IND4-4

Problem: 4

Number of pages: 8

Solution 1

We first prove three crucial things to be used frequently.

~~① P(a+1) is coprime to~~ P(a

for any $x \in \mathbb{Z} - \{0\}$.

① $P(x)$ is coprime to $P(x+1)$.

② $P(x)$ not coprime to $P(x+2) \Rightarrow x \equiv 2 \pmod 7$

③ $P(x)$ not coprime to $P(x+3) \Rightarrow x \equiv 1 \pmod 3$.

Proof:

①.
$$\big(P(x), P(x+1)\big) = (x^2+x+1,\ x^2+3x+3).$$
$$= (x^2+x+1,\ 2x+2) \rightarrow 2 \nmid x^2+x+1$$
$$= (x^2+x+1,\ x+1)$$
$$= (x^2,\ x+1) = 1 \quad \square$$

②.
$$\big(P(x), P(x+2)\big) = (x^2+x+1,\ x^2+5x+7)$$
$$= (x^2+x+1,\ 4x+6)$$
$$= (x^2+x+1,\ 2x+3).$$
$$= (2x^2+2x+2,\ 2x+3)$$
$$= \big((2x+3)x - x+2,\ 2x+3\big)$$
$$= (-x+2,\ 2x+3)$$
$$= (-2x+4,\ 2x+3) = (7,\ 2x+3)$$
$$\Rightarrow 7 \mid 2x+3 \quad \blacksquare$$

$\Rightarrow 7 \equiv 2 \pmod 7$ $\square$

③ $(P(x), P(x+3)) = (x^2+x+1,\ x^2+7x+13)$

$= (x^2+x+1,\ 6x+12)$

$= (x^2+x+1,\ 3x+6)$

Now if $3 \mid x^2+x+1 \Rightarrow x \equiv 1 \pmod 3$.

& $(x^2+x+1, x+2)$

$= (\cancel{x} - x+1, x+2)$

$= (3, x+2) \mid 3.$ $\square$

$\hookrightarrow$ hence $x \equiv 1 \pmod 3$

Now we turn our attention to the problem.

Claim 1: $b \neq 2, 3, 4$.

Proof. $b = 2$ violates ①.

$b = 3$ violates ① too, because $P(a+2)$ is coprime to $P(a+1)$, $P(a+3)$.

$b = 4$: In order not to violate ①, $P(a+2)$ has to be non coprime with $P(a+4)$. Similarly $P(a+3)$ is ~~co~~ non coprime to $P(a+1)$. $\Rightarrow a+2$ & $a+1 \equiv 2 \pmod 7$, a contradiction.

Claim 2 : $b \neq 5$.

Proof look at $P(a+3)$. in order not to violate ①, it must be non coprime to $P(a+1)$ or $P(a+5)$

WLOG assume $P(a+3)$ is non coprime to $P(a+1)$

$\Rightarrow a+1 \equiv 2 \pmod 7$.

now look at $P(a+2)$. it cannot be non coprime to $\underbrace{P(a+1), P(a+3)}_{\text{violate ①}}$, $P(a+4)$. $\hookrightarrow$ implies $a+2 \equiv 2 \bmod 7$ contra]

$\therefore P(a+2)$ non coprime to $P(a+5)$

$\therefore P(a+4)$ non coprime to $P(a+1)$ [similarly]

$\Rightarrow a+2$ & $a+4 \equiv 1 \pmod 3$, a contradiction $\square$.

Claim 3 : $b = 6$ works.

We will hunt for an a such that .

$a+1 \equiv 7 \pmod{19}$

$a+1 \equiv 1 \pmod{7}$

$a+1 \equiv -1 \pmod 3$.

a solution to the system exists by Chinese remainder theorem. let x be the solution

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$\Rightarrow \cancel{\{P(x+1), P(x+5)}$

$\Rightarrow P(x+1) \equiv 7^2+7+1 = 57 \equiv 0 \pmod{19}$

$\Rightarrow P(x+5) \equiv 11^2+11+1 = 133 \equiv 0 \pmod{19}$

hence $19 \mid P(x+1)$ & $P(x+5)$.

also,

$P(x+2) \equiv 2^2+2+1 \equiv 0 \pmod 7$

$P(x+4) \equiv \cancel{8}^2 + \cancel{5} + 1 \equiv 0 \pmod 7$

hence $7 \mid P(x+2)$ & $P(x+4)$

also,

$P(x+3) \equiv 1^2+1+1 \equiv 0 \pmod 3$.

$P(x+6) \equiv 1^2+1+1 \equiv 0 \pmod 3$.

$\Rightarrow 3 \mid P(x+3)$ & $P(x+6)$

$\Rightarrow \{P(x+1), P(x+2), P(x+3), P(x+4), P(x+5), P(x+6)\}$

is a fragrant set $\boxtimes$

$44^2+44+1 \equiv 4+2+1 \equiv 0.$ ✓ 7.
$46^2+46+1 \equiv 9-3+1 \neq 0$

$\boxed{b=6}$

$-x+1, x+2.$
$3, x+2.$

44, 48.

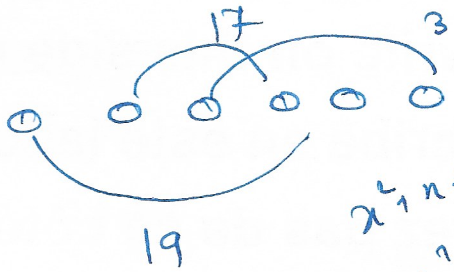
48^2

11, 7.

$19 \mid a^2+a+1 \Rightarrow a = 19k+7$ a $19k+11$

and 19.

1	1
2	4
3	9
4	16
5	11
6	17
7	11
8	
9	
10	
11	
12	
13	
14	
15	
16	
17	
18	
19	



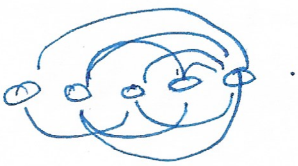
$x^2+x+1, x+2$
$-2x+1, x+3$
$2x+1$

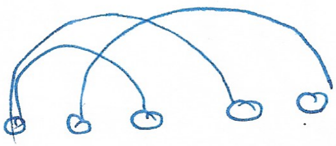
$x^2+x+1, 3x+4.$
$x^2+x+1, 3x+4.$
$3x^2+3x+3, 3x+4.$
$-3x+3$
$\cancel{-x+3}, 3x+4.$
$-3x+9, 3x+4.$
$13, 3x+4$

$a \equiv 7 \mod (19)$
$a+1 \equiv 2 \pmod{17}.$
$a+2 \equiv 1 \pmod 3).$

$a \equiv 7$
$a \equiv 1$
$a \equiv -1$
$\Rightarrow$
$11\cancel{8}a \equiv 5\cancel{6} \equiv 77 \equiv 1$
$a \equiv 1$
$2a \equiv 1$

19
17
3.





$x^2+x+1, 2x+3$ $-7, 2x^2+3$
$= 2x^2+2x+2, 2x+3.$
$= 2x^2-1, 2x+3$

a^2+2a+1, $a^2+\cancel{6a+9}$ $8a+16$.

$\therefore a^2+2a+1+\cancel{4a+8}$ $6a+15$

$\cancel{3 \mid a+1} \Rightarrow$

a^2+2a+1, $\cancel{2a+5}$, a^2-4

$2a+5$

$2a^2+4a+2$, $2a^2+5a$

$\therefore = 2-a$, $2a^2+5a$

$a+2 \Rightarrow 2-a,\ 2a+5$

$4-2a,\ 9.$

1	1	$\rightarrow$	2
4	2		6
2	3		5 ✓
2	4		6
4	5		6 ✓
1	6		7

$3 \mid n^2+n+1 \Rightarrow n = 3k+1$

$7 \mid n^2+n+1 \Rightarrow \cancel{n=7k-1}$ $7k+3$ a $7k+5$.

$3m+1 = 7n-1$ $\quad$ 34 $\longrightarrow$ 37

$7n-3m=2.$ $\quad n=5$

$n=10.$

75.

$\Rightarrow 7\cancel{|}$ $\quad 3m+1 = 7n + \cancel{2}$

$\cancel{7n-3m=4}$,

$7n-3m=3$

$\Rightarrow$

$n=3$

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$\cancel{P(a) = P(a+1)}$

$P(a-1) = a^2-2a+1+a-1+1 = a^2-a+1$

$(a^2-a+1, a^2+a+1) = (a^2-a+1), (\cancel{2}(a^2+1))$

$= a^2+1, \quad a \cdot = 1.$

$a^2+4a+4+a+2+1$

$= a^2+5a+7.$

$a^2-a+1, \quad a^2+a+1, \quad a^2+3a+3, \quad a^2+5a+7$

$(a^2-a+1, \cancel{4a+2}\ 2a+1) = (a^2+a+2), a^2-a+1$

$\frac{\cancel{a}(a-3), \ 2a+1}{2a-6, \ 2a+1} =$

$\Rightarrow a = 3.$

$= 7, 2a+1$

n	
1	3
2	7
3	13
4	21
5	31
6	43
7	57 = 3 × 19
8	73
9	91 = 7 × 13
10	111 = 3 × 37
11	133 = 7 × 19
12	157
13	183 = 3 × 61
14	211
15	241 = ~~7×3×11~~

16 = 273 = 3 × 7 × 13

17 = 307

$(a+1)^2, (a+k)^2.$

$= a^2+2a+1, \quad a^2+2k+k^2.$

$= a^2+2a+1, \quad \cancel{2(a-k)}\ 2(k-a)+k^2-1$

$= a^2+2k+k^2, \quad \cancel{2}(k-a)+k^2-1$

$a^2+a+1, \quad a^2+8a+16.$

$= a^2+a+1, \quad 7a+15$

$\cancel{7a^2+7a+15}, \cancel{7a}+15$

$7a^2+7a+7, \quad 7a^2+15a.$

$\cancel{7a^2+7a}\ 7-8a, \quad 7a^2+15a.$

$7-8a, \quad 7a+15.$

$22, \quad 15a \qquad 11 \mid a.$

$a = 11k.$

$a+1 = \cancel{3m+1}\ 7n+5$

$a+2 = 3m+1$

$a = 11k.$

$a+1 = 3m+1$

$a+2 = 7n+5.$

$a+1 = 3m = 7n+5$

$a = 66.$

$a = 33k.$

$a+1 = \cancel{3k}\ 7(3k+1)+5$

$= 21k+12.$

$33k+2 = 7n+5.$

$63, \quad k = 2, \quad n = 9.$

$33k - 7n = 3.$

$11 \mid 21k+3$

$\cancel{11 \mid 66}\ 44.$

$\cancel{2 \mid a+1}$

66, 67, 68, 69, 70, 71.

$11 \mid 21k+12$

$= k = 1$

$a+1 = 33$

$a = 32.$

65, 66, 67, 68, 69, 70,

44, 45, 46, 47, 48, 49

32, 33, 34, 35, 36, 37.