1062CH01

REAL NUMBERS 1

1.1 Introduction

In Class IX, you began your exploration of the world of real numbers and encountered irrational numbers. We continue our discussion on real numbers in this chapter. We begin with very important properties of positive integers in Sections 1.2, namely the Euclid's division algorithm and the Fundamental Theorem of Arithmetic.

Euclid's division algorithm, as the name suggests, has to do with divisibility of integers. Stated simply, it says any positive integer a can be divided by another positive integer b in such a way that it leaves a remainder r that is smaller than b. Many of you probably recognise this as the usual long division process. Although this result is quite easy to state and understand, it has many applications related to the divisibility properties of integers. We touch upon a few of them, and use it mainly to compute the HCF of two positive integers.

The Fundamental Theorem of Arithmetic, on the other hand, has to do something with multiplication of positive integers. You already know that every composite number can be expressed as a product of primes in a unique way—this important fact is the Fundamental Theorem of Arithmetic. Again, while it is a result that is easy to state and understand, it has some very deep and significant applications in the field of mathematics. We use the Fundamental Theorem of Arithmetic for two main applications. First, we use it to prove the irrationality of many of the numbers you studied in Class IX, such as $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$. Second, we apply this theorem to explore when exactly the decimal expansion of a rational number, say $\frac{p}{q} (q \neq 0)$, is terminating and when it is non-terminating repeating. We do so by looking at the prime factorisation of the denominator q of $\frac{p}{q}$. You will see that the prime factorisation of q will completely reveal the nature of the decimal expansion of $\frac{p}{q}$.

So let us begin our exploration.

1.2 The Fundamental Theorem of Arithmetic

In your earlier classes, you have seen that any natural number can be written as a product of its prime factors. For instance, $2 = 2$, $4 = 2 \times 2$, $253 = 11 \times 23$, and so on. Now, let us try and look at natural numbers from the other direction. That is, can any natural number be obtained by multiplying prime numbers? Let us see.

Take any collection of prime numbers, say 2, 3, 7, 11 and 23. If we multiply some or all of these numbers, allowing them to repeat as many times as we wish, we can produce a large collection of positive integers (In fact, infinitely many). Let us list a few :

$7 \times 11 \times 23 = 1771$ $\qquad$ $3 \times 7 \times 11 \times 23 = 5313$

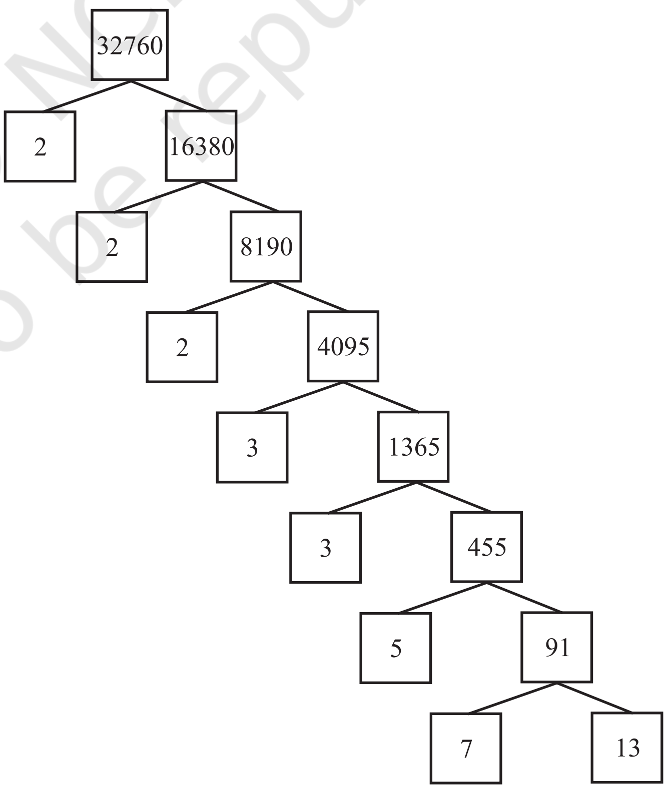
$2 \times 3 \times 7 \times 11 \times 23 = 10626$ $\qquad$ $2^3 \times 3 \times 7^3 = 8232$

$2^2 \times 3 \times 7 \times 11 \times 23 = 21252$

and so on.

Now, let us suppose your collection of primes includes all the possible primes. What is your guess about the size of this collection? Does it contain only a finite number of integers, or infinitely many? Infact, there are infinitely many primes. So, if we combine all these primes in all possible ways, we will get an infinite collection of numbers, all the primes and all possible products of primes. The question is – can we produce all the composite numbers this way? What do you think? Do you think that there may be a composite number which is not the product of powers of primes? Before we answer this, let us factorise positive integers, that is, do the opposite of what we have done so far.

We are going to use the factor tree with which you are all familiar. Let us take some large number, say, 32760, and factorise it as shown.



So we have factorised 32760 as $2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 13$ as a product of primes, i.e., $32760 = 2^3 \times 3^2 \times 5 \times 7 \times 13$ as a product of powers of primes. Let us try another number, say, 123456789. This can be written as $3^2 \times 3803 \times 3607$. Of course, you have to check that 3803 and 3607 are primes! (Try it out for several other natural numbers yourself.) This leads us to a conjecture that every composite number can be written as the product of powers of primes. In fact, this statement is true, and is called the **Fundamental Theorem of Arithmetic** because of its basic crucial importance to the study of integers. Let us now formally state this theorem.

Theorem 1.1 (Fundamental Theorem of Arithmetic) : *Every composite number can be expressed (factorised) as a product of primes, and this factorisation is unique, apart from the order in which the prime factors occur.*

An equivalent version of Theorem 1.2 was probably first recorded as Proposition 14 of Book IX in Euclid's Elements, before it came to be known as the Fundamental Theorem of Arithmetic. However, the first correct proof was given by Carl Friedrich Gauss in his *Disquisitiones Arithmeticae*.

Carl Friedrich Gauss is often referred to as the 'Prince of Mathematicians' and is considered one of the three greatest mathematicians of all time, along with Archimedes and Newton. He has made fundamental contributions to both mathematics and science.

Carl Friedrich Gauss (1777 – 1855)

The Fundamental Theorem of Arithmetic says that every composite number can be factorised as a product of primes. Actually it says more. It says that given any composite number it can be factorised as a product of prime numbers in a **'unique'** way, except for the order in which the primes occur. That is, given any composite number there is one and only one way to write it as a product of primes, as long as we are not particular about the order in which the primes occur. So, for example, we regard $2 \times 3 \times 5 \times 7$ as the same as $3 \times 5 \times 7 \times 2$, or any other possible order in which these primes are written. This fact is also stated in the following form:

The prime factorisation of a natural number is unique, except for the order of its factors.

In general, given a composite number x, we factorise it as $x = p_1 p_2 \ldots p_n$, where $p_1, p_2, \ldots, p_n$ are primes and written in ascending order, i.e., $p_1 \leq p_2 \leq \ldots \leq p_n$. If we combine the same primes, we will get powers of primes. For example,

$$32760 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 13 = 2^3 \times 3^2 \times 5 \times 7 \times 13$$

Once we have decided that the order will be ascending, then the way the number is factorised, is unique.

The Fundamental Theorem of Arithmetic has many applications, both within mathematics and in other fields. Let us look at some examples.

Example 1 : Consider the numbers 4^n, where n is a natural number. Check whether there is any value of n for which 4^n ends with the digit zero.

Solution : If the number 4^n, for any n, were to end with the digit zero, then it would be divisible by 5. That is, the prime factorisation of 4^n would contain the prime 5. This is not possible because $4^n = (2)^{2n}$; so the only prime in the factorisation of 4^n is 2. So, the uniqueness of the Fundamental Theorem of Arithmetic guarantees that there are no other primes in the factorisation of 4^n. So, there is no natural number n for which 4^n ends with the digit zero.

You have already learnt how to find the HCF and LCM of two positive integers using the Fundamental Theorem of Arithmetic in earlier classes, without realising it! This method is also called the *prime factorisation method*. Let us recall this method through an example.

Example 2 : Find the LCM and HCF of 6 and 20 by the prime factorisation method.

Solution : We have : $6 = 2^1 \times 3^1$ and $20 = 2 \times 2 \times 5 = 2^2 \times 5^1$.

You can find HCF(6, 20) = 2 and LCM(6, 20) = $2 \times 2 \times 3 \times 5 = 60$, as done in your earlier classes.

Note that HCF(6, 20) = 2^1 = **Product of the smallest power of each common prime factor in the numbers.**

LCM (6, 20) = $2^2 \times 3^1 \times 5^1$ = **Product of the greatest power of each prime factor**, **involved in the numbers**.

From the example above, you might have noticed that HCF(6, 20) × LCM(6, 20) = 6 × 20. In fact, we can verify that **for any two positive integers *a* and *b*, HCF (*a*, *b*) × LCM (*a*, *b*) = *a* × *b*.** We can use this result to find the LCM of two positive integers, if we have already found the HCF of the two positive integers.

Example 3: Find the HCF of 96 and 404 by the prime factorisation method. Hence, find their LCM.

Solution : The prime factorisation of 96 and 404 gives :

$$96 = 2^5 \times 3, \; 404 = 2^2 \times 101$$

Therefore, the HCF of these two integers is $2^2 = 4$.

Also, $$\text{LCM}(96, 404) = \frac{96 \times 404}{\text{HCF}(96, 404)} = \frac{96 \times 404}{4} = 9696$$

Example 4 : Find the HCF and LCM of 6, 72 and 120, using the prime factorisation method.

Solution : We have :

$$6 = 2 \times 3, \; 72 = 2^3 \times 3^2, \; 120 = 2^3 \times 3 \times 5$$

Here, 2^1 and 3^1 are the smallest powers of the common factors 2 and 3, respectively.

So, $$\text{HCF}(6, 72, 120) = 2^1 \times 3^1 = 2 \times 3 = 6$$

2^3, 3^2 and 5^1 are the greatest powers of the prime factors 2, 3 and 5 respectively involved in the three numbers.

So, $$\text{LCM}(6, 72, 120) = 2^3 \times 3^2 \times 5^1 = 360$$

Remark : Notice, $6 \times 72 \times 120 \neq \text{HCF}(6, 72, 120) \times \text{LCM}(6, 72, 120)$. So, the product of three numbers is not equal to the product of their HCF and LCM.

EXERCISE 1.1

1. Express each number as a product of its prime factors:
   (i) 140 (ii) 156 (iii) 3825 (iv) 5005 (v) 7429
2. Find the LCM and HCF of the following pairs of integers and verify that LCM × HCF = product of the two numbers.
   (i) 26 and 91 (ii) 510 and 92 (iii) 336 and 54
3. Find the LCM and HCF of the following integers by applying the prime factorisation method.
   (i) 12, 15 and 21 (ii) 17, 23 and 29 (iii) 8, 9 and 25
4. Given that HCF (306, 657) = 9, find LCM (306, 657).
5. Check whether 6^n can end with the digit 0 for any natural number n.
6. Explain why $7 \times 11 \times 13 + 13$ and $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 + 5$ are composite numbers.
7. There is a circular path around a sports field. Sonia takes 18 minutes to drive one round of the field, while Ravi takes 12 minutes for the same. Suppose they both start at the

same point and at the same time, and go in the same direction. After how many minutes will they meet again at the starting point?

1.3 Revisiting Irrational Numbers

In Class IX, you were introduced to irrational numbers and many of their properties. You studied about their existence and how the rationals and the irrationals together made up the real numbers. You even studied how to locate irrationals on the number line. However, we did not prove that they were irrationals. In this section, we will prove that $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and, in general, $\sqrt{p}$ is irrational, where p is a prime. One of the theorems, we use in our proof, is the Fundamental Theorem of Arithmetic.

Recall, a number 's' is called *irrational* if it cannot be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. Some examples of irrational numbers, with which you are already familiar, are :

$$\sqrt{2}, \sqrt{3}, \sqrt{15}, \pi, -\frac{\sqrt{2}}{\sqrt{3}}, 0.10110111011110\ldots, \text{ etc.}$$

Before we prove that $\sqrt{2}$ is irrational, we need the following theorem, whose proof is based on the Fundamental Theorem of Arithmetic.

Theorem 1.2 : *Let p be a prime number. If p divides a^2, then p divides a, where a is a positive integer.*

***Proof :** Let the prime factorisation of a be as follows :

$a = p_1p_2 \ldots p_n$, where $p_1, p_2, \ldots, p_n$ are primes, not necessarily distinct.

Therefore, $a^2 = (p_1p_2 \ldots p_n)(p_1p_2 \ldots p_n) = p_1^2p_2^2 \ldots p_n^2$.

Now, we are given that p divides a^2. Therefore, from the Fundamental Theorem of Arithmetic, it follows that p is one of the prime factors of a^2. However, using the uniqueness part of the Fundamental Theorem of Arithmetic, we realise that the only prime factors of a^2 are $p_1, p_2, \ldots, p_n$. So p is one of $p_1, p_2, \ldots, p_n$.

Now, since $a = p_1 p_2 \ldots p_n$, p divides a.

We are now ready to give a proof that $\sqrt{2}$ is irrational.

The proof is based on a technique called 'proof by contradiction'. (This technique is discussed in some detail in Appendix 1).

Theorem 1.3 : $\sqrt{2}$ *is irrational.*

Proof : Let us assume, to the contrary, that $\sqrt{2}$ is rational.

* Not from the examination point of view.

So, we can find integers r and s ($\neq 0$) such that $\sqrt{2} = \dfrac{r}{s}$.

Suppose r and s have a common factor other than 1. Then, we divide by the common factor to get $\sqrt{2} = \dfrac{a}{b}$, where a and b are coprime.

So, $b\sqrt{2} = a$.

Squaring on both sides and rearranging, we get $2b^2 = a^2$. Therefore, 2 divides a^2.

Now, by Theorem 1.2, it follows that 2 divides a.

So, we can write $a = 2c$ for some integer c.

Substituting for a, we get $2b^2 = 4c^2$, that is, $b^2 = 2c^2$.

This means that 2 divides b^2, and so 2 divides b (again using Theorem 1.2 with $p = 2$).

Therefore, a and b have at least 2 as a common factor.

But this contradicts the fact that a and b have no common factors other than 1.

This contradiction has arisen because of our incorrect assumption that $\sqrt{2}$ is rational.

So, we conclude that $\sqrt{2}$ is irrational.

Example 5 : Prove that $\sqrt{3}$ is irrational.

Solution : Let us assume, to the contrary, that $\sqrt{3}$ is rational.

That is, we can find integers a and b ($\neq 0$) such that $\sqrt{3} = \dfrac{a}{b}$.

Suppose a and b have a common factor other than 1, then we can divide by the common factor, and assume that a and b are coprime.

So, $b\sqrt{3} = a$.

Squaring on both sides, and rearranging, we get $3b^2 = a^2$.

Therefore, a^2 is divisible by 3, and by Theorem 1.2, it follows that a is also divisible by 3.

So, we can write $a = 3c$ for some integer c.

Substituting for a, we get $3b^2 = 9c^2$, that is, $b^2 = 3c^2$.

This means that b^2 is divisible by 3, and so b is also divisible by 3 (using Theorem 1.2 with $p = 3$).

Therefore, a and b have at least 3 as a common factor.

But this contradicts the fact that a and b are coprime.

This contradiction has arisen because of our incorrect assumption that $\sqrt{3}$ is rational. So, we conclude that $\sqrt{3}$ is irrational.

In Class IX, we mentioned that :

- the sum or difference of a rational and an irrational number is irrational and
- the product and quotient of a non-zero rational and irrational number is irrational.

We prove some particular cases here.

Example 6 : Show that $5-\sqrt{3}$ is irrational.

Solution : Let us assume, to the contrary, that $5-\sqrt{3}$ is rational.

That is, we can find coprime a and b ($b \neq 0$) such that $5-\sqrt{3}=\frac{a}{b}$.

Therefore, $5-\frac{a}{b}=\sqrt{3}$.

Rearranging this equation, we get $\sqrt{3}=5-\frac{a}{b}=\frac{5b-a}{b}$.

Since a and b are integers, we get $5-\frac{a}{b}$ is rational, and so $\sqrt{3}$ is rational.

But this contradicts the fact that $\sqrt{3}$ is irrational.

This contradiction has arisen because of our incorrect assumption that $5-\sqrt{3}$ is rational.

So, we conclude that $5-\sqrt{3}$ is irrational.

Example 7 : Show that $3\sqrt{2}$ is irrational.

Solution : Let us assume, to the contrary, that $3\sqrt{2}$ is rational.

That is, we can find coprime a and b ($b \neq 0$) such that $3\sqrt{2}=\frac{a}{b}$.

Rearranging, we get $\sqrt{2}=\frac{a}{3b}$.

Since 3, a and b are integers, $\frac{a}{3b}$ is rational, and so $\sqrt{2}$ is rational.

But this contradicts the fact that $\sqrt{2}$ is irrational.

So, we conclude that $3\sqrt{2}$ is irrational.

EXERCISE 1.2

1. Prove that $\sqrt{5}$ is irrational.
2. Prove that $3 + 2\sqrt{5}$ is irrational.
3. Prove that the following are irrationals :

   (i) $\frac{1}{\sqrt{2}}$ (ii) $7\sqrt{5}$ (iii) $6 + \sqrt{2}$

1.4 Summary

In this chapter, you have studied the following points:

1. The Fundamental Theorem of Arithmetic :

   Every composite number can be expressed (factorised) as a product of primes, and this factorisation is unique, apart from the order in which the prime factors occur.
2. If p is a prime and p divides a^2, then p divides a, where a is a positive integer.
3. To prove that $\sqrt{2}, \sqrt{3}$ are irrationals.

A NOTE TO THE READER

You have seen that :

HCF $(p, q, r) \times$ LCM $(p, q, r) \neq p \times q \times r$, where p, q, r are positive integers (see Example 8). However, the following results hold good for three numbers p, q and r :

$$\text{LCM}(p, q, r) = \frac{p \cdot q \cdot r \cdot \text{HCF}(p, q, r)}{\text{HCF}(p, q) \cdot \text{HCF}(q, r) \cdot \text{HCF}(p, r)}$$

$$\text{HCF}(p, q, r) = \frac{p \cdot q \cdot r \cdot \text{LCM}(p, q, r)}{\text{LCM}(p, q) \cdot \text{LCM}(q, r) \cdot \text{LCM}(p, r)}$$

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POLYNOMIALS 2

2.1 Introduction

In Class IX, you have studied polynomials in one variable and their degrees. Recall that if $p(x)$ is a polynomial in x, the highest power of x in $p(x)$ is called **the degree of the polynomial** $p(x)$. For example, $4x + 2$ is a polynomial in the variable x of degree 1, $2y^2 - 3y + 4$ is a polynomial in the variable y of degree 2, $5x^3 - 4x^2 + x - \sqrt{2}$ is a polynomial in the variable x of degree 3 and $7u^6 - \frac{3}{2}u^4 + 4u^2 + u - 8$ is a polynomial in the variable u of degree 6. Expressions like $\frac{1}{x-1}$, $\sqrt{x} + 2$, $\frac{1}{x^2 + 2x + 3}$ etc., are not polynomials.

A polynomial of degree 1 is called a **linear polynomial**. For example, $2x - 3$, $\sqrt{3}x + 5$, $y + \sqrt{2}$, $x - \frac{2}{11}$, $3z + 4$, $\frac{2}{3}u + 1$, etc., are all linear polynomials. Polynomials such as $2x + 5 - x^2$, $x^3 + 1$, etc., are not linear polynomials.

A polynomial of degree 2 is called a **quadratic polynomial**. The name 'quadratic' has been derived from the word 'quadrate', which means 'square'. $2x^2 + 3x - \frac{2}{5}$, $y^2 - 2$, $2 - x^2 + \sqrt{3}x$, $\frac{u}{3} - 2u^2 + 5$, $\sqrt{5}v^2 - \frac{2}{3}v$, $4z^2 + \frac{1}{7}$ are some examples of quadratic polynomials (whose coefficients are real numbers). More generally, any quadratic polynomial in x is of the form $ax^2 + bx + c$, where a, b, c are real numbers and $a \neq 0$. A polynomial of degree 3 is called a **cubic polynomial**. Some examples of a cubic polynomial are $2 - x^3$, x^3, $\sqrt{2}\,x^3$, $3 - x^2 + x^3$, $3x^3 - 2x^2 + x - 1$. In fact, the most general form of a cubic polynomial is

$$ax^3 + bx^2 + cx + d,$$

where, a, b, c, d are real numbers and $a \neq 0$.

Now consider the polynomial $p(x) = x^2 - 3x - 4$. Then, putting $x = 2$ in the polynomial, we get $p(2) = 2^2 - 3 \times 2 - 4 = -6$. The value '$-6$', obtained by replacing x by 2 in $x^2 - 3x - 4$, is the value of $x^2 - 3x - 4$ at $x = 2$. Similarly, $p(0)$ is the value of $p(x)$ at $x = 0$, which is -4.

If $p(x)$ is a polynomial in x, and if k is any real number, then the value obtained by replacing x by k in $p(x)$, is called **the value of $p(x)$ at $x = k$**, and is denoted by $p(k)$.

What is the value of $p(x) = x^2 - 3x - 4$ at $x = -1$? We have :

$$p(-1) = (-1)^2 - \{3 \times (-1)\} - 4 = 0$$

Also, note that $\quad p(4) = 4^2 - (3 \times 4) - 4 = 0.$

As $p(-1) = 0$ and $p(4) = 0$, -1 and 4 are called the zeroes of the quadratic polynomial $x^2 - 3x - 4$. More generally, a real number k is said to be a **zero of a polynomial $p(x)$**, if $p(k) = 0$.

You have already studied in Class IX, how to find the zeroes of a linear polynomial. For example, if k is a zero of $p(x) = 2x + 3$, then $p(k) = 0$ gives us $2k + 3 = 0$, i.e., $k = -\frac{3}{2}$.

In general, if k is a zero of $p(x) = ax + b$, then $p(k) = ak + b = 0$, i.e., $k = \frac{-b}{a}$.

So, the zero of the linear polynomial $ax + b$ is $\frac{-b}{a} = \frac{-\text{(Constant term)}}{\text{Coefficient of } x}$.

Thus, the zero of a linear polynomial is related to its coefficients. Does this happen in the case of other polynomials too? For example, are the zeroes of a quadratic polynomial also related to its coefficients?

In this chapter, we will try to answer these questions. We will also study the division algorithm for polynomials.

2.2 Geometrical Meaning of the Zeroes of a Polynomial

You know that a real number k is a zero of the polynomial $p(x)$ if $p(k) = 0$. But why are the zeroes of a polynomial so important? To answer this, first we will see the **geometrical** representations of linear and quadratic polynomials and the geometrical meaning of their zeroes.

Consider first a linear polynomial $ax + b$, $a \neq 0$. You have studied in Class IX that the graph of $y = ax + b$ is a straight line. For example, the graph of $y = 2x + 3$ is a straight line passing through the points (– 2, –1) and (2, 7).

x	-2	2
$y = 2x + 3$	-1	7

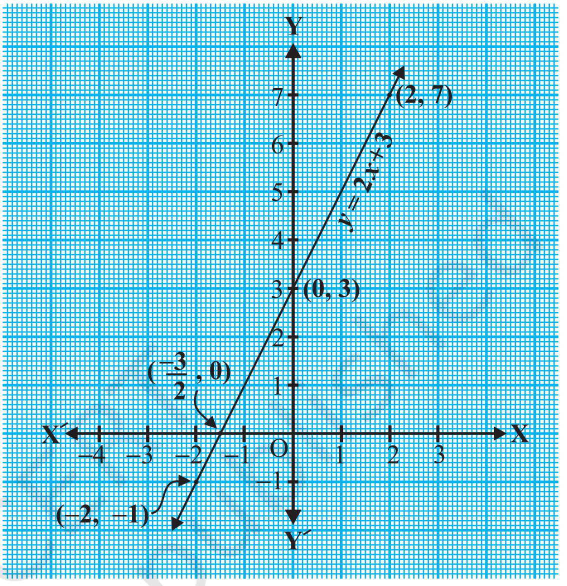


Fig. 2.1

From Fig. 2.1, you can see that the graph of $y = 2x + 3$ intersects the x-axis mid-way between $x = -1$ and $x = -2$, that is, at the point $\left(-\frac{3}{2}, 0\right)$. You also know that the zero of $2x + 3$ is $-\frac{3}{2}$. Thus, the zero of the polynomial $2x + 3$ is the x-coordinate of the point where the graph of $y = 2x + 3$ intersects the x-axis.

In general, for a linear polynomial $ax + b$, $a \neq 0$, the graph of $y = ax + b$ is a straight line which intersects the x-axis at exactly one point, namely, $\left(\frac{-b}{a}, 0\right)$. Therefore, the linear polynomial $ax + b$, $a \neq 0$, has exactly one zero, namely, the x-coordinate of the point where the graph of $y = ax + b$ intersects the x-axis.

Now, let us look for the geometrical meaning of a zero of a quadratic polynomial. Consider the quadratic polynomial $x^2 - 3x - 4$. Let us see what the graph* of $y = x^2 - 3x - 4$ looks like. Let us list a few values of $y = x^2 - 3x - 4$ corresponding to a few values for x as given in Table 2.1.

* Plotting of graphs of quadratic or cubic polynomials is not meant to be done by the students, nor is to be evaluated.

Table 2.1

x	-2	-1	0	1	2	3	4	5
$y = x^2 - 3x - 4$	6	0	-4	-6	-6	-4	0	6

If we locate the points listed above on a graph paper and draw the graph, it will actually look like the one given in Fig. 2.2.

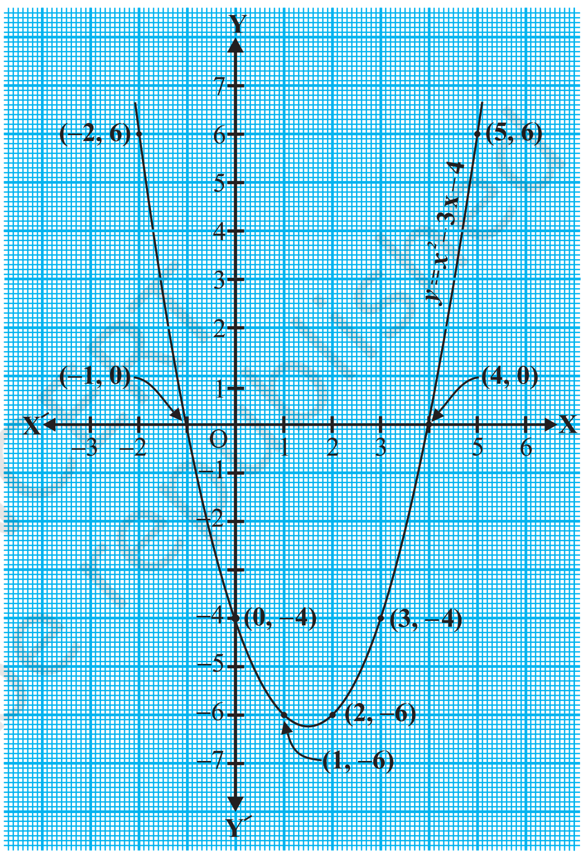


Fig. 2.2

In fact, for any quadratic polynomial $ax^2 + bx + c$, $a \neq 0$, the graph of the corresponding equation $y = ax^2 + bx + c$ has one of the two shapes either open upwards like $\cup$ or open downwards like $\cap$ depending on whether $a > 0$ or $a < 0$. (These curves are called **parabolas**.)

You can see from Table 2.1 that –1 and 4 are zeroes of the quadratic polynomial. Also note from Fig. 2.2 that –1 and 4 are the x-coordinates of the points where the graph of $y = x^2 - 3x - 4$ intersects the x-axis. Thus, the zeroes of the quadratic polynomial $x^2 - 3x - 4$ are x-coordinates of the points where the graph of $y = x^2 - 3x - 4$ intersects the x-axis.

This fact is true for any quadratic polynomial, i.e., the zeroes of a quadratic polynomial $ax^2 + bx + c$, $a \neq 0$, are precisely the x-coordinates of the points where the parabola representing $y = ax^2 + bx + c$ intersects the x-axis.

From our observation earlier about the shape of the graph of $y = ax^2 + bx + c$, the following three cases can happen:

Case (i) : Here, the graph cuts x-axis at two distinct points A and A′.

The x-coordinates of A and A′ are the **two zeroes** of the quadratic polynomial $ax^2 + bx + c$ in this case (see Fig. 2.3).

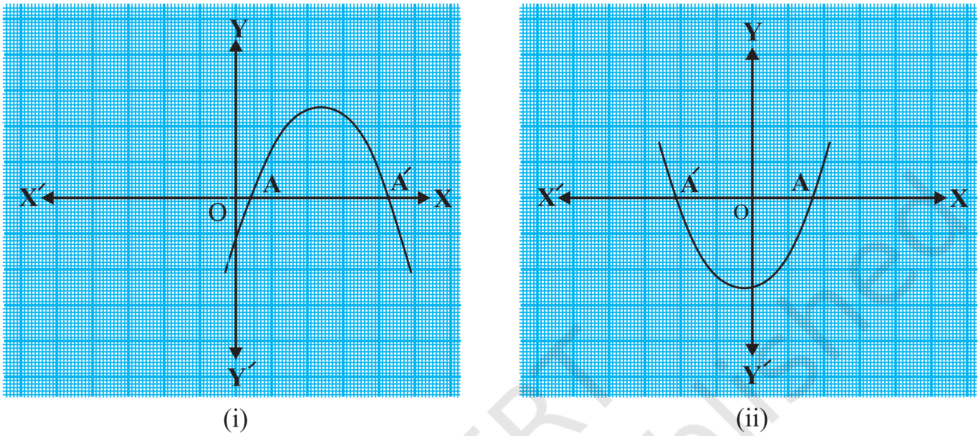


Fig. 2.3

Case (ii) : Here, the graph cuts the x-axis at exactly one point, i.e., at two coincident points. So, the two points A and A′ of Case (i) coincide here to become one point A (see Fig. 2.4).

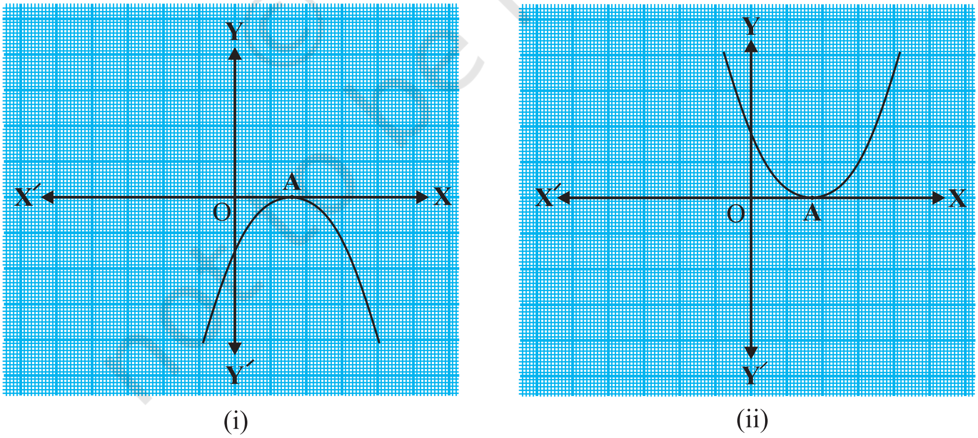
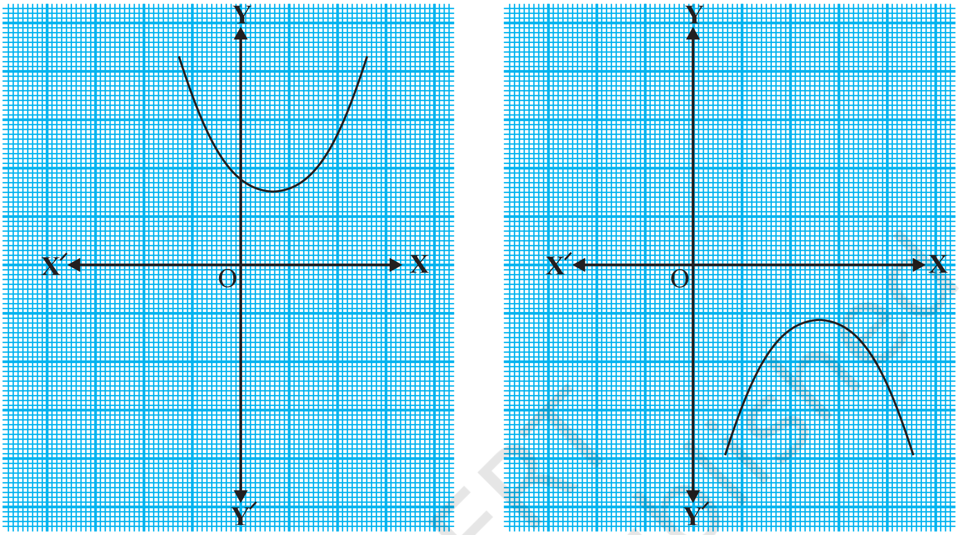


Fig. 2.4

The x-coordinate of A is the **only zero** for the quadratic polynomial $ax^2 + bx + c$ in this case.

Case (iii) : Here, the graph is either completely above the x-axis or completely below the x-axis. So, it does not cut the x-axis at any point (see Fig. 2.5).



(i) (ii)

Fig. 2.5

So, the quadratic polynomial $ax^2 + bx + c$ has **no zero** in this case.

So, you can see geometrically that a quadratic polynomial can have either two distinct zeroes or two equal zeroes (i.e., one zero), or no zero. This also means that a polynomial of degree 2 has atmost two zeroes.

Now, what do you expect the geometrical meaning of the zeroes of a cubic polynomial to be? Let us find out. Consider the cubic polynomial $x^3 - 4x$. To see what the graph of $y = x^3 - 4x$ looks like, let us list a few values of y corresponding to a few values for x as shown in Table 2.2.

Table 2.2

x	-2	-1	0	1	2
$y = x^3 - 4x$	0	3	0	-3	0

Locating the points of the table on a graph paper and drawing the graph, we see that the graph of $y = x^3 - 4x$ actually looks like the one given in Fig. 2.6.

We see from the table above that – 2, 0 and 2 are zeroes of the cubic polynomial $x^3 - 4x$. Observe that -2, 0 and 2 are, in fact, the x-coordinates of the only points where the graph of $y = x^3 - 4x$ intersects the x-axis. Since the curve meets the x-axis in only these 3 points, their x-coordinates are the only zeroes of the polynomial.

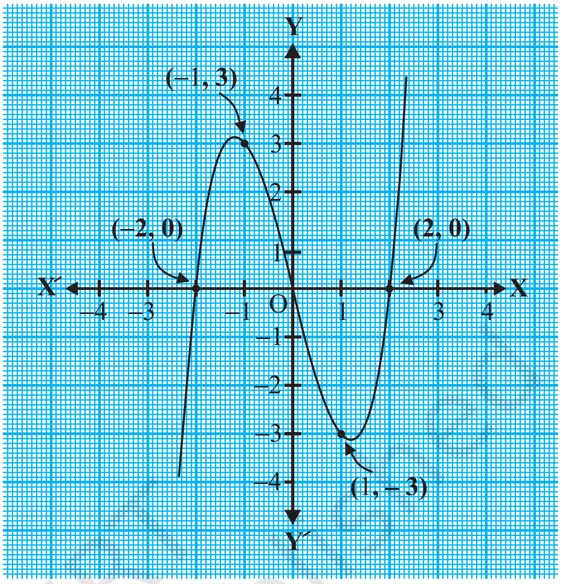


Fig. 2.6

Let us take a few more examples. Consider the cubic polynomials x^3 and $x^3 - x^2$. We draw the graphs of $y = x^3$ and $y = x^3 - x^2$ in Fig. 2.7 and Fig. 2.8 respectively.

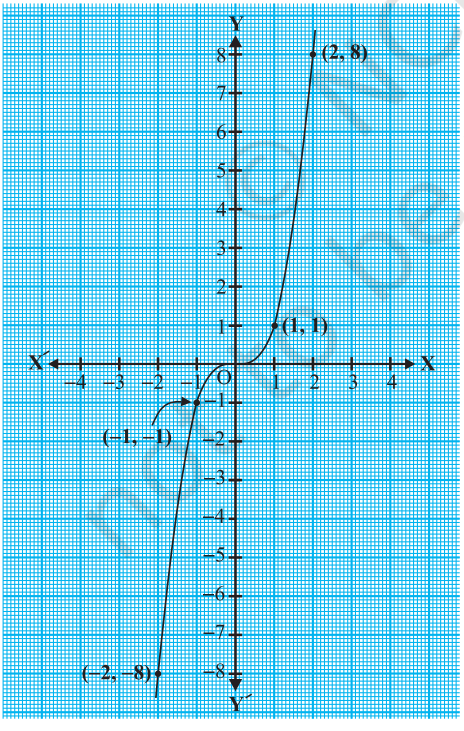


Fig. 2.7

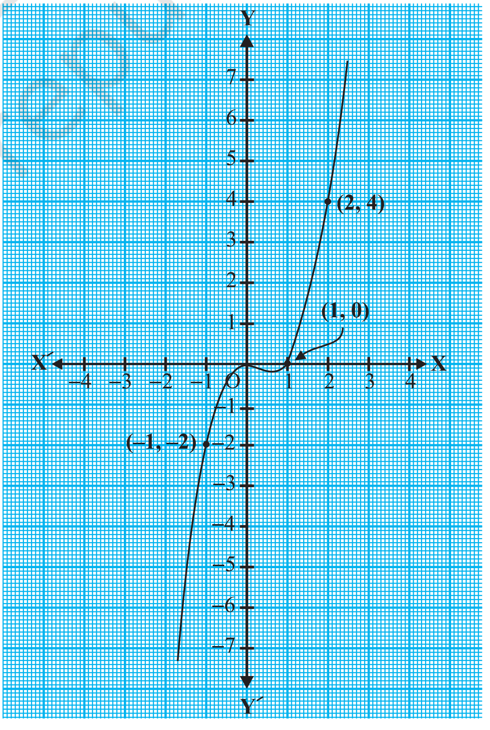


Fig. 2.8

Note that 0 is the only zero of the polynomial x^3. Also, from Fig. 2.7, you can see that 0 is the x-coordinate of the only point where the graph of $y = x^3$ intersects the x-axis. Similarly, since $x^3 - x^2 = x^2(x - 1)$, 0 and 1 are the only zeroes of the polynomial $x^3 - x^2$. Also, from Fig. 2.8, these values are the x-coordinates of the only points where the graph of $y = x^3 - x^2$ intersects the x-axis.

From the examples above, we see that there are at most 3 zeroes for any cubic polynomial. In other words, any polynomial of degree 3 can have at most three zeroes.

Remark : In general, given a polynomial $p(x)$ of degree n, the graph of $y = p(x)$ intersects the x-axis at atmost n points. Therefore, a polynomial $p(x)$ of degree n has at most n zeroes.

Example 1 : Look at the graphs in Fig. 2.9 given below. Each is the graph of $y = p(x)$, where $p(x)$ is a polynomial. For each of the graphs, find the number of zeroes of $p(x)$.

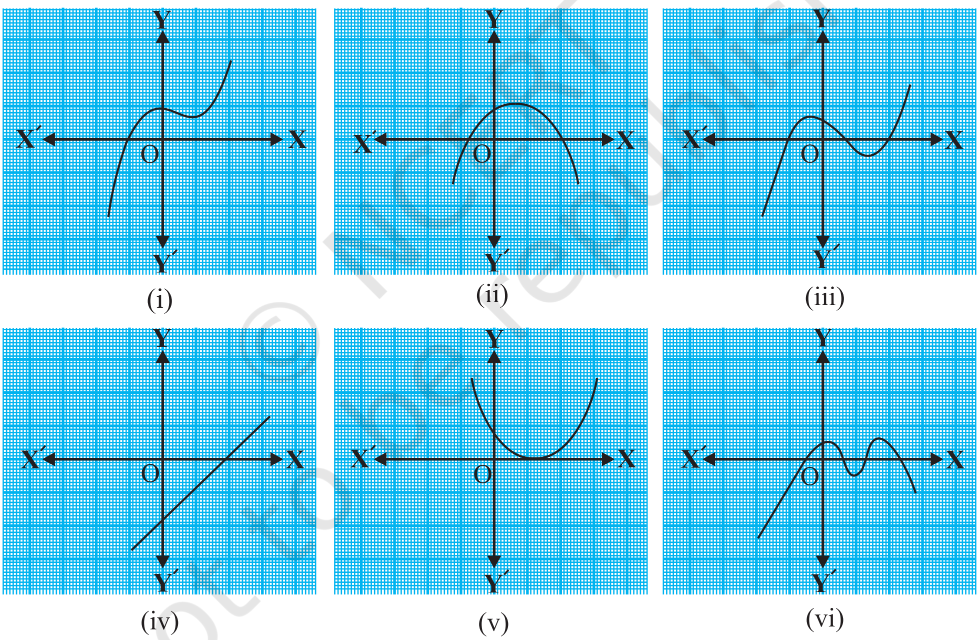


Fig. 2.9

Solution :

(i) The number of zeroes is 1 as the graph intersects the x-axis at one point only.

(ii) The number of zeroes is 2 as the graph intersects the x-axis at two points.

(iii) The number of zeroes is 3. (Why?)

(iv) The number of zeroes is 1. (Why?)

(v) The number of zeroes is 1. (Why?)

(vi) The number of zeroes is 4. (Why?)

EXERCISE 2.1

1. The graphs of $y = p(x)$ are given in Fig. 2.10 below, for some polynomials $p(x)$. Find the number of zeroes of $p(x)$, in each case.

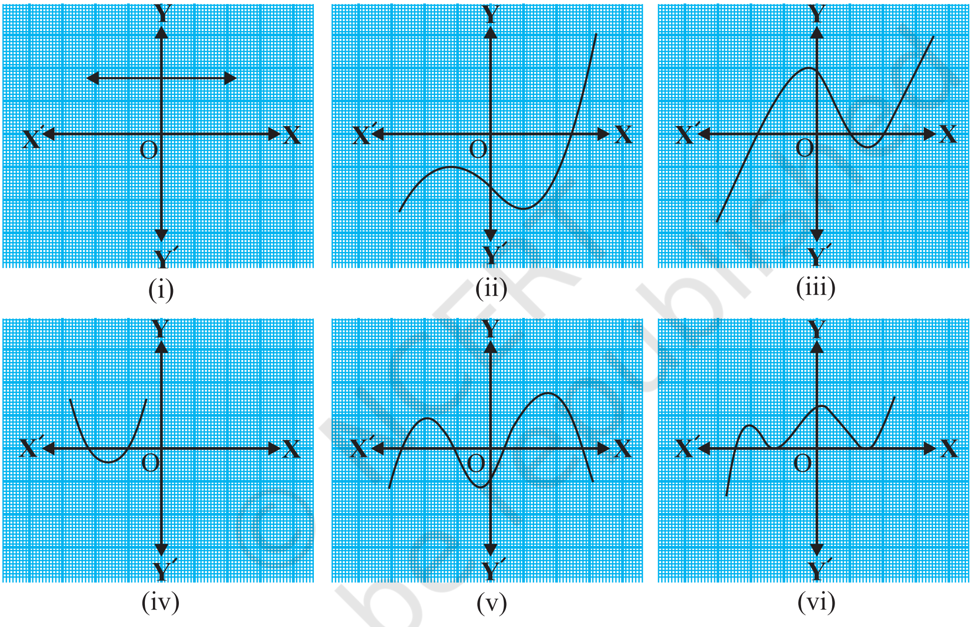


Fig. 2.10

2.3 Relationship between Zeroes and Coefficients of a Polynomial

You have already seen that zero of a linear polynomial $ax + b$ is $-\dfrac{b}{a}$. We will now try to answer the question raised in Section 2.1 regarding the relationship between zeroes and coefficients of a quadratic polynomial. For this, let us take a quadratic polynomial, say $p(x) = 2x^2 - 8x + 6$. In Class IX, you have learnt how to factorise quadratic polynomials by splitting the middle term. So, here we need to split the middle term '$-8x$' as a sum of two terms, whose product is $6 \times 2x^2 = 12x^2$. So, we write

$$2x^2 - 8x + 6 = 2x^2 - 6x - 2x + 6 = 2x(x - 3) - 2(x - 3)$$

$$= (2x - 2)(x - 3) = 2(x - 1)(x - 3)$$

So, the value of $p(x) = 2x^2 - 8x + 6$ is zero when $x - 1 = 0$ or $x - 3 = 0$, i.e., when $x = 1$ or $x = 3$. So, the zeroes of $2x^2 - 8x + 6$ are 1 and 3. Observe that :

$$\text{Sum of its zeroes} = 1 + 3 = 4 = \frac{-(-8)}{2} = \frac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2}$$

$$\text{Product of its zeroes} = 1 \times 3 = 3 = \frac{6}{2} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

Let us take one more quadratic polynomial, say, $p(x) = 3x^2 + 5x - 2$. By the method of splitting the middle term,

$$\begin{aligned} 3x^2 + 5x - 2 &= 3x^2 + 6x - x - 2 = 3x(x + 2) - 1(x + 2) \\ &= (3x - 1)(x + 2) \end{aligned}$$

Hence, the value of $3x^2 + 5x - 2$ is zero when either $3x - 1 = 0$ or $x + 2 = 0$, i.e., when $x = \frac{1}{3}$ or $x = -2$. So, the zeroes of $3x^2 + 5x - 2$ are $\frac{1}{3}$ and -2. Observe that :

$$\text{Sum of its zeroes} = \frac{1}{3} + (-2) = \frac{-5}{3} = \frac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2}$$

$$\text{Product of its zeroes} = \frac{1}{3} \times (-2) = \frac{-2}{3} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

In general, if α* and β* are the zeroes of the quadratic polynomial $p(x) = ax^2 + bx + c$, $a \neq 0$, then you know that $x - \alpha$ and $x - \beta$ are the factors of $p(x)$. Therefore,

$$\begin{aligned} ax^2 + bx + c &= k(x - \alpha)(x - \beta), \text{ where } k \text{ is a constant} \\ &= k[x^2 - (\alpha + \beta)x + \alpha\beta] \\ &= kx^2 - k(\alpha + \beta)x + k\alpha\beta \end{aligned}$$

Comparing the coefficients of x^2, x and constant terms on both the sides, we get

$$a = k,\ b = -k(\alpha + \beta) \text{ and } c = k\alpha\beta.$$

This gives

$$\boldsymbol{\alpha + \beta = \frac{-b}{a}},$$

$$\boldsymbol{\alpha\beta = \frac{c}{a}}$$

* α, β are Greek letters pronounced as 'alpha' and 'beta' respectively. We will use later one more letter 'γ' pronounced as 'gamma'.

i.e., $$\text{sum of zeroes} = \alpha + \beta = -\frac{b}{a} = \frac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2},$$

$$\text{product of zeroes} = \alpha\beta = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}.$$

Let us consider some examples.

Example 2 : Find the zeroes of the quadratic polynomial $x^2 + 7x + 10$, and verify the relationship between the zeroes and the coefficients.

Solution : We have

$$x^2 + 7x + 10 = (x + 2)(x + 5)$$

So, the value of $x^2 + 7x + 10$ is zero when $x + 2 = 0$ or $x + 5 = 0$, i.e., when $x = -2$ or $x = -5$. Therefore, the zeroes of $x^2 + 7x + 10$ are -2 and -5. Now,

$$\text{sum of zeroes} = -2 + (-5) = -(7) = \frac{-(7)}{1} = \frac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2},$$

$$\text{product of zeroes} = (-2)\times(-5) = 10 = \frac{10}{1} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}.$$

Example 3 : Find the zeroes of the polynomial $x^2 - 3$ and verify the relationship between the zeroes and the coefficients.

Solution : Recall the identity $a^2 - b^2 = (a - b)(a + b)$. Using it, we can write:

$$x^2 - 3 = \left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right)$$

So, the value of $x^2 - 3$ is zero when $x = \sqrt{3}$ or $x = -\sqrt{3}$.

Therefore, the zeroes of $x^2 - 3$ are $\sqrt{3}$ and $-\sqrt{3}$.

Now,

$$\text{sum of zeroes} = \sqrt{3} - \sqrt{3} = 0 = \frac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2},$$

$$\text{product of zeroes} = \left(\sqrt{3}\right)\left(-\sqrt{3}\right) = -3 = \frac{-3}{1} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}.$$

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Example 4 : Find a quadratic polynomial, the sum and product of whose zeroes are – 3 and 2, respectively.

Solution : Let the quadratic polynomial be $ax^2 + bx + c$, and its zeroes be α and β. We have

$$\alpha + \beta = -3 = \frac{-b}{a},$$

and
$$\alpha\beta = 2 = \frac{c}{a}.$$

If $a = 1$, then $b = 3$ and $c = 2$.

So, one quadratic polynomial which fits the given conditions is $x^2 + 3x + 2$.

You can check that any other quadratic polynomial that fits these conditions will be of the form $k(x^2 + 3x + 2)$, where k is real.

Let us now look at cubic polynomials. Do you think a similar relation holds between the zeroes of a cubic polynomial and its coefficients?

Let us consider $p(x) = 2x^3 - 5x^2 - 14x + 8$.

You can check that $p(x) = 0$ for $x = 4, -2, \frac{1}{2}\cdot$ Since $p(x)$ can have atmost three zeroes, these are the zeores of $2x^3 - 5x^2 - 14x + 8$. Now,

$$\text{sum of the zeroes} = 4 + (-2) + \frac{1}{2} = \frac{5}{2} = \frac{-(-5)}{2} = \frac{-(\text{Coefficient of } x^2)}{\text{Coefficient of } x^3},$$

$$\text{product of the zeroes} = 4 \times (-2) \times \frac{1}{2} = -4 = \frac{-8}{2} = \frac{-\text{Constant term}}{\text{Coefficient of } x^3}.$$

However, there is one more relationship here. Consider the sum of the products of the zeroes taken two at a time. We have

$$\{4 \times (-2)\} + \left\{(-2) \times \frac{1}{2}\right\} + \left\{\frac{1}{2} \times 4\right\}$$

$$= -8 - 1 + 2 = -7 = \frac{-14}{2} = \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}.$$

In general, it can be proved that if α, β, γ are the zeroes of the cubic polynomial $ax^3 + bx^2 + cx + d$, then

$$\boldsymbol{\alpha + \beta + \gamma = \frac{-b}{a},}$$

$$\boldsymbol{\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a},}$$

$$\boldsymbol{\alpha\beta\gamma = \frac{-d}{a}.}$$

Let us consider an example.

Example 5* : Verify that 3, –1, $-\frac{1}{3}$ are the zeroes of the cubic polynomial $p(x) = 3x^3 - 5x^2 - 11x - 3$, and then verify the relationship between the zeroes and the coefficients.

Solution : Comparing the given polynomial with $ax^3 + bx^2 + cx + d$, we get

$a = 3$, $b = -5$, $c = -11$, $d = -3$. Further

$p(3) = 3 \times 3^3 - (5 \times 3^2) - (11 \times 3) - 3 = 81 - 45 - 33 - 3 = 0,$

$p(-1) = 3 \times (-1)^3 - 5 \times (-1)^2 - 11 \times (-1) - 3 = -3 - 5 + 11 - 3 = 0,$

$$p\left(-\frac{1}{3}\right) = 3\times\left(-\frac{1}{3}\right)^3 - 5\times\left(-\frac{1}{3}\right)^2 - 11\times\left(-\frac{1}{3}\right) - 3,$$

$$= -\frac{1}{9} - \frac{5}{9} + \frac{11}{3} - 3 = -\frac{2}{3} + \frac{2}{3} = 0$$

Therefore, 3, –1 and $-\frac{1}{3}$ are the zeroes of $3x^3 - 5x^2 - 11x - 3$.

So, we take $\alpha = 3$, $\beta = -1$ and $\gamma = -\frac{1}{3}$.

Now,

$$\alpha + \beta + \gamma = 3 + (-1) + \left(-\frac{1}{3}\right) = 2 - \frac{1}{3} = \frac{5}{3} = \frac{-(-5)}{3} = \frac{-b}{a},$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = 3\times(-1) + (-1)\times\left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right)\times 3 = -3 + \frac{1}{3} - 1 = \frac{-11}{3} = \frac{c}{a},$$

$$\alpha\beta\gamma = 3\times(-1)\times\left(-\frac{1}{3}\right) = 1 = \frac{-(-3)}{3} = \frac{-d}{a}.$$

* Not from the examination point of view.

EXERCISE 2.2

1. Find the zeroes of the following quadratic polynomials and verify the relationship between the zeroes and the coefficients.

(i) $x^2 - 2x - 8$ (ii) $4s^2 - 4s + 1$ (iii) $6x^2 - 3 - 7x$

(iv) $4u^2 + 8u$ (v) $t^2 - 15$ (vi) $3x^2 - x - 4$

2. Find a quadratic polynomial each with the given numbers as the sum and product of its zeroes respectively.

(i) $\frac{1}{4}, -1$ (ii) $\sqrt{2}, \frac{1}{3}$ (iii) $0, \sqrt{5}$

(iv) $1, 1$ (v) $-\frac{1}{4}, \frac{1}{4}$ (vi) $4, 1$

2.4 Summary

In this chapter, you have studied the following points:

1. Polynomials of degrees 1, 2 and 3 are called linear, quadratic and cubic polynomials respectively.

2. A quadratic polynomial in x with real coefficients is of the form $ax^2 + bx + c$, where a, b, c are real numbers with $a \neq 0$.

3. The zeroes of a polynomial $p(x)$ are precisely the x-coordinates of the points, where the graph of $y = p(x)$ intersects the x-axis.

4. A quadratic polynomial can have at most 2 zeroes and a cubic polynomial can have at most 3 zeroes.

5. If α and β are the zeroes of the quadratic polynomial $ax^2 + bx + c$, then

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}.$$

6. If α, β, γ are the zeroes of the cubic polynomial $ax^3 + bx^2 + cx + d$, then

$$\alpha + \beta + \gamma = \frac{-b}{a},$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a},$$

and $$\alpha\beta\gamma = \frac{-d}{a}.$$

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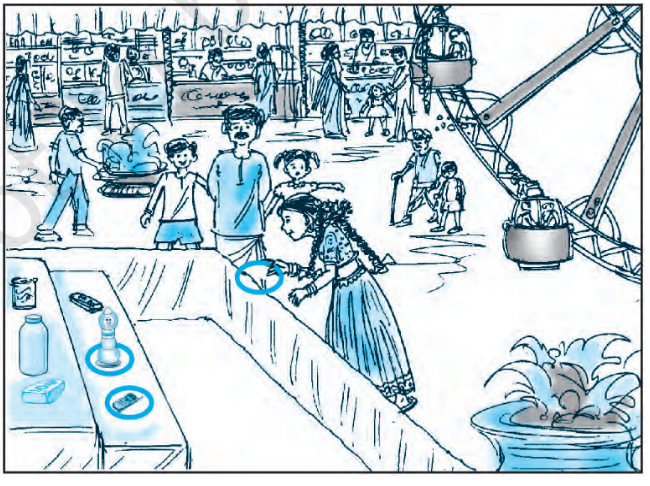
PAIR OF LINEAR EQUATIONS IN TWO VARIABLES 3

3.1 Introduction

You must have come across situations like the one given below :

Akhila went to a fair in her village. She wanted to enjoy rides on the Giant Wheel and play Hoopla (a game in which you throw a ring on the items kept in a stall, and if the ring covers any object completely, you get it). The number of times she played Hoopla is half the number of rides she had on the Giant Wheel. If each ride costs ₹ 3, and a game of Hoopla costs ₹ 4, how would you find out the number of rides she had and how many times she played Hoopla, provided she spent ₹ 20.

May be you will try it by considering different cases. If she has one ride, is it possible? Is it possible to have two rides? And so on. Or you may use the knowledge of Class IX, to represent such situations as linear equations in two variables.



Let us try this approach.

Denote the number of rides that Akhila had by x, and the number of times she played Hoopla by y. Now the situation can be represented by the two equations:

$$y = \frac{1}{2}x \qquad (1)$$

$$3x + 4y = 20 \qquad (2)$$

Can we find the solutions of this pair of equations? There are several ways of finding these, which we will study in this chapter.

3.2 Graphical Method of Solution of a Pair of Linear Equations

A pair of linear equations which has no solution, is called an *inconsistent* pair of linear equations. A pair of linear equations in two variables, which has a solution, is called a *consistent* pair of linear equations. A pair of linear equations which are equivalent has infinitely many distinct common solutions. Such a pair is called a *dependent* pair of linear equations in two variables. Note that a dependent pair of linear equations is always consistent.

We can now summarise the behaviour of lines representing a pair of linear equations in two variables and the existence of solutions as follows:

(i) the lines may intersect in a single point. In this case, the pair of equations has a unique solution (consistent pair of equations).

(ii) the lines may be parallel. In this case, the equations have no solution (inconsistent pair of equations).

(iii) the lines may be coincident. In this case, the equations have infinitely many solutions [dependent (consistent) pair of equations].

Consider the following three pairs of equations.

(i) $x - 2y = 0$ and $3x + 4y - 20 = 0$ (The lines intersect)

(ii) $2x + 3y - 9 = 0$ and $4x + 6y - 18 = 0$ (The lines coincide)

(iii) $x + 2y - 4 = 0$ and $2x + 4y - 12 = 0$ (The lines are parallel)

Let us now write down, and compare, the values of $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$ and $\frac{c_1}{c_2}$ in all the three examples. Here, a_1, b_1, c_1 and a_2, b_2, c_2 denote the coefficents of equations given in the general form in Section 3.2.

Table 3.1

Sl No.	Pair of lines	$\frac{a_1}{a_2}$	$\frac{b_1}{b_2}$	$\frac{c_1}{c_2}$	Compare the ratios	Graphical representation	Algebraic interpretation
1.	$x - 2y = 0$ $3x + 4y - 20 = 0$	$\frac{1}{3}$	$\frac{-2}{4}$	$\frac{0}{-20}$	$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$	Intersecting lines	Exactly one solution (unique)
2.	$2x + 3y - 9 = 0$ $4x + 6y - 18 = 0$	$\frac{2}{4}$	$\frac{3}{6}$	$\frac{-9}{-18}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$	Coincident lines	Infinitely many solutions
3.	$x + 2y - 4 = 0$ $2x + 4y - 12 = 0$	$\frac{1}{2}$	$\frac{2}{4}$	$\frac{-4}{-12}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$	Parallel lines	No solution

From the table above, you can observe that if the lines represented by the equation

$$a_1x + b_1y + c_1 = 0$$

and

$$a_2x + b_2y + c_2 = 0$$

are (i) intersecting, then $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.

(ii) coincident, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

(iii) parallel, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$.

In fact, the converse is also true for any pair of lines. You can verify them by considering some more examples by yourself.

Let us now consider some more examples to illustrate it.

Example 1 : Check graphically whether the pair of equations

$$x + 3y = 6 \quad (1)$$

and

$$2x - 3y = 12 \quad (2)$$

is consistent. If so, solve them graphically.

Solution : Let us draw the graphs of the Equations (1) and (2). For this, we find two solutions of each of the equations, which are given in Table 3.2

Table 3.2

x	0	6
$y = \dfrac{6-x}{3}$	2	0

x	0	3
$y = \dfrac{2x-12}{3}$	-4	-2

Plot the points A(0, 2), B(6, 0), P(0, – 4) and Q(3, – 2) on graph paper, and join the points to form the lines AB and PQ as shown in Fig. 3.1.

We observe that there is a point B (6, 0) common to both the lines AB and PQ. So, the solution of the pair of linear equations is $x = 6$ and $y = 0$, i.e., the given pair of equations is consistent.

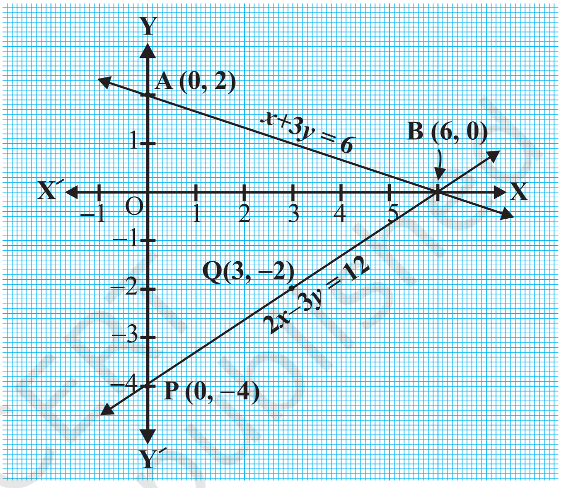


Fig. 3.1

Example 2 : Graphically, find whether the following pair of equations has no solution, unique solution or infinitely many solutions:

$$5x - 8y + 1 = 0 \quad (1)$$

$$3x - \frac{24}{5}y + \frac{3}{5} = 0 \quad (2)$$

Solution : Multiplying Equation (2) by $\frac{5}{3}$, we get

$$5x - 8y + 1 = 0$$

But, this is the same as Equation (1). Hence the lines represented by Equations (1) and (2) are coincident. Therefore, Equations (1) and (2) have infinitely many solutions.

Plot few points on the graph and verify it yourself.

Example 3 : Champa went to a 'Sale' to purchase some pants and skirts. When her friends asked her how many of each she had bought, she answered, "The number of skirts is two less than twice the number of pants purchased. Also, the number of skirts is four less than four times the number of pants purchased". Help her friends to find how many pants and skirts Champa bought.

Solution : Let us denote the number of pants by x and the number of skirts by y. Then the equations formed are :

$$y = 2x - 2 \quad (1)$$

and

$$y = 4x - 4 \quad (2)$$

Let us draw the graphs of Equations (1) and (2) by finding two solutions for each of the equations. They are given in Table 3.3.

Table 3.3

x	2	0
$y = 2x - 2$	2	-2

x	0	1
$y = 4x - 4$	-4	0

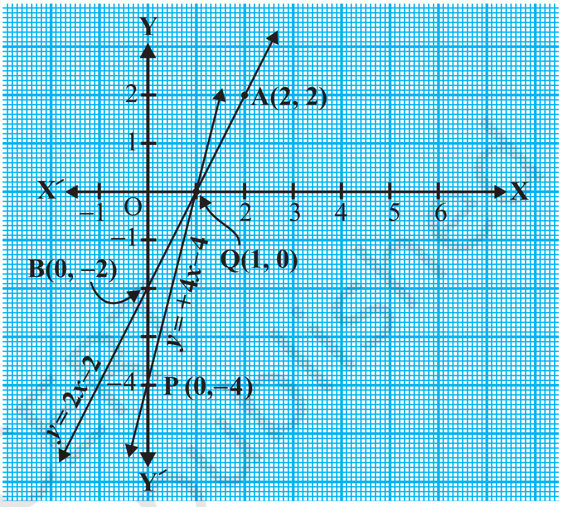


Fig. 3.2

Plot the points and draw the lines passing through them to represent the equations, as shown in Fig. 3.2.

The two lines intersect at the point (1, 0). So, $x = 1$, $y = 0$ is the required solution of the pair of linear equations, i.e., the number of pants she purchased is 1 and she did not buy any skirt.

Verify the answer by checking whether it satisfies the conditions of the given problem.

EXERCISE 3.1

1. Form the pair of linear equations in the following problems, and find their solutions graphically.

(i) 10 students of Class X took part in a Mathematics quiz. If the number of girls is 4 more than the number of boys, find the number of boys and girls who took part in the quiz.

(ii) 5 pencils and 7 pens together cost ₹ 50, whereas 7 pencils and 5 pens together cost ₹ 46. Find the cost of one pencil and that of one pen.

2. On comparing the ratios $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$ and $\frac{c_1}{c_2}$, find out whether the lines representing the following pairs of linear equations intersect at a point, are parallel or coincident:

(i) $5x - 4y + 8 = 0$
$7x + 6y - 9 = 0$

(ii) $9x + 3y + 12 = 0$
$18x + 6y + 24 = 0$

(iii) $6x - 3y + 10 = 0$
$2x - y + 9 = 0$

3. On comparing the ratios $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$ and $\frac{c_1}{c_2}$, find out whether the following pair of linear equations are consistent, or inconsistent.

(i) $3x + 2y = 5$; $2x - 3y = 7$

(ii) $2x - 3y = 8$; $4x - 6y = 9$

(iii) $\frac{3}{2}x + \frac{5}{3}y = 7$; $9x - 10y = 14$

(iv) $5x - 3y = 11$; $-10x + 6y = -22$

(v) $\frac{4}{3}x + 2y = 8$; $2x + 3y = 12$

4. Which of the following pairs of linear equations are consistent/inconsistent? If consistent, obtain the solution graphically:

(i) $x + y = 5$, $2x + 2y = 10$

(ii) $x - y = 8$, $3x - 3y = 16$

(iii) $2x + y - 6 = 0$, $4x - 2y - 4 = 0$

(iv) $2x - 2y - 2 = 0$, $4x - 4y - 5 = 0$

5. Half the perimeter of a rectangular garden, whose length is 4 m more than its width, is 36 m. Find the dimensions of the garden.

6. Given the linear equation $2x + 3y - 8 = 0$, write another linear equation in two variables such that the geometrical representation of the pair so formed is:

(i) intersecting lines

(ii) parallel lines

(iii) coincident lines

7. Draw the graphs of the equations $x - y + 1 = 0$ and $3x + 2y - 12 = 0$. Determine the coordinates of the vertices of the triangle formed by these lines and the x-axis, and shade the triangular region.

3.3 Algebraic Methods of Solving a Pair of Linear Equations

In the previous section, we discussed how to solve a pair of linear equations graphically. The graphical method is not convenient in cases when the point representing the solution of the linear equations has non-integral coordinates like $\left(\sqrt{3}, 2\sqrt{7}\right)$, (–1.75, 3.3), $\left(\frac{4}{13}, \frac{1}{19}\right)$, etc. There is every possibility of making mistakes while reading such coordinates. Is there any alternative method of finding the solution? There are several algebraic methods, which we shall now discuss.

3.3.1 Substitution Method : We shall explain the method of substitution by taking some examples.

Example 4 : Solve the following pair of equations by substitution method:

$$7x - 15y = 2 \quad (1)$$

$$x + 2y = 3 \quad (2)$$

Solution :

Step 1 : We pick either of the equations and write one variable in terms of the other. Let us consider the Equation (2) :

$$x + 2y = 3$$

and write it as $$x = 3 - 2y \quad (3)$$

Step 2 : Substitute the value of x in Equation (1). We get

$$7(3 - 2y) - 15y = 2$$

i.e., $$21 - 14y - 15y = 2$$

i.e., $$-29y = -19$$

Therefore, $$y = \frac{19}{29}$$

Step 3 : Substituting this value of y in Equation (3), we get

$$x = 3 - 2\left(\frac{19}{29}\right) = \frac{49}{29}$$

Therefore, the solution is $x = \frac{49}{29}$, $y = \frac{19}{29}$.

Verification : Substituting $x = \frac{49}{29}$ and $y = \frac{19}{29}$, you can verify that both the Equations (1) and (2) are satisfied.

To understand the substitution method more clearly, let us consider it stepwise:

Step 1 : Find the value of one variable, say y in terms of the other variable, i.e., x from either equation, whichever is convenient.

Step 2 : Substitute this value of y in the other equation, and reduce it to an equation in one variable, i.e., in terms of x, which can be solved. Sometimes, as in Examples 9 and 10 below, you can get statements with no variable. If this statement is true, you can conclude that the pair of linear equations has infinitely many solutions. If the statement is false, then the pair of linear equations is inconsistent.

Step 3 : Substitute the value of x (or y) obtained in Step 2 in the equation used in Step 1 to obtain the value of the other variable.

Remark : We have **substituted** the value of one variable by expressing it in terms of the other variable to solve the pair of linear equations. That is why the method is known as the *substitution method*.

Example 5 : Solve the following question—Aftab tells his daughter, "Seven years ago, I was seven times as old as you were then. Also, three years from now, I shall be three times as old as you will be." (Isn't this interesting?) Represent this situation algebraically and graphically by the method of substitution.

Solution : Let s and t be the ages (in years) of Aftab and his daughter, respectively. Then, the pair of linear equations that represent the situation is

$$s - 7 = 7\,(t - 7), \text{ i.e., } s - 7t + 42 = 0 \qquad (1)$$

and

$$s + 3 = 3\,(t + 3), \text{ i.e., } s - 3t = 6 \qquad (2)$$

Using Equation (2), we get $s = 3t + 6$.

Putting this value of s in Equation (1), we get

$$(3t + 6) - 7t + 42 = 0,$$

i.e.,

$$4t = 48, \text{ which gives } t = 12.$$

Putting this value of t in Equation (2), we get

$$s = 3\,(12) + 6 = 42$$

So, Aftab and his daughter are 42 and 12 years old, respectively.

Verify this answer by checking if it satisfies the conditions of the given problems.

Example 6 : In a shop the cost of 2 pencils and 3 erasers is ₹9 and the cost of 4 pencils and 6 erasers is ₹18. Find the cost of each pencil and each eraser.

Solution : The pair of linear equations formed were:

$$2x + 3y = 9 \quad (1)$$

$$4x + 6y = 18 \quad (2)$$

We first express the value of x in terms of y from the equation $2x + 3y = 9$, to get

$$x = \frac{9 - 3y}{2} \quad (3)$$

Now we substitute this value of x in Equation (2), to get

$$\frac{4(9 - 3y)}{2} + 6y = 18$$

i.e., $18 - 6y + 6y = 18$

i.e., $18 = 18$

This statement is true for all values of y. However, we do not get a specific value of y as a solution. Therefore, we cannot obtain a specific value of x. This situation has arisen because both the given equations are the same. Therefore, Equations (1) and (2) have *infinitely many solutions*. We cannot find a unique cost of a pencil and an eraser, because there are many common solutions, to the given situation.

Example 7 : Two rails are represented by the equations $x + 2y - 4 = 0$ and $2x + 4y - 12 = 0$. Will the rails cross each other?

Solution : The pair of linear equations formed were:

$$x + 2y - 4 = 0 \quad (1)$$

$$2x + 4y - 12 = 0 \quad (2)$$

We express x in terms of y from Equation (1) to get

$$x = 4 - 2y$$

Now, we substitute this value of x in Equation (2) to get

$$2(4 - 2y) + 4y - 12 = 0$$

i.e., $8 - 12 = 0$

i.e., $-4 = 0$

which is a false statement.

Therefore, the equations do not have a common solution. So, the two rails will not cross each other.

EXERCISE 3.2

1. Solve the following pair of linear equations by the substitution method.

(i) $x + y = 14$
$x - y = 4$

(ii) $s - t = 3$
$\frac{s}{3} + \frac{t}{2} = 6$

(iii) $3x - y = 3$
$9x - 3y = 9$

(iv) $0.2x + 0.3y = 1.3$
$0.4x + 0.5y = 2.3$

(v) $\sqrt{2}\,x + \sqrt{3}\,y = 0$
$\sqrt{3}\,x - \sqrt{8}\,y = 0$

(vi) $\frac{3x}{2} - \frac{5y}{3} = -2$
$\frac{x}{3} + \frac{y}{2} = \frac{13}{6}$

2. Solve $2x + 3y = 11$ and $2x - 4y = -24$ and hence find the value of 'm' for which $y = mx + 3$.

3. Form the pair of linear equations for the following problems and find their solution by substitution method.

(i) The difference between two numbers is 26 and one number is three times the other. Find them.

(ii) The larger of two supplementary angles exceeds the smaller by 18 degrees. Find them.

(iii) The coach of a cricket team buys 7 bats and 6 balls for ₹ 3800. Later, she buys 3 bats and 5 balls for ₹ 1750. Find the cost of each bat and each ball.

(iv) The taxi charges in a city consist of a fixed charge together with the charge for the distance covered. For a distance of 10 km, the charge paid is ₹ 105 and for a journey of 15 km, the charge paid is ₹ 155. What are the fixed charges and the charge per km? How much does a person have to pay for travelling a distance of 25 km?

(v) A fraction becomes $\frac{9}{11}$, if 2 is added to both the numerator and the denominator. If, 3 is added to both the numerator and the denominator it becomes $\frac{5}{6}$. Find the fraction.

(vi) Five years hence, the age of Jacob will be three times that of his son. Five years ago, Jacob's age was seven times that of his son. What are their present ages?

3.3.2 Elimination Method

Now let us consider another method of eliminating (i.e., removing) one variable. This is sometimes more convenient than the substitution method. Let us see how this method works.

Example 8 : The ratio of incomes of two persons is 9 : 7 and the ratio of their expenditures is 4 : 3. If each of them manages to save ₹ 2000 per month, find their monthly incomes.

Solution : Let us denote the incomes of the two person by ₹ $9x$ and ₹ $7x$ and their expenditures by ₹ $4y$ and ₹ $3y$ respectively. Then the equations formed in the situation is given by :

$$9x - 4y = 2000 \qquad (1)$$

and

$$7x - 3y = 2000 \qquad (2)$$

Step 1 : Multiply Equation (1) by 3 and Equation (2) by 4 to make the coefficients of y equal. Then we get the equations:

$$27x - 12y = 6000 \qquad (3)$$

$$28x - 12y = 8000 \qquad (4)$$

Step 2 : Subtract Equation (3) from Equation (4) to *eliminate* y, because the coefficients of y are the same. So, we get

$$(28x - 27x) - (12y - 12y) = 8000 - 6000$$

i.e.,

$$x = 2000$$

Step 3 : Substituting this value of x in (1), we get

$$9(2000) - 4y = 2000$$

i.e.,

$$y = 4000$$

So, the solution of the equations is $x = 2000$, $y = 4000$. Therefore, the monthly incomes of the persons are ₹ 18,000 and ₹ 14,000, respectively.

Verification : 18000 : 14000 = 9 : 7. Also, the ratio of their expenditures = 18000 – 2000 : 14000 – 2000 = 16000 : 12000 = 4 : 3

Remarks :

1. The method used in solving the example above is called the *elimination* method, because we eliminate one variable first, to get a linear equation in one variable.

In the example above, we eliminated y. We could also have eliminated x. Try doing it that way.

2. You could also have used the substitution, or graphical method, to solve this problem. Try doing so, and see which method is more convenient.

Let us now note down these steps in the elimination method :

Step 1 : First multiply both the equations by some suitable non-zero constants to make the coefficients of one variable (either x or y) numerically equal.

Step 2 : Then add or subtract one equation from the other so that one variable gets eliminated. If you get an equation in one variable, go to Step 3.

If in Step 2, we obtain a true statement involving no variable, then the original pair of equations has infinitely many solutions.

If in Step 2, we obtain a false statement involving no variable, then the original pair of equations has no solution, i.e., it is inconsistent.

Step 3 : Solve the equation in one variable (x or y) so obtained to get its value.

Step 4 : Substitute this value of x (or y) in either of the original equations to get the value of the other variable.

Now to illustrate it, we shall solve few more examples.

Example 9 : Use elimination method to find all possible solutions of the following pair of linear equations :

$$2x + 3y = 8 \quad (1)$$

$$4x + 6y = 7 \quad (2)$$

Solution :

Step 1 : Multiply Equation (1) by 2 and Equation (2) by 1 to make the coefficients of x equal. Then we get the equations as :

$$4x + 6y = 16 \quad (3)$$

$$4x + 6y = 7 \quad (4)$$

Step 2 : Subtracting Equation (4) from Equation (3),

$$(4x - 4x) + (6y - 6y) = 16 - 7$$

i.e., $0 = 9$, which is a false statement.

Therefore, the pair of equations has no solution.

Example 10 : The sum of a two-digit number and the number obtained by reversing the digits is 66. If the digits of the number differ by 2, find the number. How many such numbers are there?

Solution : Let the ten's and the unit's digits in the first number be x and y, respectively. So, the first number may be written as $10x + y$ in the expanded form (for example, $56 = 10(5) + 6$).

When the digits are reversed, x becomes the unit's digit and y becomes the ten's digit. This number, in the expanded notation is $10y + x$ (for example, when 56 is reversed, we get $65 = 10(6) + 5$).

According to the given condition.

$$(10x + y) + (10y + x) = 66$$

i.e., $$11(x + y) = 66$$

i.e., $$x + y = 6 \quad (1)$$

We are also given that the digits differ by 2, therefore,

either $$x - y = 2 \quad (2)$$

or $$y - x = 2 \quad (3)$$

If $x - y = 2$, then solving (1) and (2) by elimination, we get $x = 4$ and $y = 2$. In this case, we get the number 42.

If $y - x = 2$, then solving (1) and (3) by elimination, we get $x = 2$ and $y = 4$. In this case, we get the number 24.

Thus, there are two such numbers 42 and 24.

Verification : Here $42 + 24 = 66$ and $4 - 2 = 2$. Also $24 + 42 = 66$ and $4 - 2 = 2$.

EXERCISE 3.3

1. Solve the following pair of linear equations by the elimination method and the substitution method :
   (i) $x + y = 5$ and $2x - 3y = 4$
   (ii) $3x + 4y = 10$ and $2x - 2y = 2$
   (iii) $3x - 5y - 4 = 0$ and $9x = 2y + 7$
   (iv) $\frac{x}{2} + \frac{2y}{3} = -1$ and $x - \frac{y}{3} = 3$

2. Form the pair of linear equations in the following problems, and find their solutions (if they exist) by the elimination method :
   (i) If we add 1 to the numerator and subtract 1 from the denominator, a fraction reduces to 1. It becomes $\frac{1}{2}$ if we only add 1 to the denominator. What is the fraction?
   (ii) Five years ago, Nuri was thrice as old as Sonu. Ten years later, Nuri will be twice as old as Sonu. How old are Nuri and Sonu?
   (iii) The sum of the digits of a two-digit number is 9. Also, nine times this number is twice the number obtained by reversing the order of the digits. Find the number.

(iv) Meena went to a bank to withdraw ₹ 2000. She asked the cashier to give her ₹ 50 and ₹ 100 notes only. Meena got 25 notes in all. Find how many notes of ₹ 50 and ₹ 100 she received.

(v) A lending library has a fixed charge for the first three days and an additional charge for each day thereafter. Saritha paid ₹ 27 for a book kept for seven days, while Susy paid ₹ 21 for the book she kept for five days. Find the fixed charge and the charge for each extra day.

3.4 Summary

In this chapter, you have studied the following points:

1. A pair of linear equations in two variables can be represented, and solved, by the:

(i) graphical method

(ii) algebraic method

2. Graphical Method :

The graph of a pair of linear equations in two variables is represented by two lines.

(i) If the lines intersect at a point, then that point gives the unique solution of the two equations. In this case, the pair of equations is **consistent**.

(ii) If the lines coincide, then there are infinitely many solutions — each point on the line being a solution. In this case, the pair of equations is **dependent (consistent)**.

(iii) If the lines are parallel, then the pair of equations has no solution. In this case, the pair of equations is **inconsistent**.

3. Algebraic Methods : We have discussed the following methods for finding the solution(s) of a pair of linear equations :

(i) Substitution Method

(ii) Elimination Method

4. If a pair of linear equations is given by $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then the following situations can arise :

(i) $\frac{a_1}{a_2} \neq \frac{b_1}{b_1}$: In this case, the pair of linear equations is consistent.

(ii) $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$: In this case, the pair of linear equations is inconsistent.

(iii) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$: In this case, the pair of linear equations is dependent and consistent.

5. There are several situations which can be mathematically represented by two equations that are not linear to start with. But we alter them so that they are reduced to a pair of linear equations.