Auction Design

Jakob Ahrens

Spectrum Licenses

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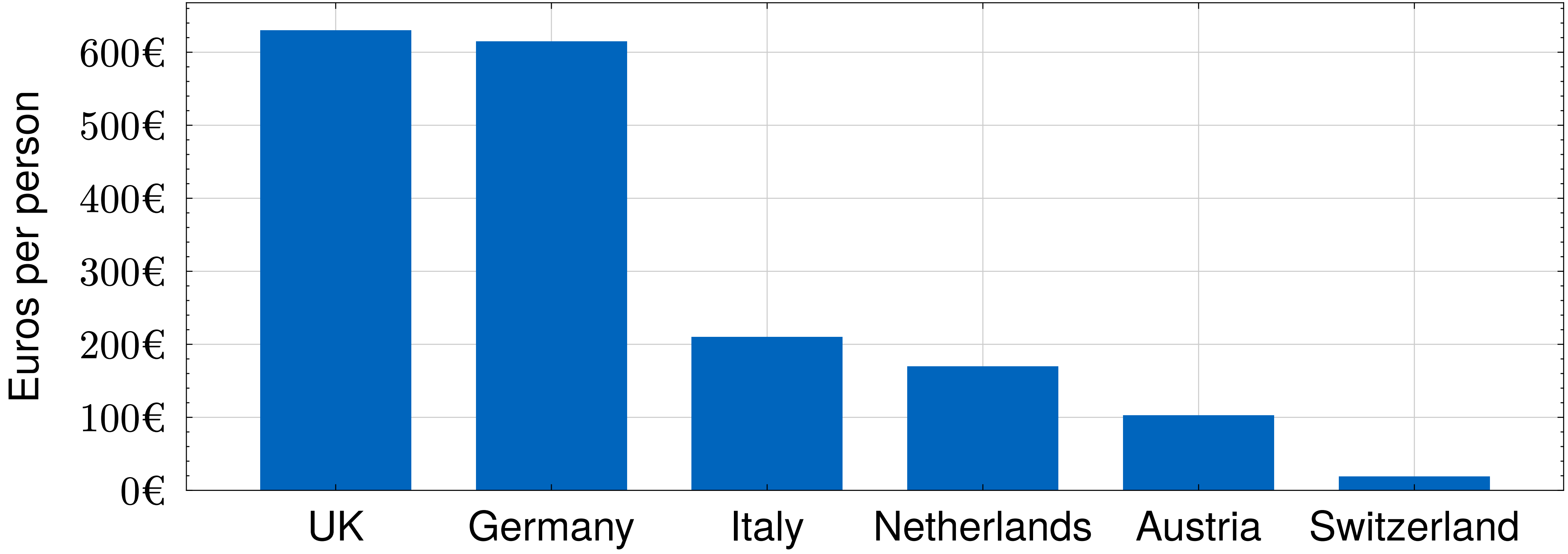

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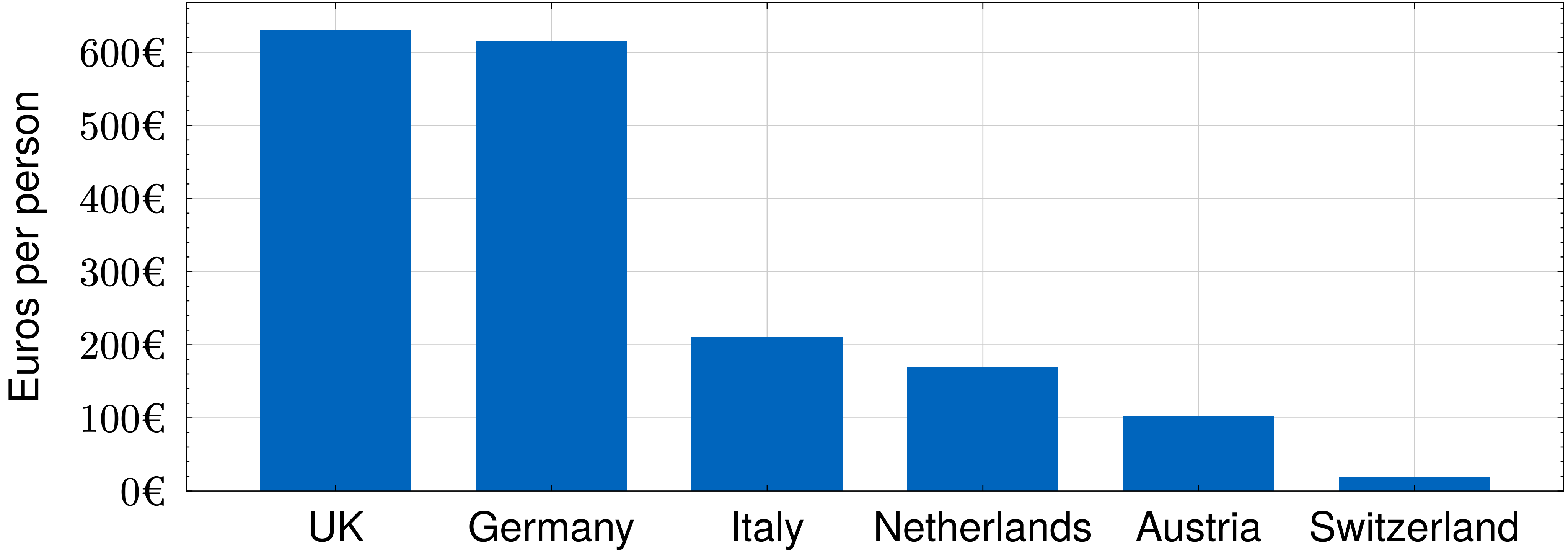

Auction design matters[1–4]. What makes a **successful** auction?

Modeling sealed-bid, single-item auctions

Sell single advertisement in newspaper.

Bidders $N := \{1, ..., n\}$ have **value** v_i and submit **sealed bids** $b_i \geq 0$.

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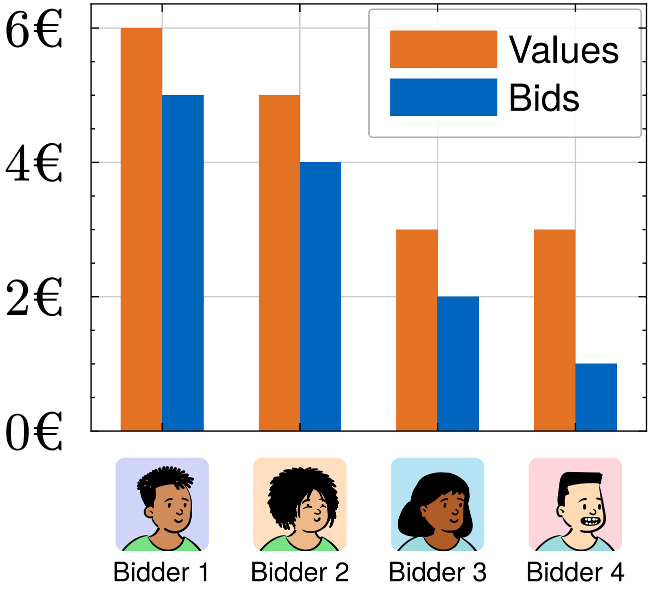

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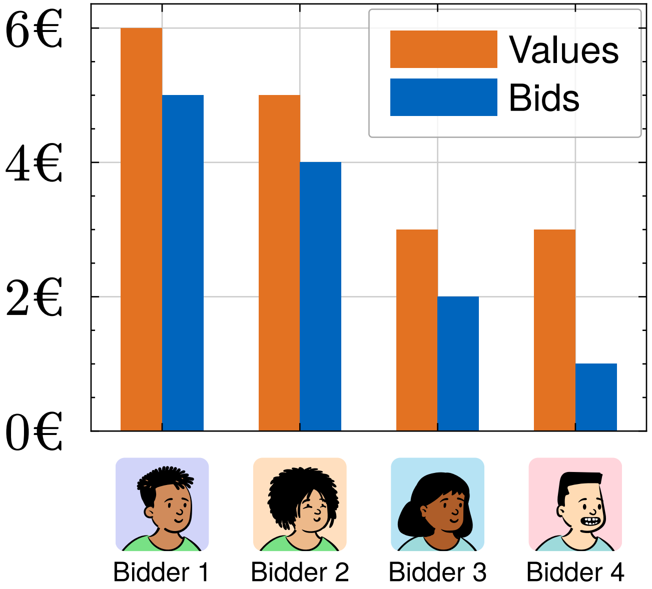

- **Bid profile** $b := (b_1, \ldots, b_n) = (5, 4, 2, 1)$
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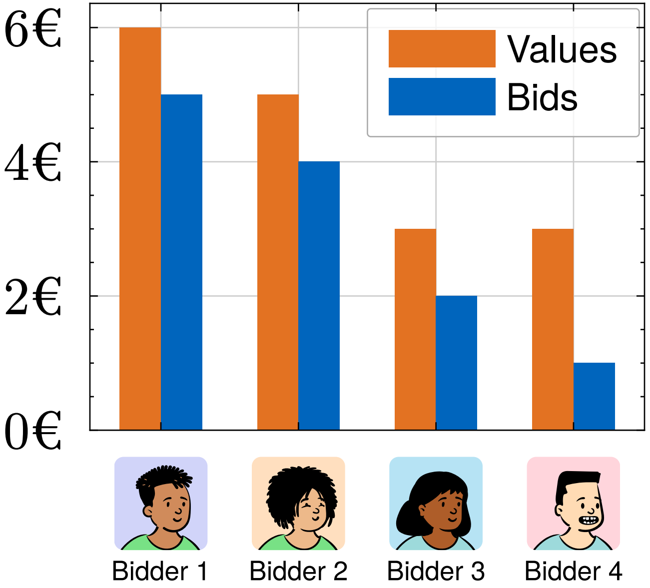

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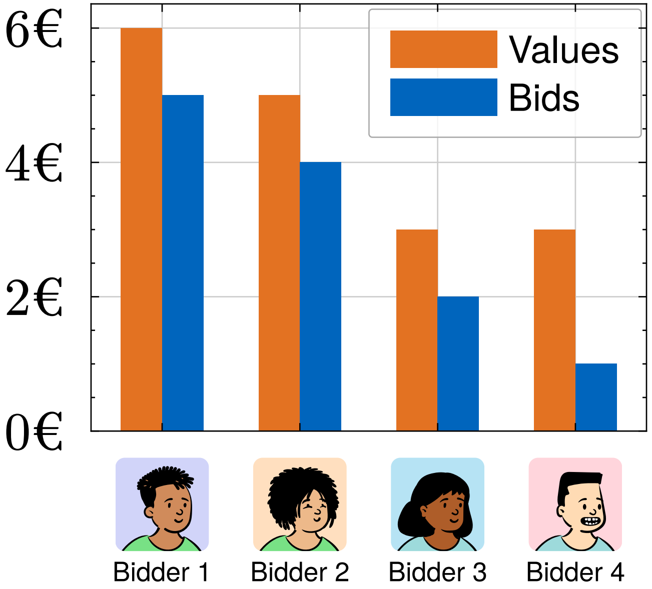

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- Payments? **Payment rule** $t(b) \in \mathbb{R}^n$.
  - We let  pay their bid, so $t(b) = (5, 0, 0, 0)^\mathsf{T}$

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Which **effects** are desirable? **Design goals**.

Auction design goals

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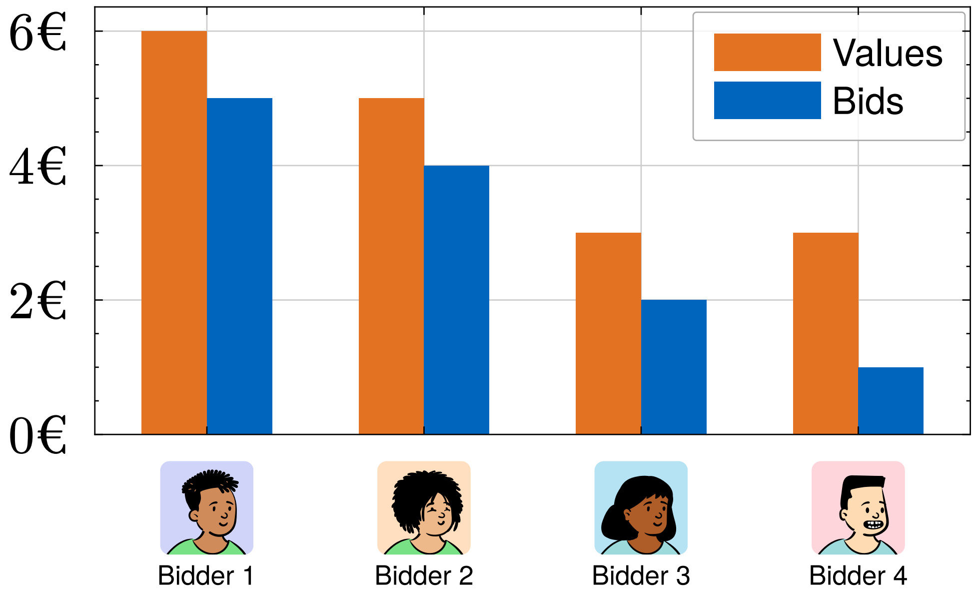

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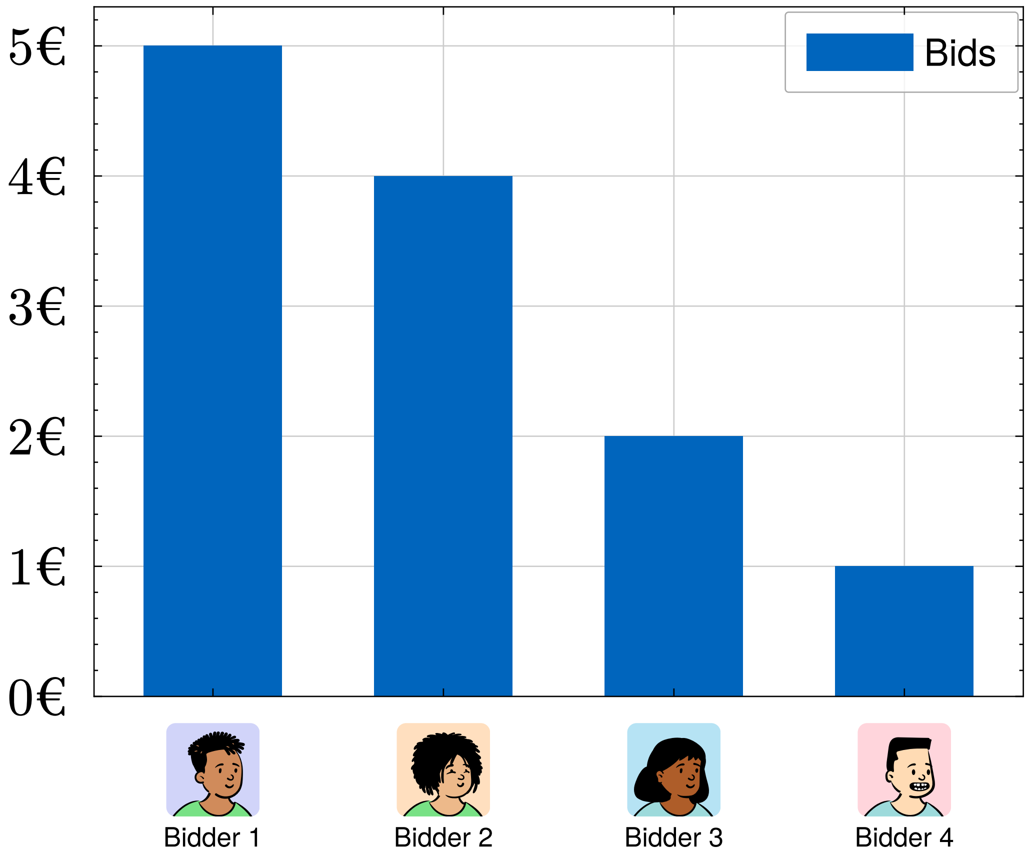
Interdependent value

- Private part: Individual values
- Common part: Value of artist's past work

FPSB and SPSB[6]

Let's design our first auctions.

Assume $b_1 \geq b_2 \geq \ldots \geq b_n$.



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First-price sealed-bid (FPSB)

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- $x(b) = (1, 0, \ldots, 0)^\mathsf{T}$
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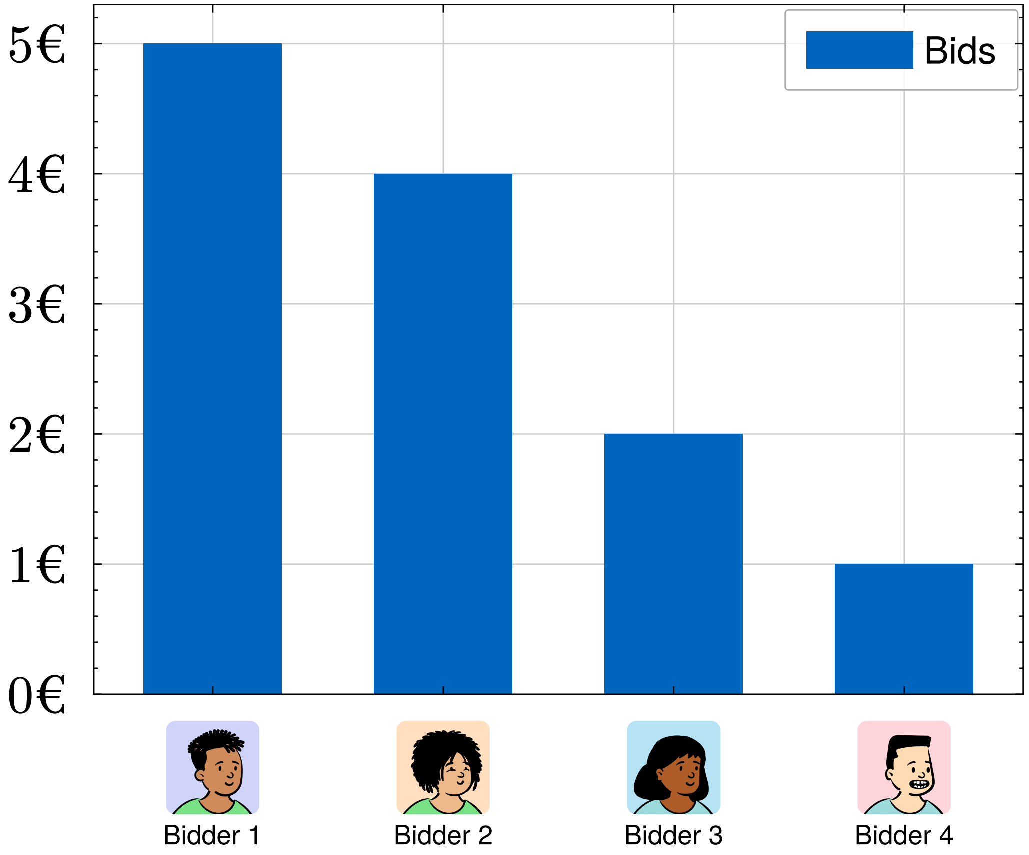

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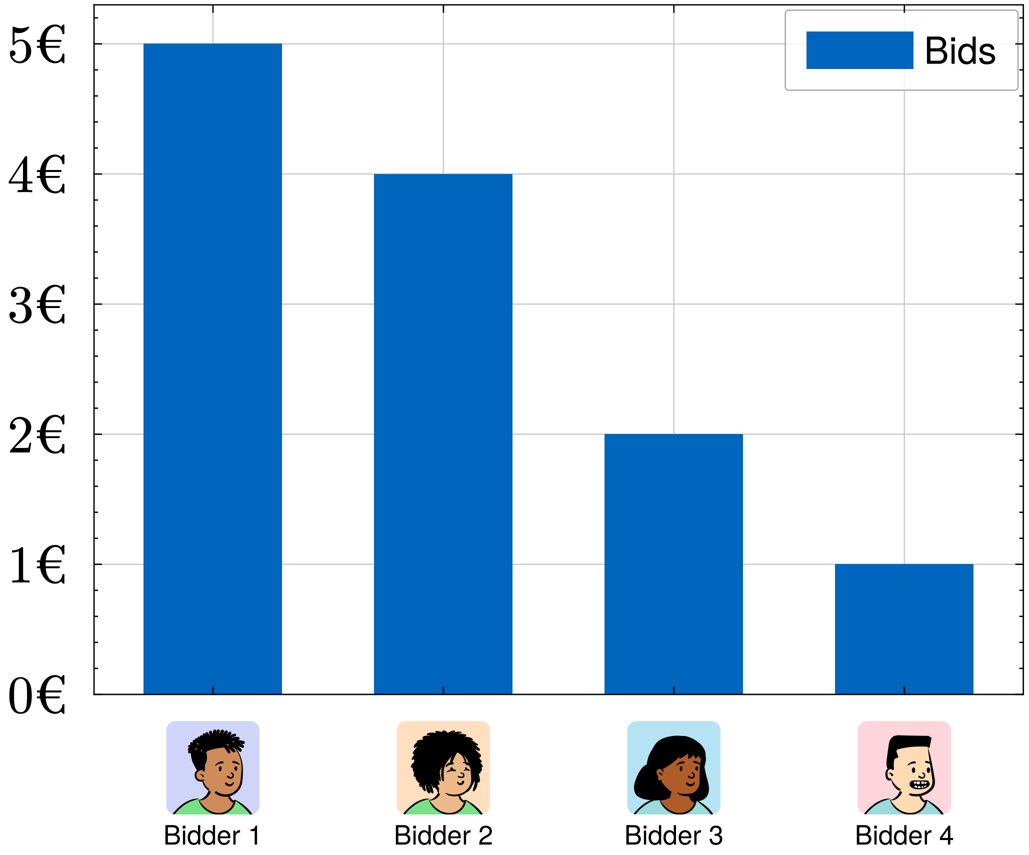
Second-price sealed-bid (SPSB)[7]

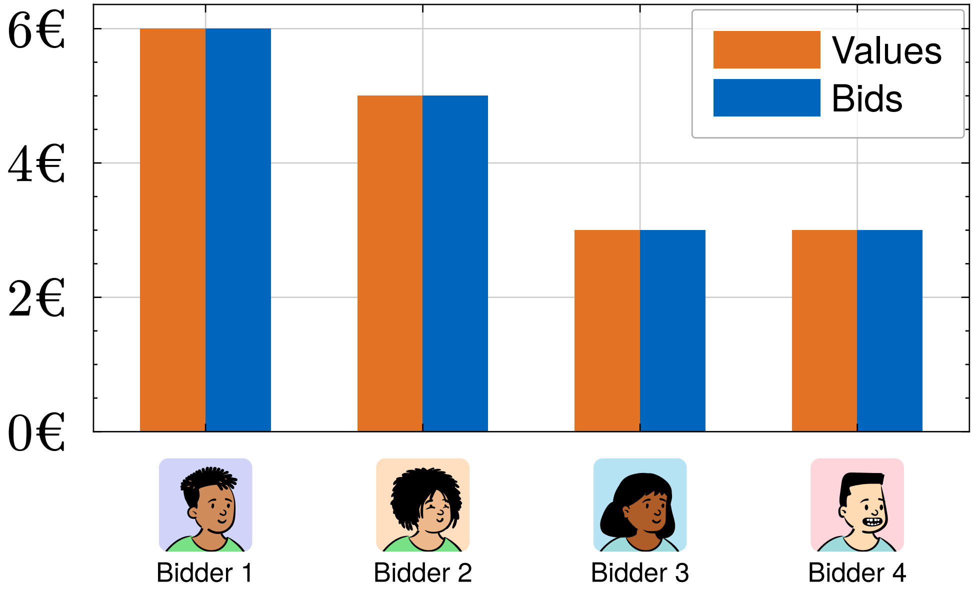
-  wins and pays bid of 
- $x(b) = (1, 0, \ldots, 0)^\mathsf{T}$
- $t(b) = (b_2, 0, \ldots, 0)^\mathsf{T}$



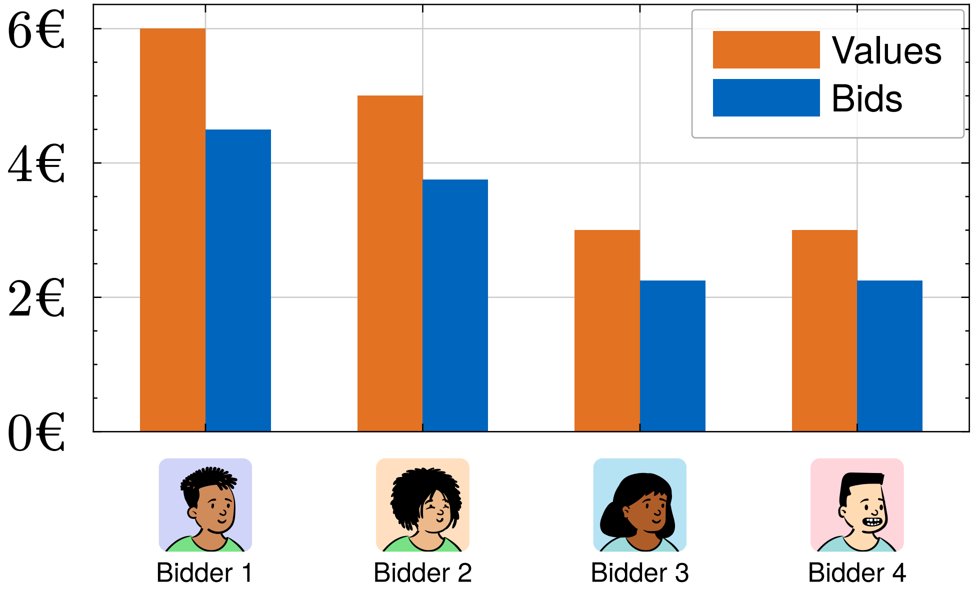
Why charge the second highest bid?

Auction's are **games** for bidders to play.

Design rules influence actions, **equilibrium bids**, and therefore revenue.



Equilibrium (DSE) strategy in **SPSB**



Equilibrium (BNE) strategy in **FPSB**

Quasi-linear utility

Given bid-profile b, the **utility** for bidder i is $u_i(b) := \overbrace{x_i(b)}^{\in\{0,1\}} \cdot v_i - \overbrace{t_i(b)}^{\in\mathbb{R}}$.

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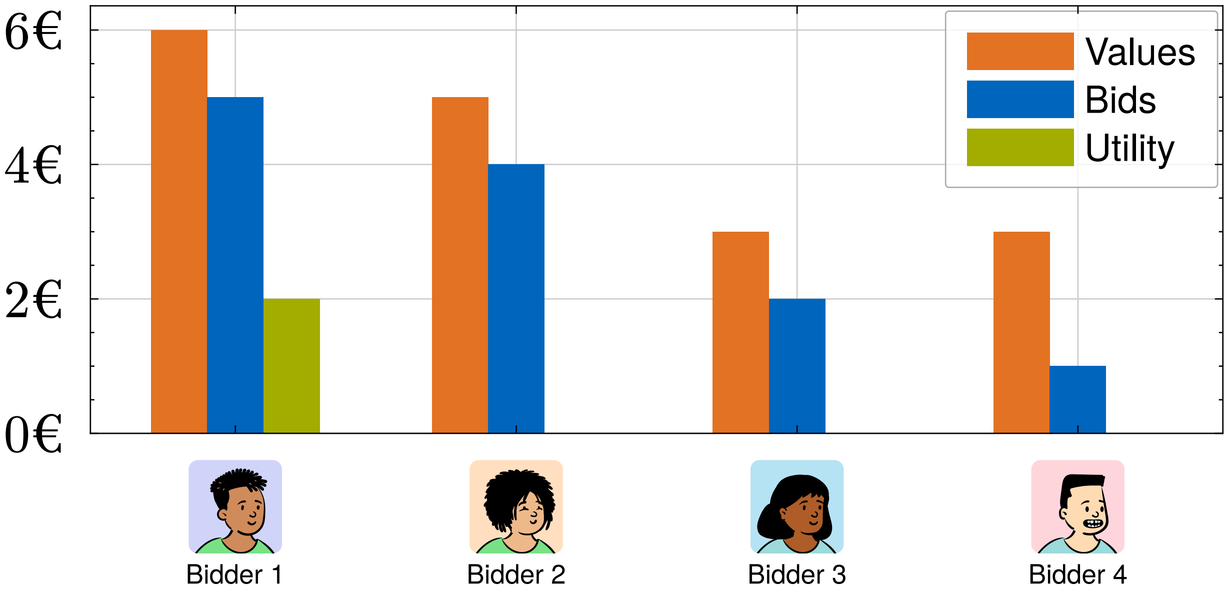
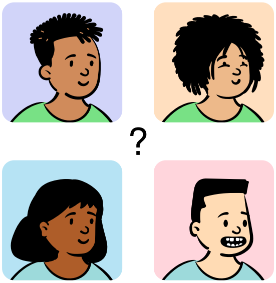


Utility in an **SPSB** auction

Auctions as games of *incomplete information*

Prisoner's dilemma: Payoff matrix is common knowledge.

Sealed-bid auction: True values are private.



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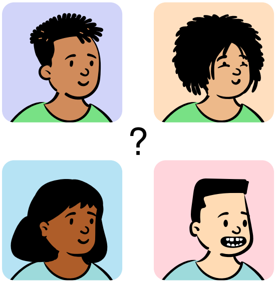

We will see:

Analyzing an equilibrium under uncertainty about values requires **strategies** to specify a bid for **every possible value** of a bidder.

Auction strategy

Strategy $s_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ defines bid $s_i(v_i)$ for all v_i of bidder i in a sealed-bid auction.

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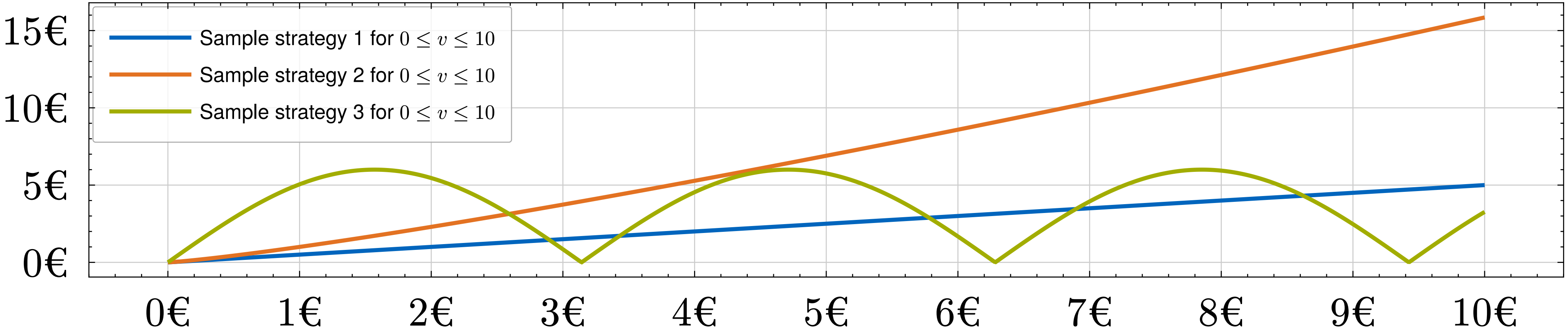

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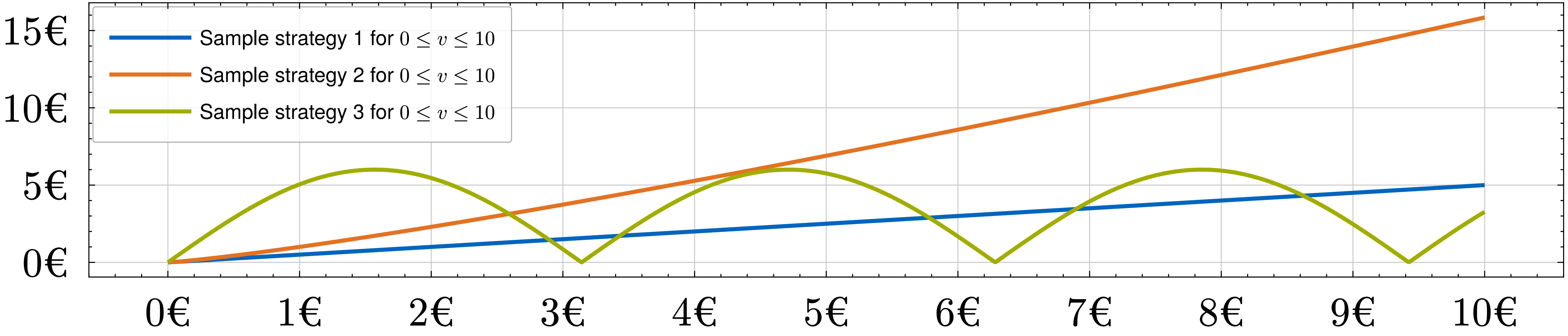

Notation

- $s(v) := (s_1(v_1), \ldots, s_n(v_n))$ for *value profile* $v := (v_1, \ldots, v_n)$
- $s_{-i}(v_{-i}) := (s_1(v_1), \ldots, s_{i-1}(v_{i-1}), s_{i+1}(v_{i+1}), \ldots, s_n(v_n))$
- s^* for an *equilibrium strategy profile*

Dominant-strategy equilibrium (DSE)

$s^* = (s_1^*, ..., s_n^*)$ is a **DSE** in a sealed-bid auction, iff, for every bidder i,

$$u_i(s_i^*(v_i), s_{-i}(v_{-i})) \geq u_i(b_i, s_{-i}(v_{-i})) \ \forall v_i, b_i, v_{-i}, s_{-i}.$$

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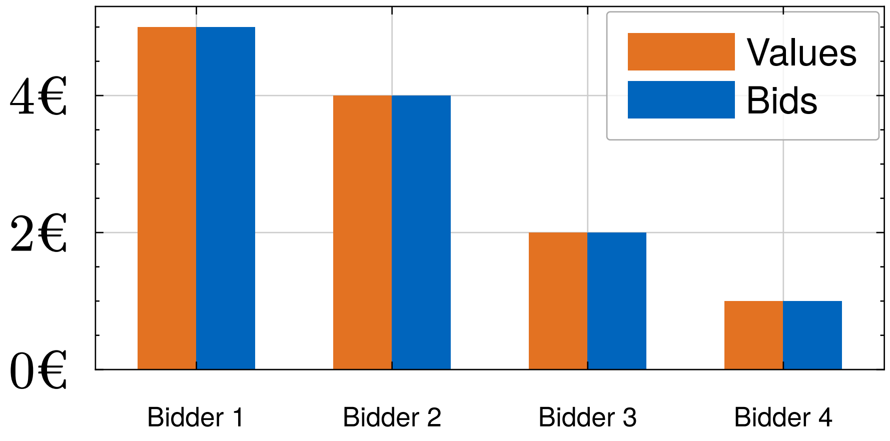
A sealed-bid auction is **strategy-proof**, if truthful bidding is a DSE.

Truthful bidding: $s_i(v_i) = v_i$ for bidder i.



SPSB is strategy-proof and efficient[7]

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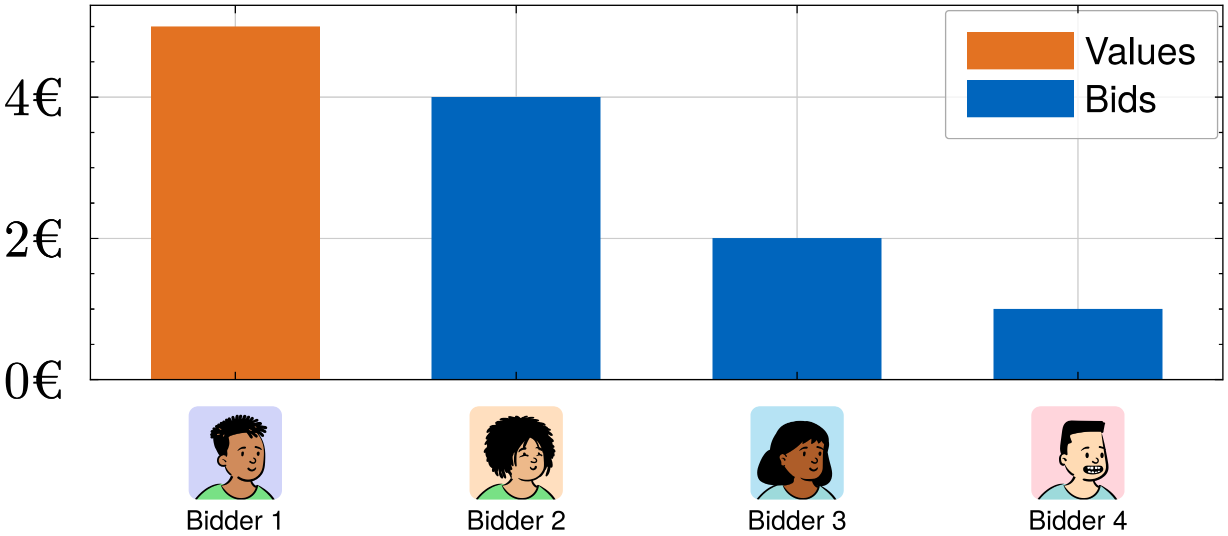

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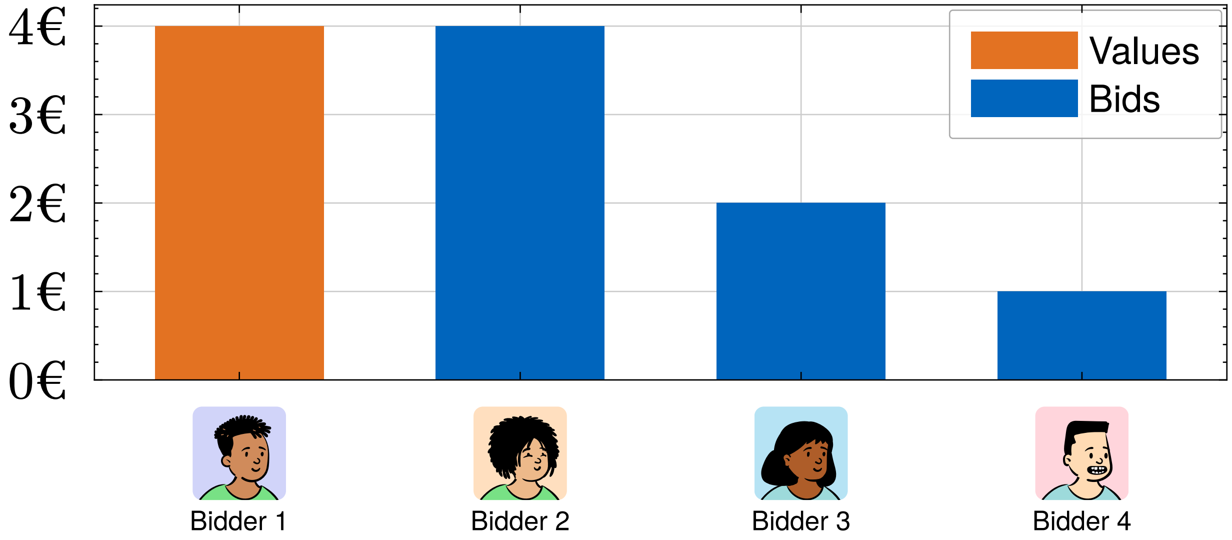

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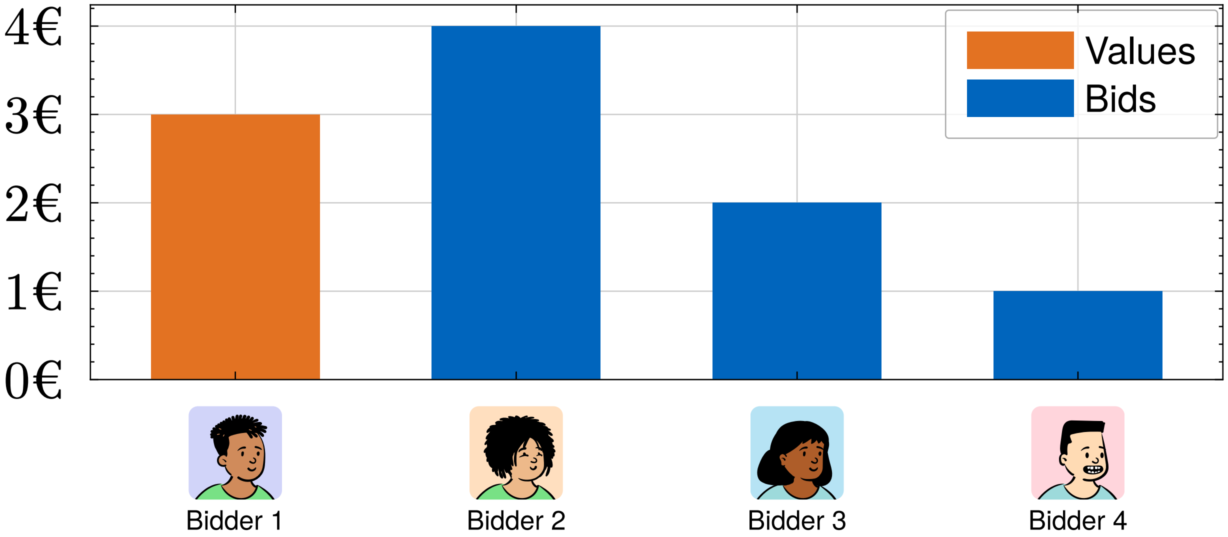
Case 3: $v_1 < b'$: Bid anything less than b' to avoid utility $v_1 - b' < 0$ in favor of utility 0 when loosing.



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In all cases, truthful bid $b_1 = v_1$ is a best response $\iff$ **strategy-proof**.

SPSB is **efficient**, as truthful bidding is a DSE. The item is allocated to the bidder with the highest true value.

Uncertainty of other’s values

Bids of others remain uncertain, but in an FPSB auction, bidders prefer to bid a smaller amount while still winning.

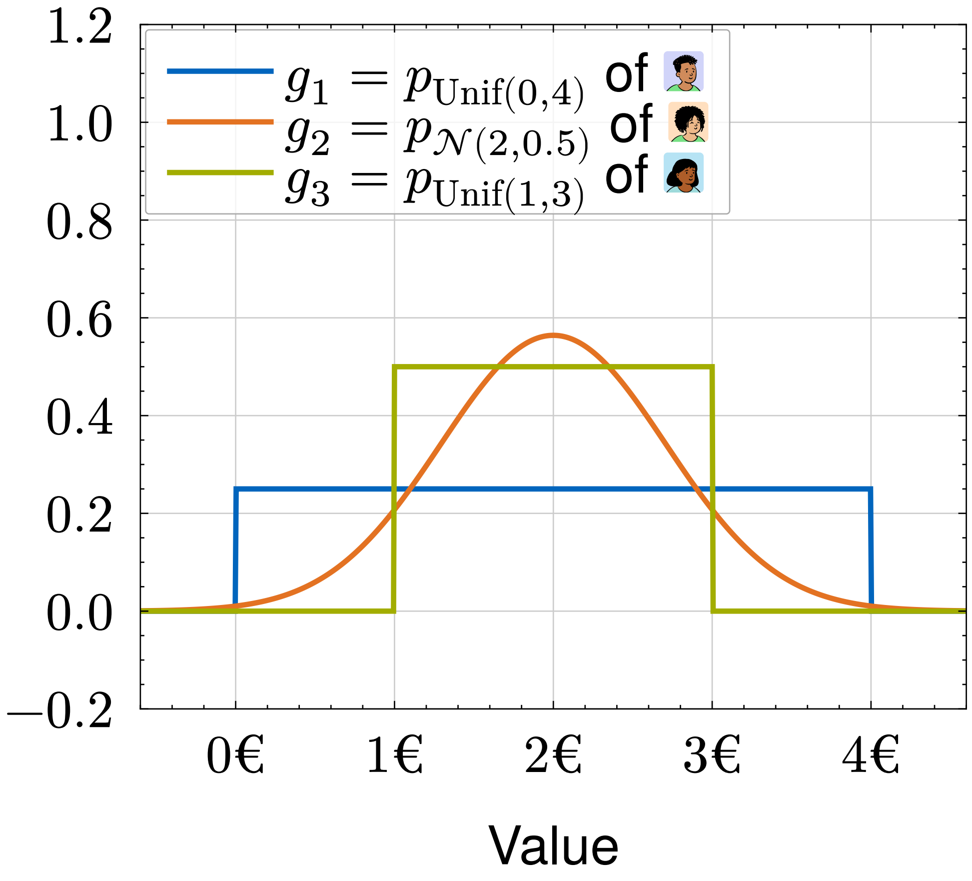
Simulation of shading

Independent private value (IPV) environment

Model uncertainty of values as an **IPV** environment:

- Values v_i sampled independently from **probability distributions** G_i
- G_i continuously differentiable with full support on $[0, v_{\max}]$, such that pdf $g_i(z) > 0$ for $z \in [0, v_{\max}]$
- $G_1, \ldots, G_n$ are common knowledge among bidders

If G_i are the same $\forall i$, we say **IID private values**.



Bayes-Nash equilibrium (BNE)

Idea: Extend Nash equilibrium to account for uncertainty. Assume *risk neutral* bidders.

$s^* = (s_1^*, ..., s_n^*)$ is a **BNE** in a sealed-bid auction, iff, for every bidder i,

$$\underbrace{\mathbb{E}_{v_{-i}}[u_i(s_i^*(v_i), s_{-i}^*(v_{-i}))]}_{\substack{\text{Expected utility to bidder } i \\ \text{given bid } s_i^*(v_i)}} \geq \underbrace{\mathbb{E}_{v_{-i}}[u_i(b_i, s_{-i}^*(v_{-i}))]}_{\substack{\text{Expected utility when} \\ \text{deviating to bid } b_i}} \ \forall v_i, b_i.$$

Expectations taken w.r.t. distribution of values v_{-i} and, therefore, bids of others.

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DSE	**BNE**
Maximize utility regardless the values and strategies of others	Maximize expected utility given beliefs about other players' values

BNE might break if a player assumes non-rational behavior or different distributions.

BNE of an FPSB auction

For IID private values with bounded support, the FPSB auction has a **unique symmetric** and **increasing BNE**[8].

- *Symmetric* equilibrium: Every bidder has the same strategy.
- *Increasing* equilibrium: Bids are strictly increasing with value.

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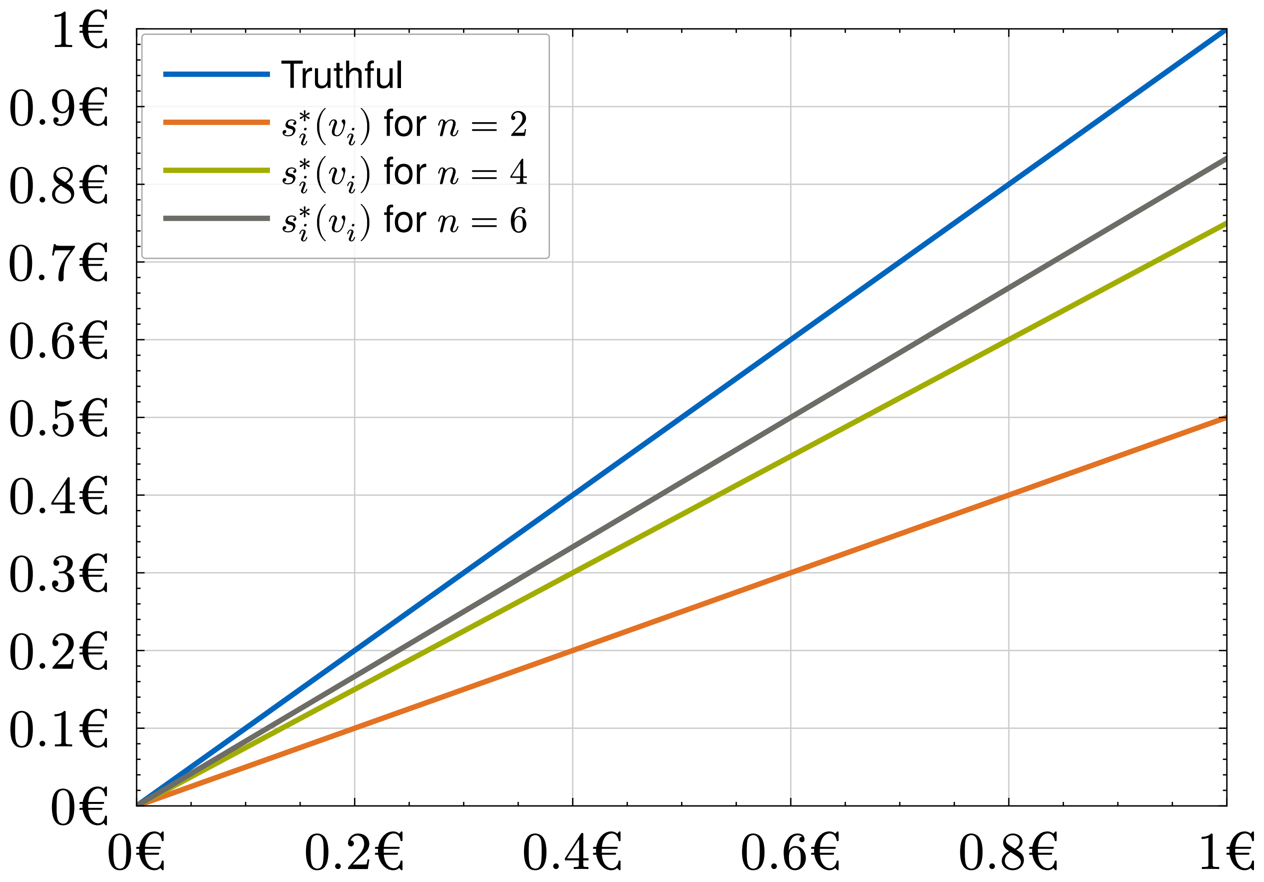
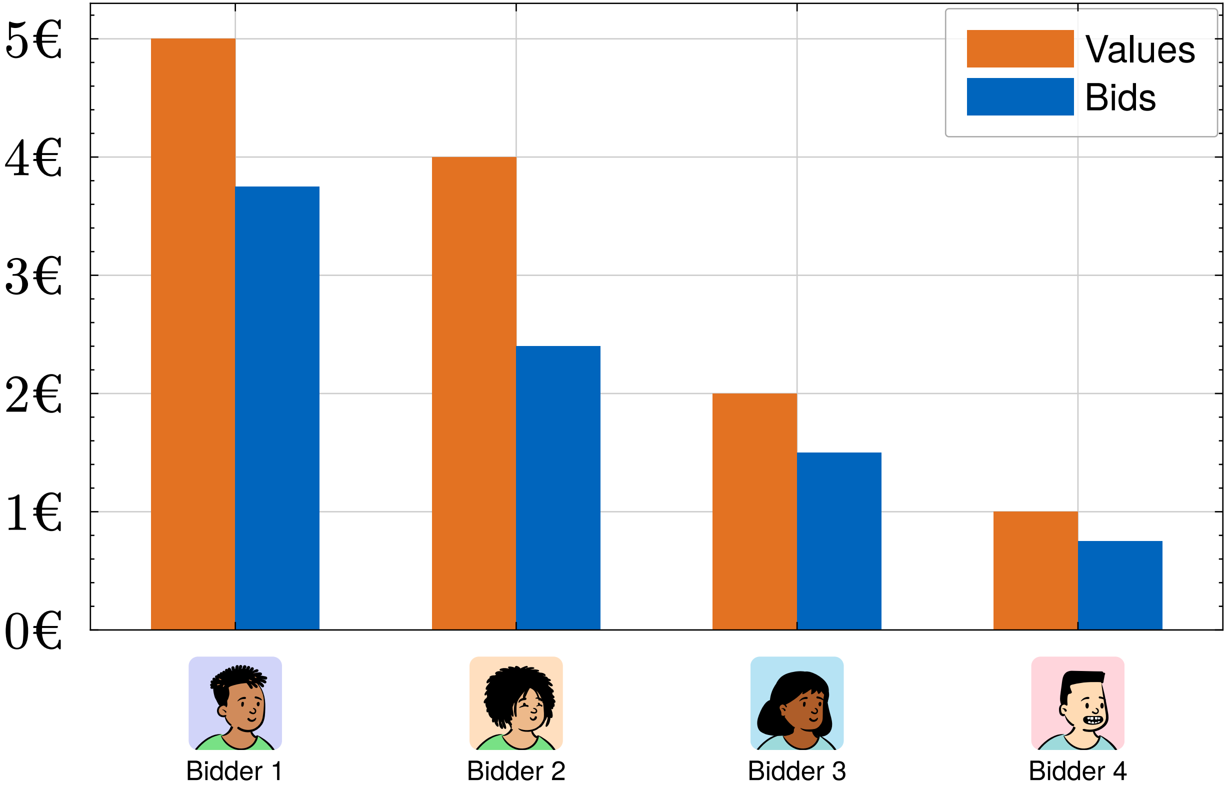
BNE in an FPSB auction with IID values $v_i \sim \mathrm{Unif}(0, 1)$ is for each bidder to play strategy

$$s_i^*(v_i) = \frac{n-1}{n} v_i.$$

BNE of an FPSB auction with IID values $v_i \sim \mathrm{Unif}(0,1)$



Every DSE is a BNE, but not every BNE is a DSE



Bidder 1 maximizes their expected utility. Not a DSE, since they could have bid less.

SPSB vs. FPSB

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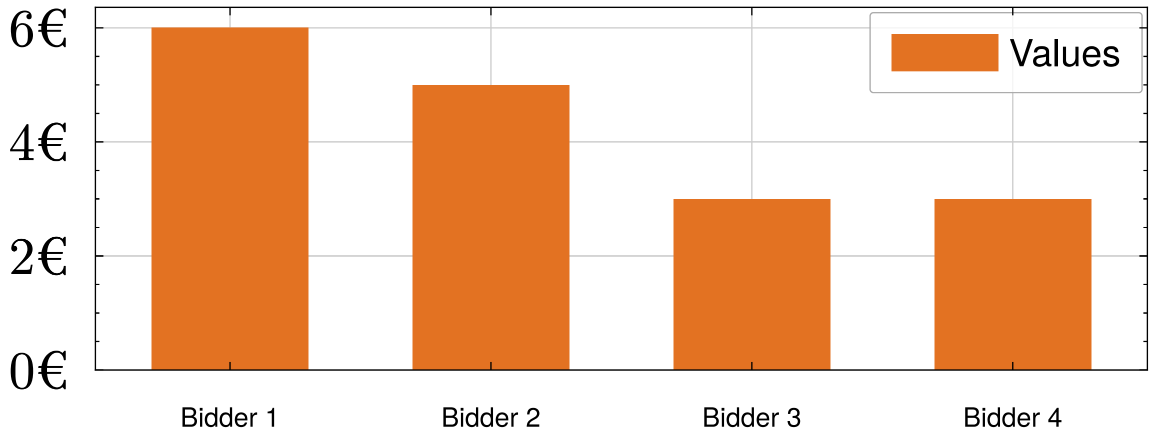

Intermediates represent bidders $\{1, 2\}$ and bidders $\{3, 4\}$, submit bids $b_1 = 6$ and $b_2 = 3$. Revenue becomes $3€$ instead of $5€$.

SPSB vs. FPSB

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- **Not censor-proof**
- **Not credible**
  - Auctioneer can add a *shill-bid* just below the highest bid

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What does revenue look like in single-item settings for SPSB and FPSB auctions?

SPSB collects **less revenue** than FPSB for the same bid profile b *but* DSE in SPSB: **Truthful**, BNE in FPSB: Bid **less than value**

Simulation of revenue

Interim qualities[9]

Allocation, payment, utility for bidder i when knowing own v_i, but not v_{-i}.

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We want to reason about the **expected revenue** of the FPSB and SPSP auctions:

$$\text{Rev} = \sum_{i=1}^{n} \underbrace{\mathbb{E}_{v_i}[t_i^*(v_i)]}_{\substack{\text{Expected payment} \\ \text{without knowledge of values}}}$$

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Adding these inequalities and cancelling out terms gives $(v_i - v_i')(x_i^*(v_i) - x_i^*(v_i')) \geq 0.$

W.l.o.g. let $v_i > v_i'$. Then $x_i^*(v_i) \geq x_i^*(v_i')$.

Thus, $x_i^*(v_i)$ must be **monotone weakly increasing** in v_i.

Interim payment identity

Assume bidder i with true value z considers following the equilibrium strategy for a **slightly larger value**.

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$$\underbrace{\left.\frac{\mathrm{d}t_i^*(v_i)}{\mathrm{d}v_i}\right|_{v_i=z}}_{\substack{\text{Rate of increase}\\ \text{in interim payment}}} = \underbrace{z\left.\frac{\mathrm{d}x_i^*(v_i)}{\mathrm{d}v_i}\right|_{v_i=z}}_{\substack{\text{Rate of increase}\\ \text{in interim value}}} \quad \forall z.$$

- $\mathrm{LHS} < \mathrm{RHS}$: Bidding as if having a **larger value** than z would be **useful**.
- $\mathrm{LHS} > \mathrm{RHS}$: Bidding as if having a **smaller value** than z would be **useful**.

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Integrating this and writing t_i' and x_i' for the derivatives, we get

$$\int_{z=v_{\min}}^{v_i} t_i'(z)\,\mathrm{d}z = \int_{z=v_{\min}}^{v_i} z x_i'(z)\,\mathrm{d}z$$

$$\iff t_i^*(v_i) - t_i^*(v_{\min}) = [z x_i^*(z)]_{v_{\min}}^{v_i} - \int_{z=v_{\min}}^{v_i} x_i^*(z)\,\mathrm{d}z$$

$$\iff t_i^*(v_i) = v_i x_i^*(v_i) - \int_{z=v_{\min}}^{v_i} x_i^*(z)\,\mathrm{d}z - \underbrace{(v_{\min} x_i^*(v_{\min}) - t_i^*(v_{\min}))}_{:=C_i \text{ constant}}$$

Bayes-Nash characterization

In any BNE of any sealed-bid auction, we must have, for bidder i with value v_i,

- **Interim monotonicity**: $x_i^*(v_i)$ is monotone weakly increasing in v_i.
- **Interim payment identity**: We must have

$$t_i^*(v_i) = v_i x_i^*(v_i) - \int_{z=v_{\min}}^{v_i} x_i^*(z)\,\mathrm{d}z - C_i$$

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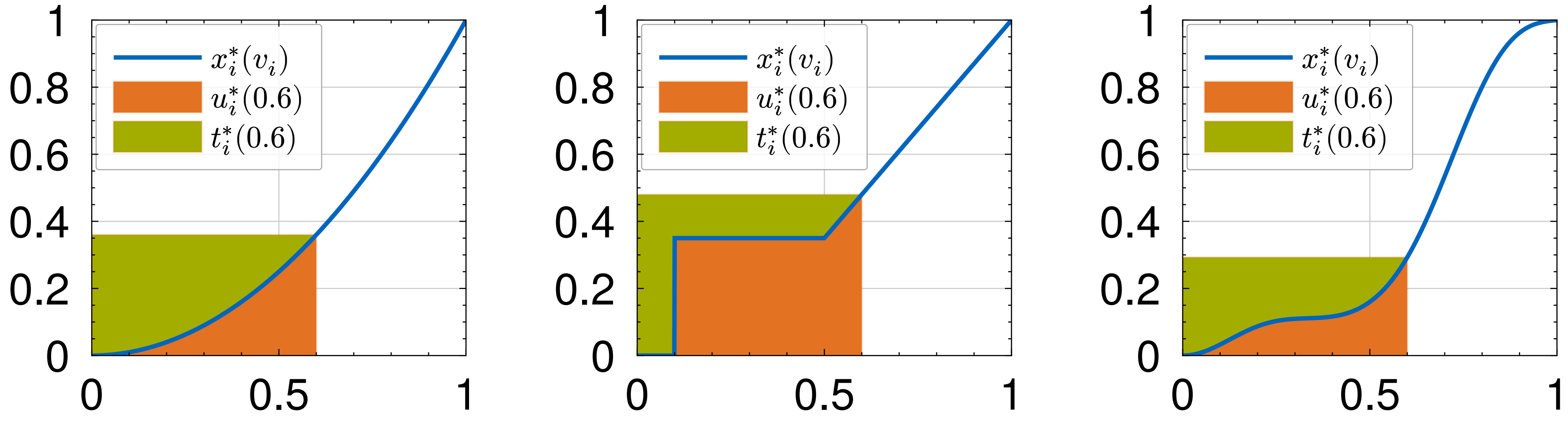
  for constant $C_i := v_{\min} x_i^*(v_{\min}) - t_i^*(v_{\min}) = u_i^*(v_{\min})$.

An auction is called **normalized**, if $u_i^*(v_{\min}) = 0$.

Visualizing the interim payment identity



Interim allocation, payment, and utility for a bidder i

Interim payment identity for a normalized auction:

$$t_i^*(v_i) = v_i x_i^*(v_i) - \int_{z=v_{\min}}^{v_i} x_i^*(z)\,\mathrm{d}z.$$

Revenue equivalence[9]

Interim payment identity in a normalized auction is

$$t_i^*(v_i) = v_i x_i^*(v_i) - \int_{z=v_{\min}}^{v_i} x_i^*(z)\,\mathrm{d}z.$$

Expected revenue is $\mathrm{Rev} = \sum_{i=1}^{n} \mathbb{E}_{v_i}[t_i^*(v_i)]$.

Any two normalized auctions with the same interim allocations have the same interim payment (interim payment identity) and, thus, the same expected revenue.

Implication: All normalized efficient auctions have the same expected revenue.

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Higher revenue needs a different allocation, e.g., by introducing a **reserve price**.

Overview

- Allow for **responses** to bids
- Important for **common** or **interdependent value** environments
- Provide **transparency** and trust in auction rules

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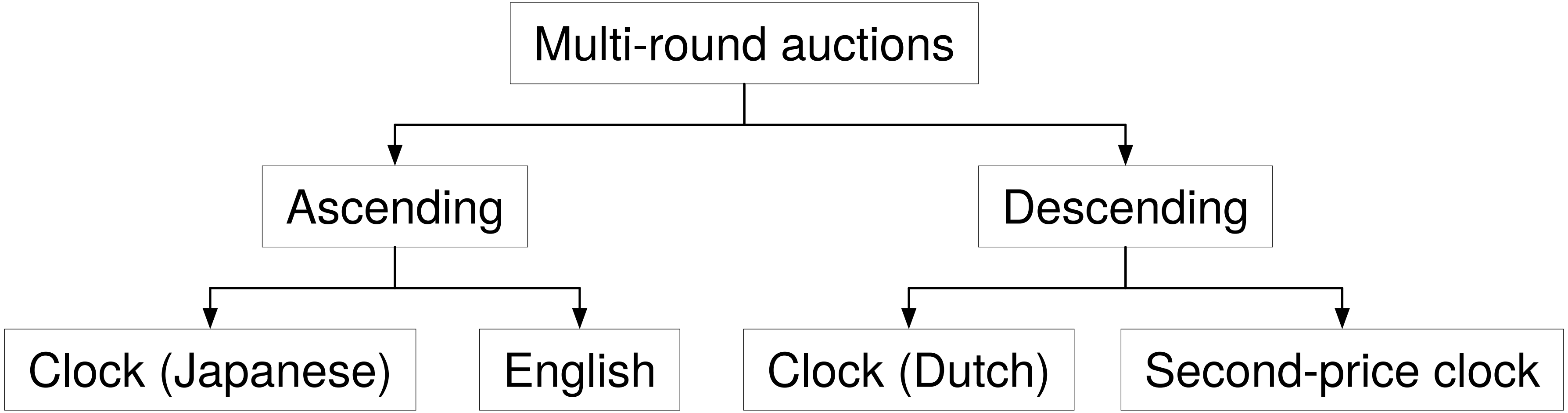
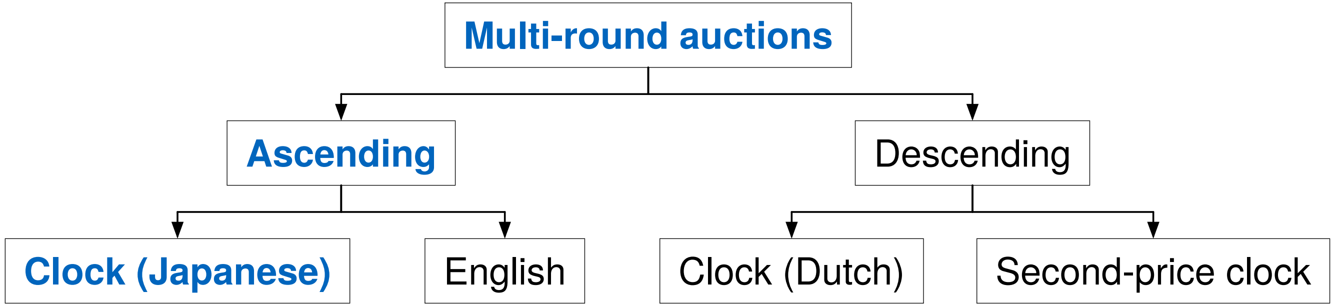
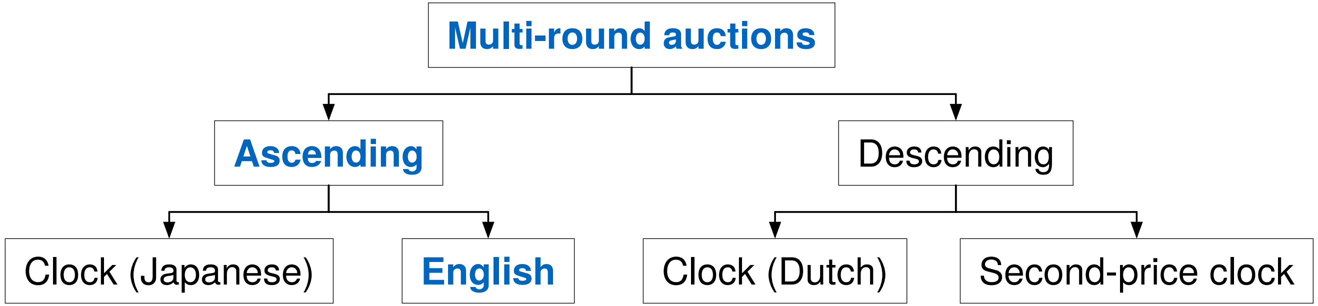


Ascending-clock auction[10]



- Price starts low and increases continuously with time
- Bidders can privately drop out at any time
- Last bidder remaining wins and pays final price
- Ties are broken at random

English auction[6]



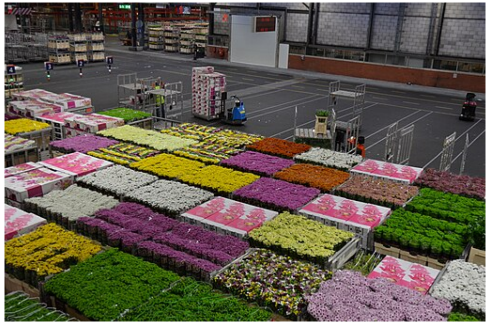
- Price starts low
- Public bids at or above current price
- Current highest bidder is *provisional winner*
- Auction price at *minimal bid increment* above highest bid
- Closes after period with no bidding and item is sold to provisional winner at their last bid

Auction at Christie's[11]

Descending-clock auction[6]

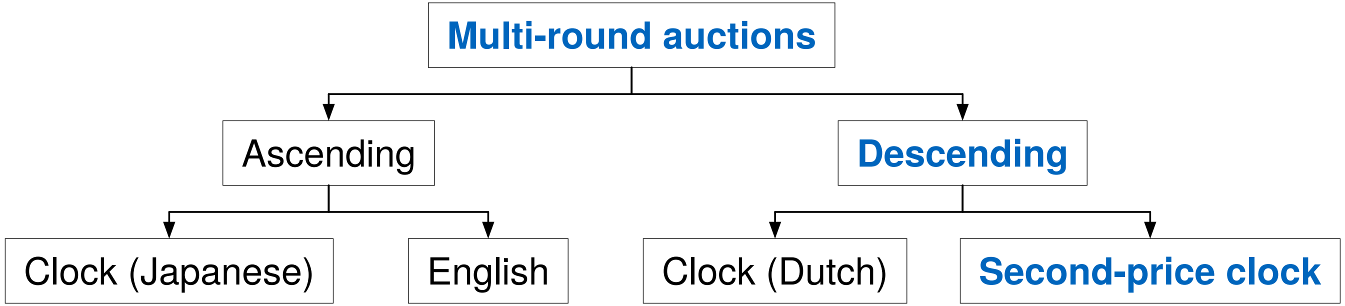
- **Multi-round auctions**
  - Ascending
    - Clock (Japanese)
    - English
  - **Descending**
    - **Clock (Dutch)**
    - Second-price clock

- Price starts high and decreases continuously with time
- Bidders can place bid at any time at current price
- Stops at first bid, with this bidder paying the final price



Dutch flower auction[12]

Second-price clock auction



- Similar to descending-clock auction
- First bid only visible to auctioneer
- Closes at second bid
- First bidder wins and pays the price at the time of the second bid

Strategic equivalence

Pair of sealed-bid and multi-round auctions are **strategically equivalent**, if, for any s in one auction, there exists s' in the other such that outcomes are the same for all v.

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Falling price, first bid wins

Second-price descending-clock auction
Falling price, first bid wins at second price

Ascending-clock auction
Rising price, last remaining wins

English auction
Rising price, public bids, closes after inactivity

SPSB auction

FPSB auction

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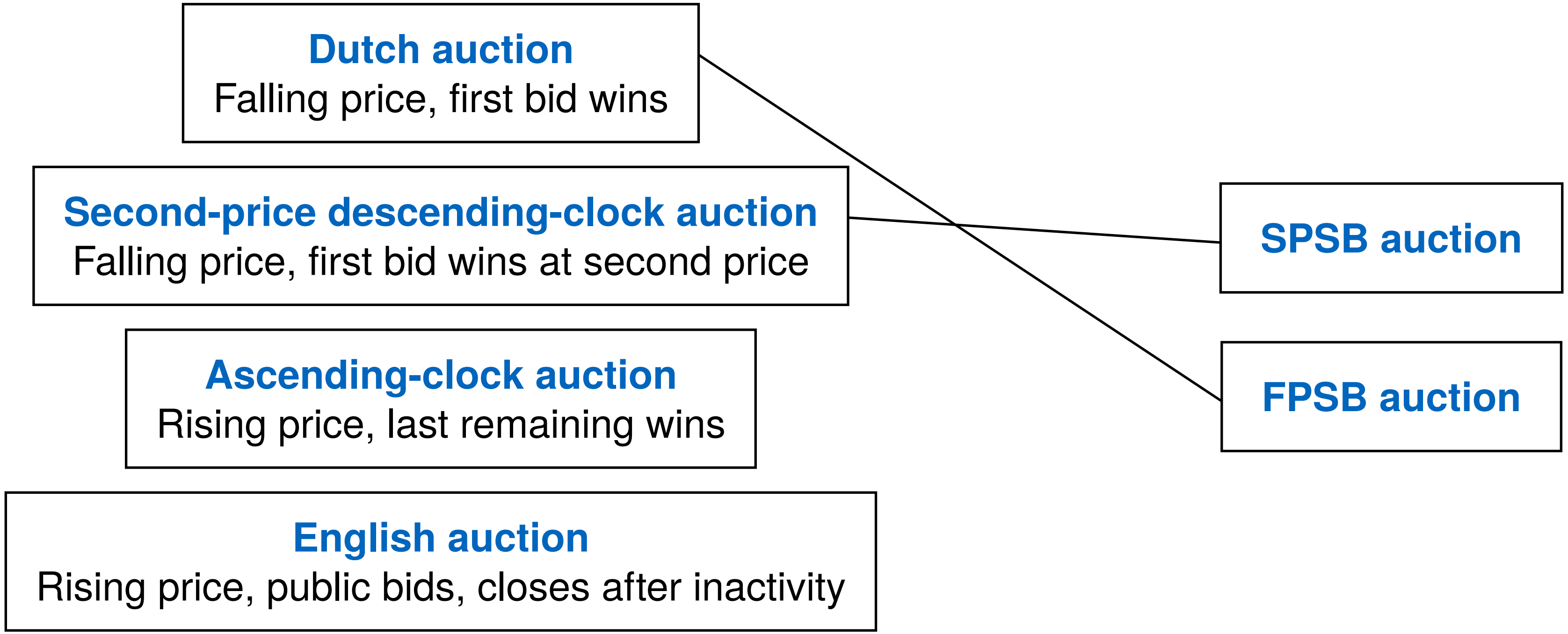

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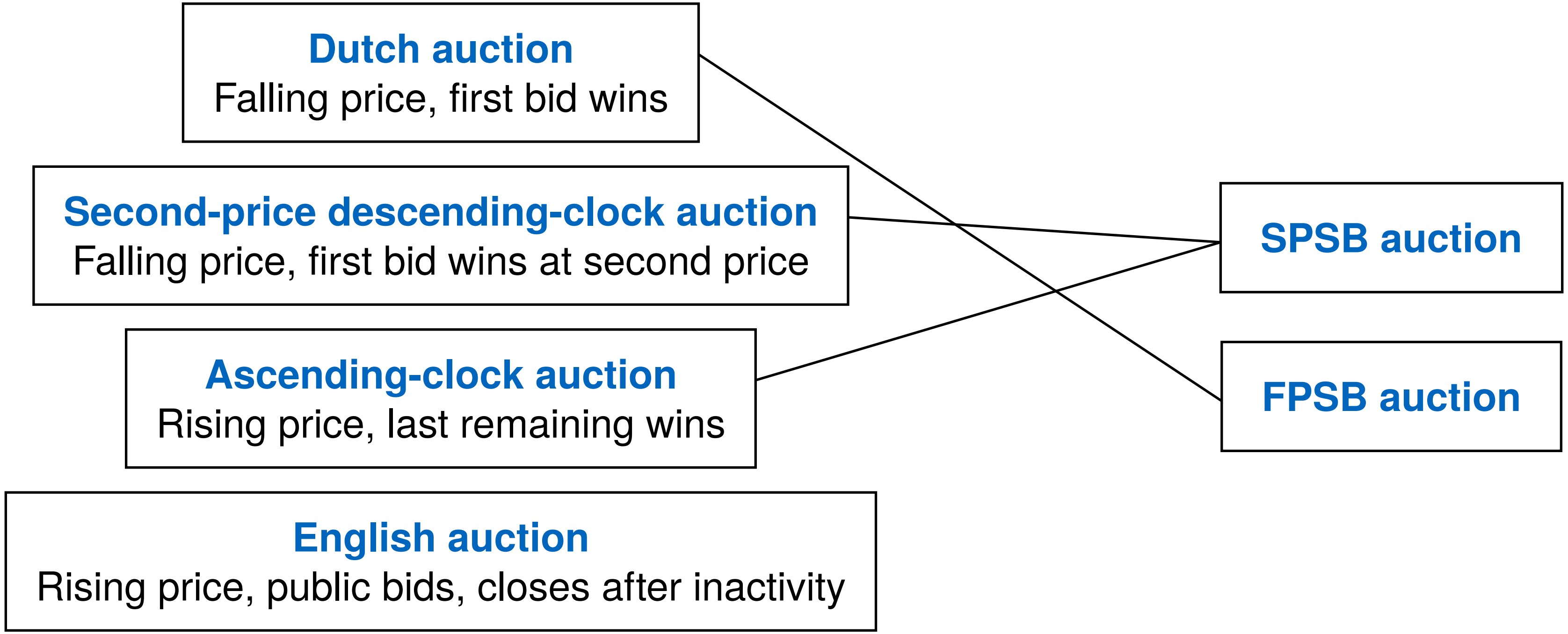

Summary

Auctions are characterized by their:

- **Format**: Rules, number of items, participants.
- **Information**: Bids of others, values (*private*, *common*, *interdependent*).

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Sealed-bid, single-item auctions:

- **SPSB**
  - Strategy-proof and efficient
  - Not censor-proof and not credible
- **FPSB** *(for IID private values with bounded support)*
  - Unique symmetric, increasing BNE makes it efficient
  - BNE more brittle than DSE
  - Incentivizes bid shading

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Revenue equivalence: Any two normalized, sealed-bid auctions having a BNE with an identical interim allocation share the *same expected revenue* in these BNE.

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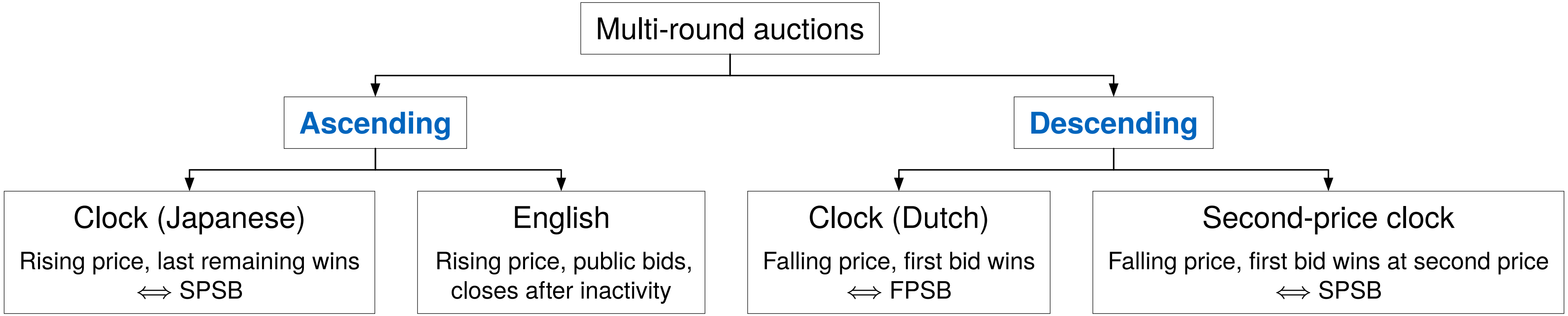
Multi-round auctions:



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Concepts are additionally referenced from[13]. Illustrations of the bidders are from[14]. Uncited images have been generated with[15].[12] is licensed under CC BY-SA 2.0[16].

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