$, $\angle(Xf(A), b) = \beta$ for $A \in a$, is a line.

Proof. Let l_α and l_β be two lines such that $\angle(l_\alpha, a) = \alpha$, $\angle(l_\beta, b) = \beta$, and let ∞_α and ∞_β be their points at infinity, respectively. Then, $\infty_\alpha X \mapsto A \mapsto f(A) \mapsto \infty_\beta X$ is projective, and since the line at infinity is mapped to itself by the problem statement, it follows from theorem 2.1 that the locus of points X is a line. $\square$

Example 2.3. Given a $\triangle ABC$ and its orthocenter H. ω is the circumcircle of triangle BHC. We take a point P on ω. Lines BP and CP intersect lines AC and AB at points X and Y respectively. Show that when P moves on ω, circumcircles of quadrilaterals $AXPY$ pass through one point different from A.

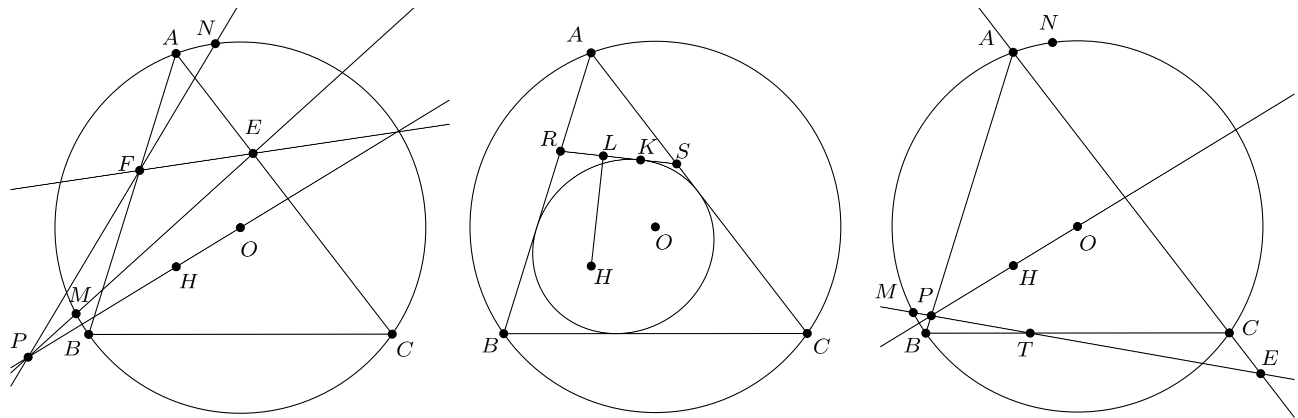


Solution. Let γ be circumcircle of $AXPY$, and O be its circumcenter. Note that the problem statement is equivalent to all points O lie on a fixed line, and the point all γ pass through is a reflection of A over this line. Let E and F be the midpoints of AY, AX, respectively, so that O is the intersection of perpendiculars through E, F to AY, AX, respectively. Making a homothety centered at A with coefficient 2, we obtain $E \mapsto Y$, $F \mapsto X$, $O \mapsto O'$, and it's left to prove that the points O' lie on a fixed line.

While P is moving on ω, we have $X \mapsto P \mapsto Y$ is projective. Since $\angle(AX, XO') = \angle(AY, YO') = 90°$, it follows from lemma 2.3 that we need to prove ∞_{AB} is mapped to ∞_{AC}. Consider P being the point symmetric to A wrt. the midpoint of BC. Clearly, it lies on ω, $BP \parallel AC$, and $CP \parallel AB$, thus $X = \infty_{AC}$, $Y = \infty_{AB}$. Hence ∞_{AB} is indeed mapped to ∞_{AC}, and the conclusion follows. $\square$

The next example shows how Steiner conic can be applied in solving geometry problems.

Example 2.4. Given an acute $\triangle ABC$ and its Euler line k. Points M and N are reflections of B and C with respect to k. P is an arbitrary point on k. Point E is the intersection of lines PM and AC and F is the intersection of lines PN and AB. Let H be the orthocenter of $\triangle ABC$ and S be its reflection with respect to the line EF. Prove that S lies on the circumcircle of $\triangle ABC$.



Solution. Move P on k. We have that $E \mapsto P \mapsto F$ is projective, thus line EF is tangent to a conic $\mathcal{C}$ tangent to AB and AC (A isn't mapped to itself since P, M, A, N cannot lie on a line). Let's check some positions of EF to get more information about that conic. Note that since MN is symmetric to BC wrt. k, for $P = MN \cap BC$ we get that FE coincides with MN. By the same reason, $MC \cap BN \in k$, hence for $P = MC \cap BN$ the line FE coincides with CB. So, we know that $\mathcal{C}$ is tangent to AB, BC, BA, and MN.

Let's think about some conic which can satisfy all the above conditions and the statement of the problem. The first idea is an ellipse $\mathcal{C}'$ with foci H, O inscribed in $\triangle ABC$, which exists since O, H are isogonally conjugate to each other in $\triangle ABC$. Note that it's symmetric to itself wrt. k, hence is tangent to MN.

Let's check the problem statement about this ellipse. Let RS be a tangent to it, $R \in AB$, $S \in AC$ (spoiler: it will happen that $R = F$ and $S = E$ for some $P \in k$), and L be a foot of perpendicular through H on RS. Note that the ellipse $\mathcal{C}'$ is inscribed in both $\triangle ARS$ and $\triangle ABC$, thus H, O are also isogonally conjugate to each other in $\triangle ARS$, and hence the pedal circles of H, O in $\triangle ABC$ and $\triangle ARS$ coincide. This implies L lies on the nine-point circle of $\triangle ABC$, and since it's homothetic to $\odot(ABC)$ wrt. H with coefficient 2, the point symmetric to H wrt. RS indeed lies on $\odot(ABC)$.

So, we just need to prove that $\mathcal{C} = \mathcal{C}'$. Since a conic is uniquely determined by 5 lines it's tangent to, we need to find one more line is tangent to $\mathcal{C}'$. Let $P = k \cap AB$. Then $F = P$, $FE = MP$, and let $T = MP \cap BC$. Note that since M is symmetric to B wrt. k, k is the angle bisector of $\angle APT$. Thus $\angle APH = \angle TPO$, $\angle PAH = \angle CAO$,$\angle ACH = \angle BCO$, hence H, O are isogonally conjugate to each other in $APTC$, which means that PT is also tangent to $\mathcal{C}'$, which means $\mathcal{C} = \mathcal{C}'$, and the conclusion follows. $\square$

As you've seen in this example, sometimes it can be useful to seek for a conic which generates a projective map. I strictly recommend to read [2] and [3] to learn useful facts about conics. Here I'll give some tips that may help you in problems like the previous one:

- Circles tangent to two lines determine a projective map between these lines. It comes especially useful while working with incircles, excircles, mixtilinear circles, etc.
- Ellipse and circle have no points at infinity.
- Parabola is tangent to the line at infinity, so if some of the lines tangent to Steiner conic goes to infinity, then this conic is parabola.
- Hyperbola has two points at infinity, so if you note that there are at least two points at infinity of Steiner conic generated by projective map between two pencils of lines, then this conic is a hyperbola. If you can prove that there is exactly one such point, then it's a parabola.
- The Steiner conic generated by projective map between two lines is a parabola if and only if this projective map is a spiral similarity, the center of which is the focus of the parabola.
- When Steiner conic is an ellipse or a hyperbola, it's good to remember about isogonal conjugation and pedal circles.
- The conic is uniquely determined by:
  - 5 points it passes through, no three of which are collinear;
  - 5 lines it is tangent to, no three of which are concurrent;
  - 4 lines it is tangent to, no three of which are concurrent, and 1 point of tangency;
  - 3 non-concurrent lines it is tangent to and 2 points of tangency on different lines.

2.1 Problems for practice

1. Given an acute triangle ABC such that $AB > AC$. H is the orthocenter of ABC. D is inside $\triangle ABC$ and $DB = DC$. BD, CD meet CA, AB at E, F,respectively. $EF \cap BC = K$, X is the orthocenter of $\triangle DBC$. Prove that $HX \perp AK$.
2. Given two circles ω_1 and ω_2 intersecting each other in two distinct points A and B. Let $P \in \omega_1$ and $Q \in \omega_2$ be points such that $\angle PAB = \angle QAB$. Prove that for every such pair P, Q circles $\odot(APQ)$ pass through a fixed point other than A.
3. Let AA_1 be the bisector of a triangle ABC. Points D and F are chosen on the line BC such that A_1 is the midpoint of the segment DF. A line l, different from BC, passes through A_1 and intersects the lines AB and AC at points B_1 and C_1, respectively. Find the locus of the points of intersection of the lines B_1D and C_1F for all possible positions of l.
4. Given a triangle ABC and a point D on the side AB. Let I be a point inside $\triangle ABC$ on the angle bisector of $\angle ACB$. The second intersections of lines AI and CI with the circumcircle of $\triangle ACD$ are P and Q, respectively. Similarly, the second intersections of lines BI and CI with the circumcircle of $\triangle BCD$ are R and S, respectively. Show that if $P \neq Q$ and $R \neq S$, then the lines AB, PQ, and RS pass through one point or are parallel.

5. Let ABC be a triangle and let P be a point on the plane. Let DEF be the pedal triangle of P wrt ABC. Let X_A be a variable point in BC. Let X_AY_B, X_AY_C be the parallels to DE, DF with $Y_B, Y_C \in AB, AC$ respectively. Then the circumcircles of AY_BY_C passes through a fixed point, which is $\odot(BPC) \cap \odot(AEF)$.

3 The third side of the MMP: the Polynomial Method

In this section, we will look at moving points from absolutely different point of view, but very natural. To understand everything here, i recommend to look at the books presented in References and to be familiar with the previously described methods.

We work in the real projective plane. We say that if a point A has *homogeneous coordinates* $(x : y : z)$, where x, y, z are not all zero, then

- if $z \neq 0$, then A has Cartesian coordinates $\left(\frac{x}{z}, \frac{y}{z}\right)$;
- if $z = 0$, then A is a point on infinity corresponding to a line passing through the origin and a point with Cartesian coordinates (x, y).

Note that $(ap : bp : cp) = (a : b : c)$ for every real nonzero p, so the points $(a_1 : b_1 : c_1)$ and $(a_2 : b_2 : c_2)$ are the same if and only if $(a_1b_2 - b_1a_2, b_1c_2 - c_1b_2) = (0, 0)$.

Similarly, the equation of a line will be $ax + by + cz = 0$ for points $(x : y : z)$ and fixed a, b, c, not all of which are zero. If $a = b = 0$, then this is the line at infinity.

Now, state some well-known theorems about homogeneous coordinates and projective transformations which will be very helpful for us soon.

Theorem 3.1. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two lines or two conics (possibly, coinciding) in the plane, and $f : \mathcal{C}_1 \to \mathcal{C}_2$ be a projective map. Then, there exists a projective transformation of the plane sending $\mathcal{C}_1$ to $\mathcal{C}_2$ such that every point $A \in \mathcal{C}_1$ is sent to $f(A) \in \mathcal{C}_2$.*

Sketch of proof. It's well-known that there exists a projective transformation sending any three points to any three. For lines, take $A, B, C \in \mathcal{C}_1$ and send them to $f(A), f(B), f(C) \in \mathcal{C}_2$. For conics, send three points $A, B, C \in \mathcal{C}_1$ to $(0 : 0 : 1), (1 : 0 : 0), (0 : 1 : 0)$. Then $\mathcal{C}_1$ is sent to hyperbola, and making affine transformation $(x : y : z) \mapsto (ax : by : cz)$ we get a hyperbola $xy + yz + zx = 0$ with three points $(0 : 0 : 1), (1 : 0 : 0), (0 : 1 : 0)$. Make such map for $\mathcal{C}_2$ and $f(A), f(B), f(C)$, and make a composition of the first map and the second reversed. $\square$

Theorem 3.2. *Every projective transformation from plane to itself can be expressed in the following way:*

$$(x : y : z) \mapsto (a_1x + b_1y + c_1z : a_2x + b_2y + c_2z : a_3x + b_3y + c_3z) \tag{1}$$

for some $a_i, b_i, c_i \in \mathbb{R}$.

In fact, every transformation of the form (1) with some additional restrictions to a_i, b_i, c_i is projective. For move information, read [4, Part two, Chapter II].

Definition 3.2.1. Given a moving point A, and let its homogeneous coordinates be of the form $(P(t) : Q(t) : R(t))$, where P, Q, R are coprime polynomials, i.e. there is no polynomial $S(x) \in \mathbb{R}[x]$ with degree at least 1 dividing all of them. We define the *degree* $\deg A$ of A to be the max degree of P, Q, R. Similarly we define the degree of a moving line $P(t)x + Q(t)y + R(t)z = 0$ in the same way.

Now, we are ready to state the most important results related to this topic. Here we will also speak about the point-line duality, since the polar transformation wrt. circle $x^2 + y^2 - z^2 = 0$ swaps a point $(a : b : c)$ with the line $ax + by + cz = 0$, so in addition it preserves degrees of lines and points.

Theorem 3.3. *Suppose points A, B have degree d_1, d_2, and there are k values of t for which $A = B$. Then line AB has degree at most $d_1 + d_2 - k$. Similarly, if lines ℓ_1, ℓ_2 have degrees d_1, d_2, and there are k values of t for which $\ell_1 = \ell_2$, then the intersection $\ell_1 \cap \ell_2$ has degree at most $d_1 + d_2 - k$.*

Proof. We show the first statement, and the second follows from point-line duality.

Note that the line through the points $A = (P_1(t) : Q_1(t) : R_1(t))$ and $B = (P_2(t) : Q_2(t) : R_2(t))$ is the line

$$(Q_1R_2 - Q_2R_1)x + (R_1P_2 - R_2P_1)y + (P_1Q_2 - P_2Q_1)z = 0,$$

and you may check that clearly A and B lie on this line. Then, for every value t_0 for which $A = B$, $(t - t_0)$ factors out of each term. So, degree of the line is at most $d_1 + d_2 - k$. $\square$

Theorem 3.4. *Given a conic $\mathcal{C}$, a point $A \in \mathcal{C}$, and a moving line l through A. Let $B = l \cap \mathcal{C} \neq A$. Then,* $\deg B \leq 2 \deg l$.

Proof. Make a projective transformation sending $\mathcal{C}$ to a circle $x^2 + y^2 - z^2 = 0$, and a point A to $(0 : 1 : 1)$. Then, if l has equation $P(t)x + Q(t)y + R(t)z = 0$, we have to solve the following system:

$$\begin{cases} X^2 + Y^2 - Z^2 = 0 \\ P(t)x + Q(t)y + R(t)z = 0 \\ P(t) \cdot 0 + Q(t) \cdot 1 + R(t) \cdot 1 = 0 \end{cases}$$

Solving this, we obtain that B has coordinates $(Q(t)^2 - P(t)^2 : 2Q(t)P(t) : Q(t)^2 + P(t)^2)$, thus $\deg B \leq 2 \deg l$. $\square$

Theorem 3.5. *Let $\mathcal{C}_1$ and C_2 be two lines or two conics (possibly, coinciding). Let A, B be moving along $\mathcal{C}_1, \mathcal{C}_2$, respectively, such that the map $f : A \mapsto B$ is projective. Then,* $\deg A = \deg B$.

Proof. By theorems 3.1 and 3.2, f can be expressed in the form (1). Since (1) preserves linearity, we obtain that $\deg A \leq \deg B$. Making the same observations with f^{-1}, we get that $\deg B \leq \deg A$, hence $\deg A = \deg B$. $\square$

Theorem 3.6. *Let $\mathcal{C}$ be a conic, and let A, B be moving along $\mathcal{C}$ such that* $\deg A = \deg B$. *Then,* $\deg A = \deg B = \deg AB$.

Proof. Let $d = \deg A = \deg B$. Map $\mathcal{C}$ into a unit circle centered at the origin. First, we need the following lemma:

Lemma. Given coprime polynomials $P, Q, R \in \mathbb{R}[x]$ such that $P(x)^2 + Q(x)^2 = R(x)^2$. Then, $(P, Q, R) = (A^2 - B^2, 2AB, A^2 + B^2)$ for some polynomials $A, B \in \mathbb{R}[x]$.

Proof. Note that $P(x)^2 = (R(x) - Q(x))(R(x) + Q(x))$, and since $R(x) - Q(x)$ and $R(x) + Q(x)$ are coprime (if not, then R, Q are not coprime, hence P, Q, R are mot coprime), it follows that they are the squares of polynomials, i.e.

$$\begin{aligned} R(x) - Q(x) &= A(x)^2 \\ R(x) + Q(x) &= B(x)^2 \\ P(x) &= A(x)B(x). \end{aligned}$$

Solving this and changing $(A, B) \mapsto (A\sqrt{2}, B\sqrt{2})$ we get the desired. $\square$

Now, according to lemma we may suppose that A, B are moving along $\mathcal{C}$ with coordinates $(P^2 - Q^2 : 2PQ : P^2 + Q^2)$ and $(R^2 - S^2 : 2RS : R^2 + S^2)$, where $P, Q, R, S \in \mathbb{R}[x]$ (their coordinates are coprime as polynomials since in opposite way we could factor their common term and decrease $\deg A$ or $\deg B$). Then, the line AB has equation

$$(2PQ(R^2+S^2)-2RS(P^2+Q^2) : (P^2+Q^2)(R^2-S^2)-(P^2-Q^2)(R^2+S^2) : 2RS(P^2-Q^2)-2PQ(R^2-S^2)).$$

Note that every coordinate is divisible by $2(PS - QR)$, and after division we obtain

$$(QS - PR : -(QR + PS) : PR + QS). \tag{2}$$

Suppose that three coordinates have a common factor $E \in \mathbb{R}[x]$. Then, E divides $(QS - PR) + (PR + QS) = 2QS$ and $(PR + QS) - (QS - PR) = 2PR$. Since $(P, Q) = 1$ and $(R, S) = 1$, E divides (P, S) or (Q, R), wlog $E \mid (P, S)$. But then it follows from $E \mid (QR + PS)$ that $E \mid QR$, which contradicts the fact that $(P, Q) = (R, S) = 1$. Therefore, the coordinates of AB in (2) are coprime and $\deg AB$ equals max degree of $QS - PR, QR + PS, PR + QS$.

Obviously, degrees of A and B equal to degrees of $P^2 + Q^2$ and $R^2 + S^2$, respectively. Then, $\max(\deg P, \deg Q) = \max(\deg R, \deg S) = d/2$. If not all degrees of P, Q, R, S are equal, then the coordinate in (2) containing the product of two polynomials with degree $d/2$ has degree d. If degrees of P, Q, R, S are equal, then $\max(\deg(QS - PR), \deg(PR + QS)) = d$. So, in both cases, $\deg AB = d$. $\square$

Theorem 3.7. *Given three moving points A_1, A_2, A_3. Then, if they are collinear in at least $\deg A_1 + \deg A_2 + \deg A_3 + 1$ cases, then they are always collinear.*

Proof. Let A_i be moving according to a rule $(P_i(t) : Q_i(t) : R_i(t))$. Then, the equation of moving line A_1A_2 is

$$(Q_1R_2 - Q_2R_1)x + (R_1P_2 - R_2P_1)y + (P_1Q_2 - P_2Q_1)z = 0,$$

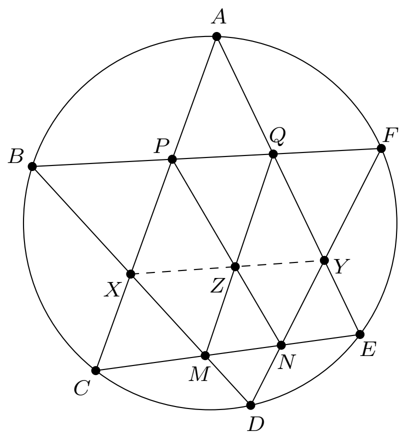
and the point A_3 lies on A_1A_2 is equivalent to

$$(Q_1R_2 - Q_2R_1)P_3 + (R_1P_2 - R_2P_1)Q_3 + (P_1Q_2 - P_2Q_1)R_3 = 0.$$

Since this is a polynomial of degree at most $\deg A_1 + \deg A_2 + \deg A_3$, it equals zero polynomial if there exist $\deg A_1 + \deg A_2 + \deg A_3 + 1$ distinct values of t for which it equals zero. $\square$

With the last five theorems, let's look at some examples.

Example 3.1. Given a 6-gon $ABCDEF$ inscribed in a circle. Let $AC \cap BD = X$, $AE \cap FD = Y$, $AC \cap BF = PX$, $AE \cap BF = Q$, $BD \cap CE = M$, $FD \cap CE = N$. Lines MQ and NP intersect at Z. Prove that X, Y, Z are collinear.

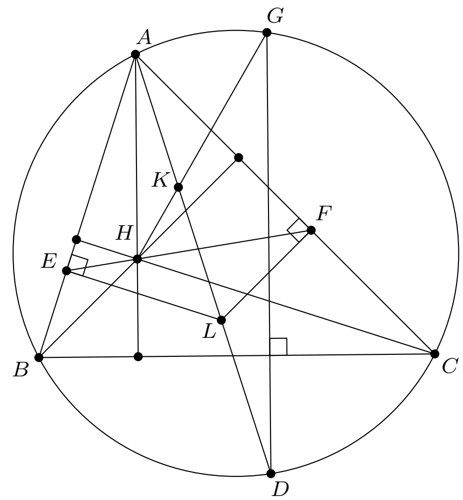


Solution. Let Γ be the circumcircle of $ABCDEF$, and let A be moving on it. Then $X \mapsto P \mapsto A \mapsto Q \mapsto Y$ is projective. according to theorem 3.5, points X, Y, P, Q have equal degrees, so let X be moving linearly on BD to make the degrees of X, Y, P, Q equal 1. Then, $\deg MP = \deg P = \deg Q = \deg MQ = 1$, hence by theorem 3.2 $\deg Z \leq 2$. Thus, according to theorem 3.7 it's left to check $1 + 2 + 1 + 1 = 5$ cases to prove that X, Y, Z are collinear.

- $A = B$. Here $P = B$, $X = B$, $Q = B$. Hence, $Z = B = X$, and X, Y, Z are indeed collinear.
- $A = F$. Symmetric to the previous case.
- $A = D$. Here $X = Y = D$, so X, Y, Z are indeed collinear.
- $A = C$. Then AC is tangent to Γ at C, $Q = CE \cap BF \Rightarrow MQ = CE \rightarrow Z = N$. Also $Y = N$, thus X, Y, Z are collinear.
- $A = E$. Symmetric to the previous case.

Remark. Beautiful, but not a good example since the problem is purely projective. Just make a projective transformation preserving Γ and sending the line through $BD \cap AE$ and $AC \cap DF$ to infinity. You will get $PQYNMX$ is symmetric wrt. XY, so done.

Example 3.2. Let ABC be an acute triangle with orthocenter H and circumcircle Γ. A line through H intersects segments AB and AC at E and F, respectively. Let K be the circumcenter of $\triangle AEF$, and suppose line AK intersects Γ again at a point D. Prove that line HK and the line through D perpendicular to $\overline{BC}$ meet on Γ.



Solution 1. Let's restate the problem in a bit easier way. Take a point D on Γ, and let $G \in \Gamma$ such that $DG \perp BC$. Let $K = HG \cap AD$, and the circle $\omega(K, KA)$ intersect sides AB, AC at E, F, respectively. Note that since K lies on the perpendicular bisectors of AE and AF, we may define E, F as follows: consider $L \in AD$ such that K is the midpoint of AL. Then E, F are the feet of perpendiculars from L on AB, AC, respectively. Our goal is to prove that E, H, F are collinear.

Let D be moving on Γ with degree 2. Since there is a case $D = A$, AD has degree at most $\deg D + \deg A - 1 = 2 + 0 - 1 = 1$. Then, $D \mapsto G$ is projective, so $\deg G = 2$. Thus HG has degree 2, hence $\deg L = \deg K \le 2 + 1 - 1 = 2$ since there is a case $AD = HD$ when $D = AH \cap \Gamma$. Hence, the lines $\infty_{HB}L$ and $\infty_{HC}L$ both have degrees at most $0 + 2 = 2$. But note that when $DC \perp BC$, we have $AD \parallel HC = HG$, hence $K = L = \infty_{HC}$ in this case. So, it follows by theorem 2.2 again that $\deg \infty_{HC}L \le 0 + 2 - 1 = 1$. Similarly, $\deg \infty_{HB}L \le 1$, and since E and F are moving, we have $\deg \infty_{HB}L = \deg \infty_{HC}L = 1$. Thus, $\deg E = \deg F = 1$, and therefore $E \mapsto F$ is projective (an exercise for you to prove it). It's left to check three cases to prove all lines EF pass through H.

- $D = A$. Then $GH = AH$, and $K = AD \cap AH = A$ (AD is tangent to Γ as the limit case). So $E = F = A$ and the statement is true.
- $D = B$. Then G is the antipode of C on Γ. Then $K = HG \cap AD$ is the midpoint of AB, so $L = B$. Then $E = B$ and $F = BH \cap AC$, so E, H, F collinear.
- $D = C$. Similar to the previous case.

Solution 2. Consider initial problem statement, and define $L = \infty_{HC}E \cap \infty_{HB}F$. Let E be moving on AB with degree 1. Note that $\infty_{HC}E \mapsto E \mapsto F \mapsto \infty_{HB}F$ is projective, hence L is moving along a Steiner conic or a line, hence has degree at most 2, hence $\deg K = 2$. Then, note that as E varies, the angle between EF and AB changes by the same value as $\angle(KA, AC)$, thus $E \mapsto AK \mapsto D \mapsto G$ is projective, so G has degree 2. Now, $\deg GK \le \deg G + \deg K = 4$, so according to theorem 3.7 it's left to check $0 + 2 + 2 + 1 = 5$ points E to show that H, K, G are always collinear. I leave to the reader to reformulate the three cases from the previous solution in terms of this one. Let's check two new points:

- $EFCB$ is concyclic. Here $K \in AH$, $D = AH \cap \Gamma \Rightarrow G = A$, so H, K, G are collinear.
- $E = \infty_{AB}$. Then $L = \infty_{BH}$, so $K = \infty_{BH}$. D is the antipode of C, thus $G = B$, and H, K, D are collinear in this case.

□

The problems on this method appear rarer than on the methods described in the first two sections. But it's always good to keep it in mind when it's difficult to prove that some map is projective, or the set of moving points splits in several classes, in every of which there is a projective map between points, but there is no projective map between any two different classes. On the other hand, sometimes this method will require checking too many cases, so use it wisely.

3.1 Problems for practice

1. M is the midpoint of arc BC of the circumcircle of triangle ABC. D is also on arc BC but different from M. AB meets CD at E. AC meets BD at F. EM, FM meet BC at P, Q, respectively. Prove that A, P, D, Q are cyclic.
2. Given triangle ABC inscribed in a circle with center O, with orthocenter H. Point D is such that the quadrilateral $ABDC$ is a parallelogram. Let BH intersect circle (O) at E and OE intersect AC at X. Take K to be the perpendicular projection of H on DX and M to be the midpoint of BC. Prove that E, K, M are collinear.
3. Let ABC be a triangle with circumcircle ω and ℓ a line without common points with ω. Denote by P the foot of the perpendicular from the center of ω to ℓ. The side-lines BC, CA, AB intersect ℓ at the points X, Y, Z different from P. Prove that the circumcircles of the triangles AXP, BYP and CZP have a common point different from P or are mutually tangent at P.

Hints

Since the Method of Moving Points doesn't always work, one should be able to solve a geometry problem synthetically. So, it's great if you solve some of the practice problems without using this technique. Read a hint only if you are stuck for a long time and don't have any ideas how to solve the problem.

Section 1

1. Move E.

2. Move X.

3. Move C or D.

4. Move P and use inversion to show that $P \mapsto \odot(AX_1Y_1) \cap (O)$ is projective.

5. Fix line AP and move P on it.

Section 2

1. Move D along the perpendicular bisector of BC. Prove that the Steiner conic of $E \mapsto F$ is tangent to BC.

2. Inversion at A.

3. Move l and find projective maps showing that the locus of considered points is a line. Then, try some positions of l to determine this line.

4. Move D. Is it true that $Q \mapsto P$ and $R \mapsto S$ are projective? What happens with I during these maps?

5. The same strategy as in Example 2.3.

Section 3

1. Note that the concyclity of A, P, D, Q is equivalent to $\angle CAP = \angle BDQ$. Move P and define Q_2 to be such that $\angle BDQ_2 = \angle CAP$. Find the degrees of Q and Q_2 and prove $Q = Q_2$. For finding the degree of rotated line, use the line at infinity.

2. Let E, K, M be collinear and we prove that X, K, D are collinear. Move A with degree 2 and find the degrees of X and line KD. Use theorem 3.6 for line KD.

3. Perform inversion at P and move B. Show that $X \mapsto B \mapsto Y$ is projective, so if $\deg X = = \deg Y = 1$, then $\deg B = 2$ (why?). From dual of theorem 3.7, show that it's left to check 5 cases. By the way, it's IMO SL 2012/G8.

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