Auction Design

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Auctions are everywhere, and their design matters intensely. Whether Google is auctioning an ad slot in search results[1] or a government is selling its debt[2,3], the structure of these auctions influences key properties such as revenue, incentives, and the behavior of participants[4].

Characterization of auctions

Auctions differ in *format* and *information*[5].

The **format** of an auction includes[4,5]:

- *Auction rules*: Announcement/update of prices, placement of bids, allocation of items, payments.
- *Number of items*: Single-item vs. multi-item auctions.
- *Participants*: Forward (one sells to many) vs. reverse (one buys from many) vs. double auctions (multiple sellers and buyers).

Meanwhile, **information** in an auction can be characterized by[4,5]:

- Models of bidders' *values*: Private vs. common vs. interdependent value.
- Information on *bids* by other participants.

We will focus on single-item, forward auctions by first examining sealed-bid and then multi-round variants.

Sealed-bid auctions

Mathematical model[4]

In a sealed-bid auction, bidders $N := \{1, ..., n\}$ are interested in buying an item for which they have a value v_i. The value describes the maximum amount a bidder is willing to pay for the item.

Bidders submit **sealed bids** $b_i \geq 0$ without knowledge of other bids. A bid must not necessarily equal the value of a bidder. In fact, we will see that for specific auction designs, it proves useful in equilibrium to deviate from bidding one's actual value.

The bids of all bidders comprise the **bid-profile** $b := (b_1, ..., b_n)$. The format of a sealed-bid auction is then defined by:

- An **allocation rule** $x(b) \in \{0, 1\}^n$. This decides who wins the auction.
- A **payment rule** $t(b) \in \mathbb{R}^n$. This defines the payment made by each bidder.

> **Example** Let's say four bidders compete in a sealed-bid auction for a new pair of shoes and submit bids $b = (85, 75, 60, 30)$. As the auctioneer, we decide to let bidder one win and have them pay their bid. The allocation would then be $x(b) = (1, 0, 0, 0)^\mathsf{T}$ with payment $t(b) = (85, 0, 0, 0)^\mathsf{T}$.

Design goals

When deciding on these rules of an auction, two design goals are typically desirable[4].

- **Revenue maximization**: Maximize the expected revenue to the seller. A revenue-maximizing auction is called *optimal*.
- **Allocative efficiency**: In an equilibrium, allocate the item to the bidder with the highest value for all value profiles. Such an auction is referred to as *efficient*.

Modeling bidders' values

When discussing values of bidders, we need a way to precisely model what others know about them and how they are affected by information on others' values. We distinguish between the following settings[4,6].

- **Private value**: Bidders know their own private value, which is unaffected by knowledge of the values of other bidders.

> **Example** The shoe auction from the previous example could be a private value auction.

- **Common value**: Bidders are uncertain about a common value. If they had the same information, they would have the same value.

> **Example** Imagine an auction of an oil field, where the value of the oil would be equal for all, but the amount of oil present is not known.

- **Interdependent value**: Bidders are uncertain about their own value. Values may differ between bidders. In such settings, bidders' values are comprised of a private and a common value part.

> **Example** This could be the case in an art auction, where bidders have idiosyncratic tastes, the private value component, but a common value is ascribed to the piece, e.g., by the popularity of the artist causing the entire collection to be inherently valuable.

We will focus on private value environments and discuss common and interdependent ones as part of multi-round auctions.

FPSB and SPSB auction[4,6]

Let's design our first sealed-bid, single-item auctions.

- **First-price sealed-bid** (FPSB): The item is allocated to the highest bidder, who pays their bid.
- **Second-price sealed-bid** (SPSB)[7]: The item is allocated to the highest bidder, who pays the second highest bid.

It may seem unintuitive to opt for only charging the second highest bid in the SPSB auction. Evidently, it decreases revenue for the same bid profile b.

But auctions are **games of incomplete information**. When analyzing their equilibriums, we will notice the impact of these rules on the behavior of bidders.

In particular, truthful bidding will turn out to be an equilibrium strategy for SPSB auctions, while rational bidders would want to bid less than their value in an FPSB equilibrium. This makes it unclear, at first, how expected revenue differs between these auction designs. We aim to find out by discussing their equilibriums.

To do so, we need to introduce how we define utility and strategies for the game that is an auction.

Quasi-linear utility[4]

Given a bid-profile b, the **utility** for bidder i is $u_i(b) := x_i(b) \cdot v_i - t_i(b)$.

It is called *quasi*-linear, as we will only require linearity in the payment. A change of price always has the same linear effect on quasi-linear utility no matter a bidder's value.

Strategies

As mentioned, auctions are games of incomplete information. To analyze equilibriums under uncertainty about values, we require strategies to specify how to bid for every possible value of a bidder. We will focus on pure strategies.

A **strategy** $s_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defines a bid $s_i(v_i)$ for every value v_i of bidder i in a sealed-bid auction[4].

> **Example** A simple strategy for all bidders i would be to always bid their value, i.e., $s_i(v_i) = v_i$. This is referred to as **truthful bidding**[4].

We adopt the following notation.

- $s(v) := (s_1(v_1), ..., s_n(v_n))$ for value profile $v := (v_1, ..., v_n)$.
- $s_{-i}(v_{-i}) := (s_1(v_1), ..., s_{i-1}(v_{i-1}), s_{i+1}(v_{i+1}), ..., s_n(v_n))$. In general, we will let the subscript v_{-i} denote dropping the i-th entry from a vector or tuple v.
- s^* for an equilibrium strategy profile.

Equilibrium of an SPSB auction

Dominant-strategy equilibrium (DSE)[4]

Strategy profile $s^* = (s_1^*, ..., s_n^*)$ is a **DSE** in a sealed-bid auction, iff, for every bidder i,

$$u_i(s_i^*(v_i), s_{-i}(v_{-i})) \geq u_i(b_i, s_{-i}(v_{-i})) \ \forall v_i, b_i, v_{-i}, s_{-i}.$$

In a DSE, each bidder has a strategy that maximizes their utility no matter the values and strategies of others.

Strategy-proofness

An auction is called **strategy-proof**, if truthful bidding is a DSE.[4]

SPSB auctions are strategy-proof and efficient

We will see that an SPSB auction is strategy-proof, i.e., truthful bidding is a DSE[7]. Since this implies that items are allocated to the bidder with the highest value in equilibrium, an SPSB auction is also efficient.

Equilibrium of an FPSB auction

In an FPSB auction, we will notice that rational bidders prefer to bid less than their value. But bids remain uncertain.

Independent private value (IPV) environment

We will model this uncertainty about values as an **IPV environment**[4]:

- Values v_i are assumed to be sampled independently from probability distributions G_i.
- We require the G_i to be continuously differentiable with full support on $[0, v_{\max}]$, such that the probability density functions $g_i(z) > 0$ for $z \in [0, v_{\max}]$.
- We let $G_1, ..., G_n$ be common knowledge among bidders.

If the G_i are the same for all bidders i, we speak of **IID private values**.

Bayes-Nash equilibrium (BNE)[4]

We extend the Nash equilibrium to account for uncertainty of values. Assuming bidders are risk neutral, such that they want to maximize their expected utility, this leads to the concept of the **Bayes-Nash equilibrium**.

Strategy profile $s^* = (s_1^*, ..., s_n^*)$ is a BNE in a sealed-bid auction, iff, for every bidder i,

$$\mathbb{E}_{v_{-i}}[u_i(s_i^*(v_i), s_{-i}^*(v_{-i}))] \geq \mathbb{E}_{v_{-i}}[u_i(b_i, s_{-i}^*(v_{-i}))] \ \forall v_i, b_i.$$

In a BNE, there is no useful deviation for any bidder from the equilibrium strategy in expectation, given knowledge of one's own value, the distributions of others' values, and assuming other bidders also use s^*. A BNE might break if a player assumes non-rational behavior or different distributions of others, making it less robust than the DSE.

Note that any DSE is a BNE, but not every BNE is a DSE.

BNE of an FPSB auction

It can be shown that for IID private value environments with bounded support, the FPSB auction has a unique symmetric *(every bidder has the same strategy)* and increasing *(bids are strictly increasing with value)* BNE[8]. This implies that the item is allocated to the bidder with the highest value in equilibrium, making the FPSB auction efficient.

Comparing SPSB and FPSB auctions

We will notice that the properties of the SPSB and FPSB auctions result in distinct strengths and weaknesses. However, comparing the expected revenue in a simulation will lead us to an interesting observation. Expected revenue seems equal. This will enable us to hypothesize revenue equivalence, which we will formally demonstrate.

Revenue equivalence

Interim quantities[4,9]

We will need to reason about interim quantities, such as allocation, payment, and utility of a bidder i, when knowing v_i, but being uncertain about v_{-i}.

- **Interim allocation** $x_i^*(v_i) = \mathbb{E}_{v_{-i}}[x_i(s_i^*(v_i), s_{-i}^*(v_{-i}))]$: The probability of being allocated the item in equilibrium.
- **Interim payment** $t_i^*(v_i) = \mathbb{E}_{v_{-i}}[t_i(s_i^*(v_i), s_{-i}^*(v_{-i}))]$: The expected payment in equilibrium.
- **Interim utility** $u_i^*(v_i) = v_i x_i^*(v_i) - t_i^*(v_i)$.

Bayes-Nash characterization[4,9]

By analyzing how these interim quantities should behave in a BNE, we will see that we can require two properties of any BNE of any sealed-bid auction for a bidder i with value v_i:

- **Interim monotonicity**: $x_i^*(v_i)$ is monotone weakly increasing in v_i.
- **Interim payment identity**: We must have

$$t_i^*(v_i) = v_i x_i^*(v_i) - \int_{z=v_{\min}}^{v_i} x_i^*(z)\,\mathrm{d}z - C_i$$

  for the constant $C_i := v_{\min} x_i^*(v_{\min}) - t_i^*(v_{\min}) = u_i^*(v_{\min})$.

We call an auction **normalized**, if $u_i^*(v_{\min}) = 0$.

Revenue equivalence

When analyzing the expected revenue

$$\mathrm{Rev} = \sum_{i=1}^{n} \underbrace{\mathbb{E}_{v_i}[t_i^*(v_i)]}_{\substack{\text{Expected payment} \\ \text{without knowledge of values}}}$$

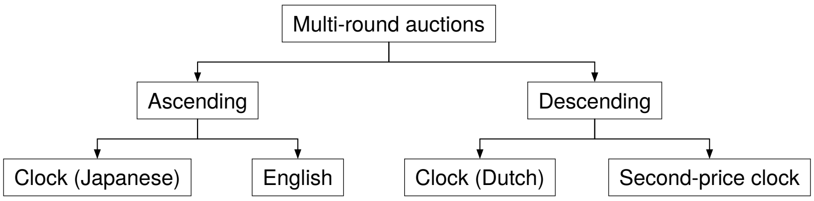
of any two normalized sealed-bid auctions that each have a BNE with an identical interim allocation, we observe that, by the *interim payment identity*, their interim payment must also be the same. Thus, their expected revenue is equivalent[4,9].

In particular, all normalized efficient auctions, such as the FPSB and the SPSB auction, have an identical interim allocation to the bidder with the highest value. Therefore, their revenue is equivalent[4].

Multi-round auctions

Allowing for responses to other bids opens the door to multi-round formats. They are particularly relevant for common or interdependent value environments. In providing transparency, they may aid with building trust that rules are correctly followed[4].

Thinking of auctions as games, a multi-round auction resembles a **sequential-move game**, while a sealed-bid auction would be akin to a **simultaneous-move game**.



Ascending multi-round auctions[4,10]

Ascending-clock[4]
- Price starts low and increases continuously with time
- Bidders can drop out at any time
- Last bidder remaining wins and pays final price
- Ties are broken at random

English[4,6]
- Price starts low
- Public bids at or above current price
- Current highest bidder is *provisional winner*
- Auction price at *minimal bid increment* above highest bid
- Closes after period of inactivity and item is sold to provisional winner at their last bid

Descending multi-round auctions[4]

Descending-clock[4,6]
- Price starts high and decreases continuously with time
- Bidders can place bid at any time at current price
- Stops at first bid, with this bidder paying the final price

Second-price clock[4]
- Similar to descending-clock auction
- First bid only visible to auctioneer
- Closes at second bid
- First bidder wins and pays the price at the time of the second bid

Strategic equivalence

A pair of sealed-bid and multi-round auctions are **strategically equivalent**, if, for any strategy profile s in one auction, there exists s' in the other such that outcomes are the same for all value profiles[4].

We will find that some of the presented multi-round auctions are strategically equivalent to an SPSB or FPSB auction. Feel free to already think about which that might be.

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