IND4-1

Problem: 1

Number of pages: 5

57th IMO HONG KONG 2016

第五十七屆國際數學奧林匹克

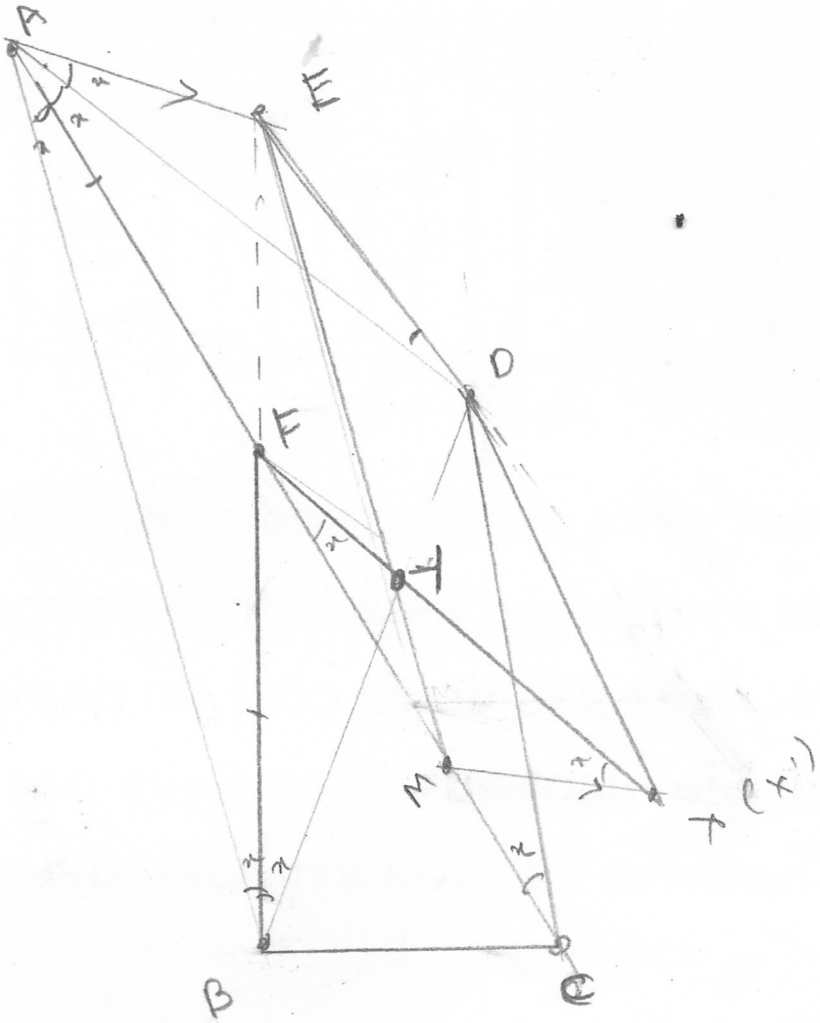
57th International Mathematical Olympiad

Name Kapil Pause

Contestant Code IND 4

Problem 1 Page 1.

Sol^n P1



[Sorry for the bad diagram ☹]

☆ We will first show that F is the incentre of ABD

– Let us take $\angle FAB = x$. Then $AF = MB \Rightarrow$

~~$\angle BFA$~~ $\angle BFC = 2x$.

& $AD = DC \Rightarrow \angle DCF = x$.

Now In ΔAFB & ADC, they are both isosceles with base angle x

$\Rightarrow$ $\cancel{\Delta}$ $AFB \sim ADC$

$\Rightarrow \dfrac{AB}{AC} = \dfrac{AF}{AD}$ . . . ①.

Now In ΔABC, ΔAFD, $\angle BAF = \angle FAD$ and ①, implies $\Delta ABC \sim \Delta AFD$.

$\Rightarrow \angle AFD = \angle ABC = 90 + x$.

$\Rightarrow \angle BFD = 2x + \angle DFC = 2x + 90 - x = 90 + x$.

Now, F lies on angle bisector of $\angle BAD$ and directed angle $\measuredangle BFD = 90 + x$. Hence it follows that F is the incentre of ΔABD. [Because incentre is the unique point satisfying both criteria). $\square$

★ Now we shall show $E \in \odot ABD$.

— $EA = ED \Rightarrow$ ~~E lies on perp. bisector of AD.~~
$\angle AED = 180 - 2x = 180 - \angle ABD$. $\square$

★ Also, $M \in \odot ABD$. [Incentre – Excentre lemma from evan chen handout]. (since $\angle FBC = 90°$, $C \in AF$ C is excentre of ABD].

Now, let ME & BD intersect at Y. We shall finish the problem by showing that if a line parallel to AE through M intersects FY at X', then MX' = AE, thus, AEMX' would be a parallelogram and X = X'.

– As $\angle MDB = \angle MED = x$, $\Rightarrow$ ⊙ EDY tangent to MD $\Rightarrow MY \cdot MD = MD^2$

but $MD^2 = MB^2 = MF^2 = MY \cdot MD$

M midpt of arc BD [small] M centre of ⊙FBC

$\Rightarrow$ ⊙ EFY tangent to MF

$\Rightarrow \angle MFY = \angle MEB = x$.

Now, angle $\angle X'MC = \angle EAC = 2x$

$\Rightarrow$ ~~∠X'~~ $\angle MX'F = x$.

$\Rightarrow MF = MX$

$\Rightarrow MX = \frac{1}{2} FC$.

$\therefore$ it suffices to show $AE = \frac{1}{2} FC$.

But this is easy, as since AE subtends $\angle x$ and MB subtends $\angle x$ in ⊙ ABMDE, $AE = MB$

But $MB = MF = \frac{1}{2} FC$.

Hence done!! YaY.

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$$\frac{\frac{a+\sqrt{a^2+b^2}}{2} \times \frac{1}{\cos x}}{2} \times \frac{1}{\cos x}.$$

$$= \frac{a+\sqrt{a^2+b^2}}{4} \cdot \cancel{\frac{a}{\sqrt{a^2+b^2}}}\ \cancel{\sqrt{a^2+b^2}}\ \frac{2 \times \sqrt{a^2+b^2}}{\sqrt{a^2+b^2}+a}.$$

$$= \boxed{\frac{\sqrt{a^2+b^2}}{2}} = AE = MX.$$

BFE DXM.

$\boxed{FD = CX}$

X ~~BE~~

XBE F DM.

F → Incentre of ABD.

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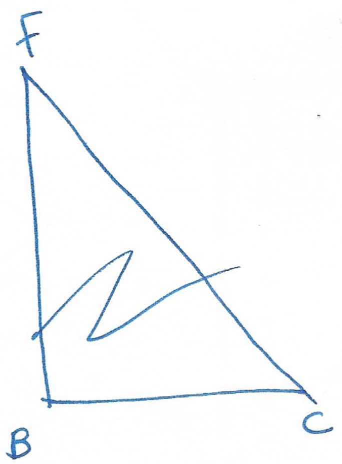
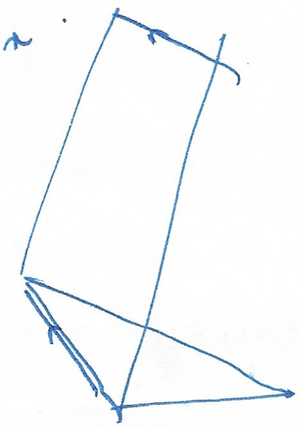
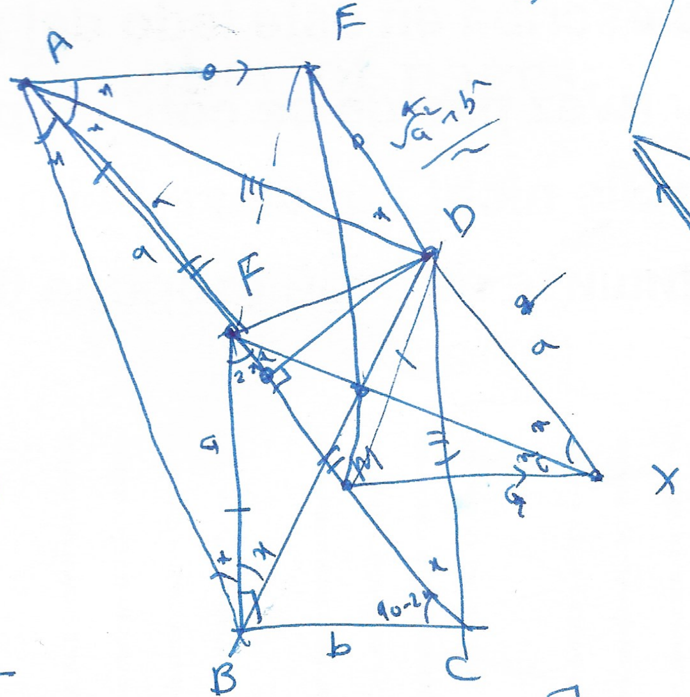


$\sin x \cdot \cos x$

$\frac{\sqrt{a^2+b^2}+a}{2} \times \cos x$

$\frac{\sqrt{a^2+b^2}+a}{4} = \cos^2 x$





$\cos^2 x - \sin^2 x = \cos 2x$

$\Rightarrow \cos^2 x = \frac{\cos 2x + 1}{2}$

$\therefore \frac{\sqrt{a^2+b^2}+a}{4} \times \sqrt{a^2+b^2}$

$\sqrt{a^2+b^2}+a-\left(\frac{\sqrt{a^2+b^2}}{2}+\frac{a}{2}\right)-\sqrt{\frac{a^2+b^2}{2}}$

$= \frac{1}{2}$

$= \frac{a}{2}$.