IND4-2

Problem: 2

Number of pages: 10.

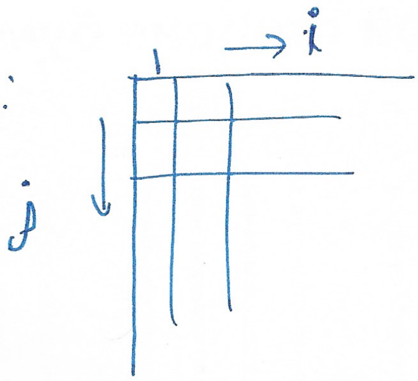
Solution 2.

We will present the crux of problem first; namely that a ~~a~~ required $n \times n$ table exists iff $9 | n$.

We replace the ~~problem~~ I, M, O in the problem by $1, \omega, \omega^2$ resp. Then the problem becomes that sum of nos in rows, columns and diagonals is zero.

We give the construction for such $n \times n$ board when $9|n$.

Construction:



in each cell write the no. $\omega^{\left(i+j+\lfloor \frac{i}{3} \rfloor - \lfloor \frac{j}{3} \rfloor\right)}$

Proof:

1) Along row and column.

Sum of ~~rows~~ k^{th} row will be.

$$\sum_{i=1}^{n} \omega^{i+k-\lfloor \frac{i}{3}\rfloor - \lfloor \frac{k}{3}\rfloor}$$

$$=\left[\sum_{i=1}^{n} \omega^{i-\lfloor \frac{i}{3}\rfloor}\right] \cdot \omega^{k-\lfloor \frac{k}{3}\rfloor}$$

$\Rightarrow \cancel{\sum_{m=1}}$ let $n = 3m$.

$\Rightarrow$ ~~Sum~~ but $\displaystyle\sum_{i=1}^{n} \omega^{i-\lfloor \frac{i}{3}\rfloor}$

$$= \sum_{r=0}^{3m-1} \omega^{3r+1-\lfloor \frac{3r+1}{3}\rfloor} + \sum_{r=0}^{3m-1} \omega^{3r+2-\lfloor \frac{3r+2}{3}\rfloor} + \sum_{r=1}^{3m} \omega^{3r-\lfloor \frac{3r}{3}\rfloor}$$

$$= \sum_{r=0}^{3m-1} \omega^{2r+1} + \sum_{r=0}^{3m-1} \omega^{2r+2} + \sum_{r=1}^{3m} \omega^{2r}$$

~~$= 0$ [as $r \to 2r+1$, $r \to 2r+2$, $r \to 2r$ are all bijections in $\mathbb{Z}$ mod 3]~~

~~$= \sum_{r=0}^{3m-1} \omega^{r} = 0$ as $r \to 2r+1$, $r \to 2r+2$, $r \to$~~

$$= \sum_{r=1}^{3m} \omega^{r} + \sum_{r=1}^{3m} \omega^{r+1} + \sum_{r=1}^{3m} \omega^{-r}$$

$$\cancel{= 0 + \omega\left(\sum_{r=1}^{3m} \omega\right)}$$

$= 0$ (well known), as $r \to r$, $r \to r+1$, $r \to -r$ are invertible maps in $\mathbb{Z}/3\mathbb{Z}$)

2) <u>along diagonals</u>.

i) diagonal with $i+j$ constant.

Note that the only diagonals we need to sum up are with $i+j \equiv 1 \pmod 3$)

$$\therefore \text{required sum} = \sum_{\substack{i+j=3k+1 \\ i,j \geq 1}} \omega^{i+j+\lfloor \frac{i}{3} \rfloor - \lfloor \frac{j}{3} \rfloor}$$

$$= \omega^{i+j}\left(\sum_{i=1}^{3k} \omega^{\lfloor \frac{i}{3} \rfloor - \lfloor \frac{3k-i}{3} \rfloor}\right)$$

$$\cancel{= \omega^{i+j}\left(\sum_{i=1}^{3k} \omega^{2\lfloor \frac{i}{3} \rfloor - k}\right)}$$

$$\cancel{= \omega^{i+j}\left(\sum_{i=1}^{3k} \omega^{2\lfloor \frac{i}{3} \rfloor}\right) \cdot \omega^{-3k^2}}$$

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~~We will show~~

$$\cancel{\sum_{i=1}^{3k} 2\left\lfloor \frac{i}{3} \right\rfloor}$$

$$= \omega^{i+j} \left(\sum_{i=1}^{3k} \omega^{\left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor + k} \right)$$

$$= \omega^{i+j} \left(\sum_{i=1}^{3k} \omega^k \right) = 0.$$

↳ This is wrong.

ii) along diags with $i-j$ constant.

Note that ~~the~~ we need to sum only along diags with $i-j \equiv 0 \pmod 3$.

$$\therefore \text{Sum} = \sum_{\substack{i-j=3k \\ i,j \geq 1}} \omega^{i-j+\left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + 2j}.$$

$$= \sum \omega^{i+j} \left(\sum_{j=1}^{3k} \left\lfloor \frac{3k+j}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + 2j \right)$$

$$= \omega^{i-j} \left(\sum_{j=1}^{3k} \omega^{k+2j} \right)$$

$$= \omega^{i-j} \cdot \omega^{k(3k-1)} \left(\sum_{j=1}^{3k-1} \cdot \omega^{2j} \right)$$

$$= 0.$$

i) along diagonal with $i+j$ constant.

→ Note that we only need to sum up for diags with $i+j \equiv 1 \pmod 3$.

$$\therefore \text{Sum} = \sum_{\substack{i+j=3k+1 \\ i,j \geq 1}} \omega^{\,i+j+\lfloor \frac{i}{3} \rfloor - \lfloor \frac{j}{3} \rfloor}$$

$$= \cancel{\omega^{i+j} \left(\sum_{j=1}^{3k} \left\lfloor \frac{3k+1-j}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor \right)}$$

$$\cancel{= \omega}$$

$$= \sum \omega^{i+j} \left[\sum_{j=1}^{3k} \omega^{\left\lfloor \frac{3k+1-j}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor} \right]$$

$$= \cancel{\sum \omega^{i+j} \left[\sum_{j=1}^{3k} \omega^{k-2\left\lfloor \frac{j}{3} \right\rfloor} \right]}$$

$$= \omega^{i+j} \left[\sum_{j=1}^{3k} \omega^{\left\lfloor \frac{3k+1-j}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor} \right]$$

$$= \omega^{3k} \cdot \omega^{k} \cdot \omega^{i+j} \left[\sum_{j=1}^{3k} \omega^{\left\lfloor \frac{1-j}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor} \right]$$

$$= \boxed{} \left[\sum_j = 0 \right.$$

(can be shown)

(time is less ☹)

Now we shall show that if $9 \nmid n$ then required construction does not exist.

~~I will~~

Sum along all diagonals and then sum along the rows $2, 5 \ldots$ and columns $2, 5 \ldots$.

We get that all nos in table are summed up, but only nos in $[2, 5 \ldots] \times [2, 5 \ldots]$ are summed 4 times.

hence sum of all nos in $[2, 5 \ldots] \times [2, 5 \ldots]$ $= 0$.

$\Rightarrow$ no. of cells in $[2, 5 \ldots] \times [2, 5 \ldots]$ is divisible by 3.

$\Rightarrow$ n is divisible by 9.

(Time is 1 min left)

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$j \backslash i$	1	2	3	4	5	6	7	8	9
1	ω		ω^2	ω		ω^2	ω		ω^2
2		1	ω^2		ω^2			ω^2	
3				ω^2		ω	ω		ω^2
4		ω^2	ω^2			ω^2	ω	ω	
5		1			1			ω^2	
6				ω			ω	1	ω
7		ω^2	ω^2	1			1	ω	
8		ω		ω	ω^2		ω		
9	ω	1		ω^2			ω^2	1	ω

$\omega^{i+j} \rightarrow i$

$\frac{3\times 2+2}{9}$

$8+1$

$i+j = i-j$

$\Rightarrow 2i$

$j+1-j-1$

$\omega^{i+j}($

$\underline{\omega}(i$

$\omega^{i+j}(i-j-1)$

$\frac{\omega^{2i+j}}{}$ $= \omega^{i+j}(\quad - \quad)$

$\omega^{i^2-j^2+i}$

$\Sigma\omega^{i+j}$

$\omega^{i-j} \cdot \omega^{2j+i}$

i^2+2j^2

$\omega^{i+j} \quad \omega^k = \omega^{i-j} \quad \omega^{l}$

$(i-j)(i+j)-j^2$

$\omega^{i+2j} \quad \omega^{i-j} \quad \omega^{8j}$

$\omega^{i^2-j^2}$

$\Sigma\omega^{i^2-j^2+i}$

$\omega^{i+j} + i^2+j^2$

$\Sigma\omega^{i+j}(\omega^{2i+j})$

$(i-1)^2+(j+1)^2$

i^2-2i-j^2-2j

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$n = 3^2 x.$

$2+0$

$1+0$

$1+1$

$i+0=4.$

$\left\lfloor \frac{1}{3} \right\rfloor - \left\lfloor \frac{4-1}{3} \right\rfloor$

$2+0$

$2+0$

$3+1$

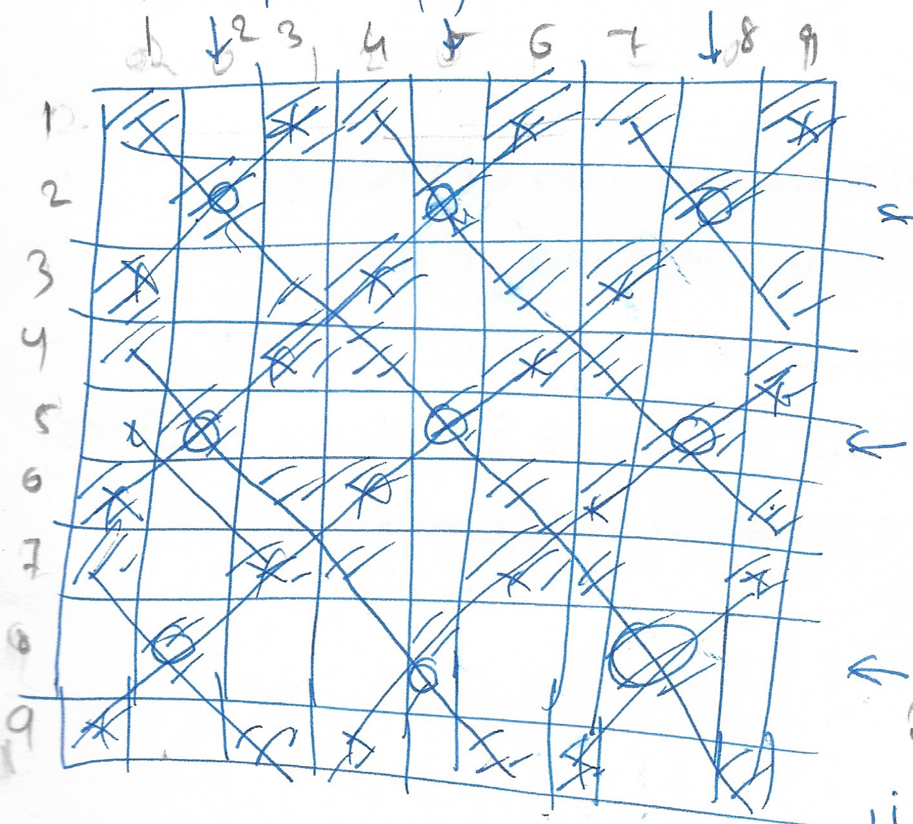
$7+1+$

$2-6$

$\frac{1}{3}$

$8+2+2-6$

$9+3+3-$



$0-2$

$0-2$

$1-3$

$0-1$

$0-1$

$1-2$

$1-2.$

$2j$

$1-2$

$2-3$

$\delta = k - \left\lfloor \frac{i}{3} \right\rfloor$

$1 \to 0$

$2 \to 0$

$3 \to 1$

$4 \to 1$

$5 \to$

$0-2$

$0-1$

$\boxed{\omega^{\,i+j + \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor}}$

$\omega^{\,i-j - \left\lfloor \frac{i}{3} \right\rfloor + 2j}$

-1

$1-1$

$1-0$

$2-0.$

$\omega^{\left\lfloor \frac{i}{3} \right\rfloor k + 2j}$

$1 \to 1$

$2 \to 2$

$3 \to 2$

$4 \to 3$

$5 \to 4$

$6 \to 4$

$7 \to 2$

$8 \to 3$

$1 \to 8$

$\boxed{\omega^{\left\lfloor \frac{i}{3} \right\rfloor + 2\left\lfloor \frac{j}{3} \right\rfloor}}$

$1 \to 1$

$2 \to 2$

$3 \to 1$

$4 \to 2$

$5 \to 3$

$6 \to 2$

$7 \to 3$

$8 \to 1$

$9 \to 3$

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$\omega^{(i-j-1)(i+j)}$ $i+1$ $j-1$

$\omega^{(i-j-1)(\quad)}$

$\omega^{(i+j)(i-j-2)}$ $8-j^2$

$\cancel{\omega^{i^2-j^2}-1}$ $3-j^2$

$\omega^{(i+j)(i-j-1)}$ $2i-j$

$i-j+i$

$\cancel{\omega^{i^2-j^2}-(i+j)}$ $(i+j)-j$ $i+j+i-2j$

ω^{2i+j}

$-j^2-j$ $\omega^{i^2-j^2+2i}$ $\boxed{\omega^{i-j}}+i+2j$

$i+1+2(j+1)$

$\omega^{(i+1-j)(i+j)}$ $\omega^{i^2+j^2}=\omega^{(i+j)^2-2ij}$

$\omega^{i^2-j^2+i+j}$ $2i+j$ i^2-2j

$2j+i$

$1 \to 1$ $2([i+j]+i-2j$ j^2-j^2+ $1 \to -1$

$2 \to 1$ $2 \to 0$

$6 \to 0$ $0 \to 0$

$j \cdot j^2$ ij

$1, x, xy,$

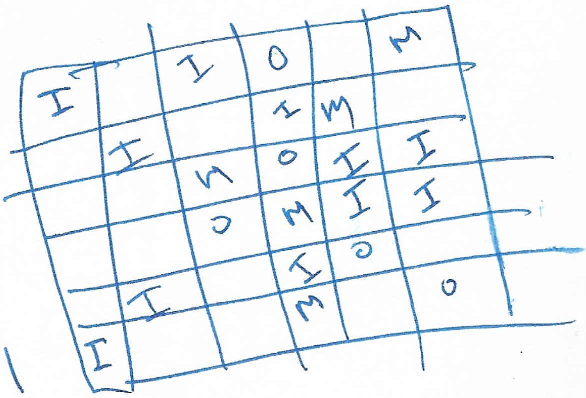
	1	2	3
1	1	2	3
2	4	5	6
3	7	8	9

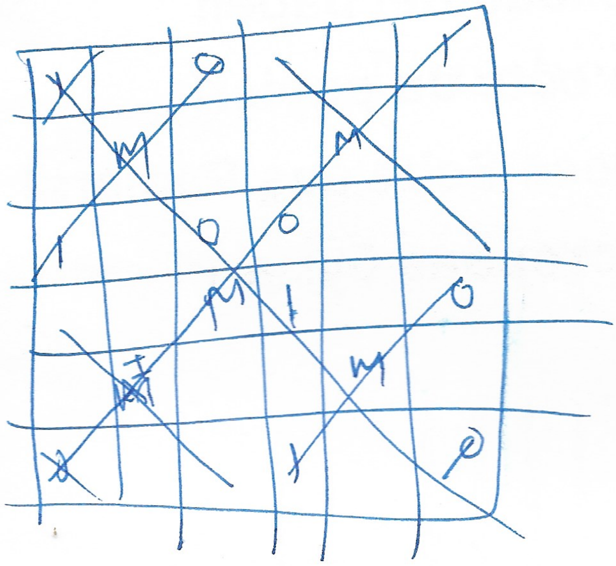
1	ω	ω^2
ω	$M\omega^2$	ω
ω^2	ω	1

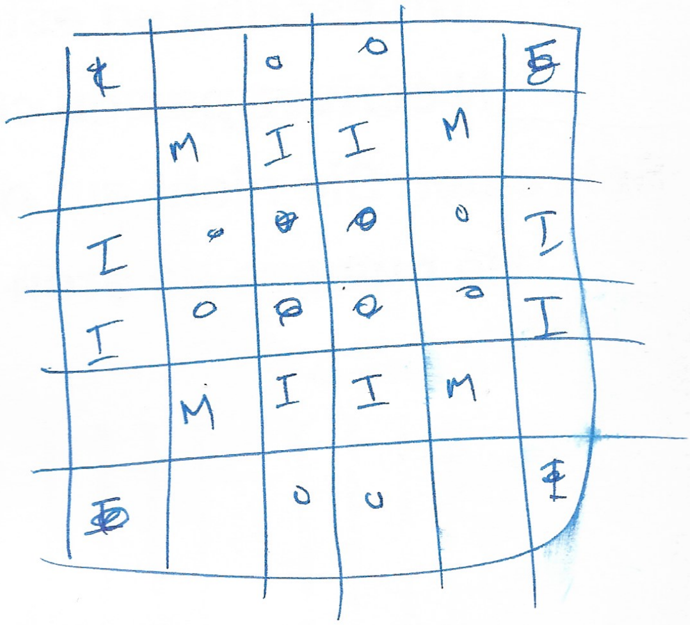
$1 \quad \omega \quad \omega^2$

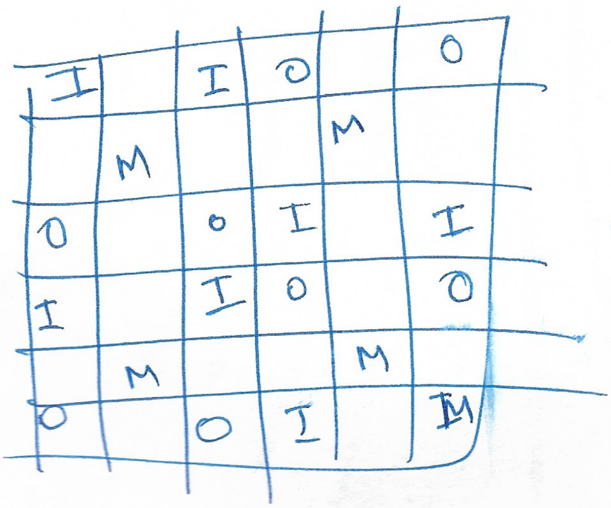
$\omega^2 \quad 1 \quad \omega$

$\omega^2 \quad \omega^2 \quad 1$









$1 \quad \omega \quad \omega^2.$

$6 \rightarrow$ impossible