Contents

CHAPTER 2

Coordinate Frames, Kinematics, and the Earth

This chapter provides the mathematical and physical foundations of navigation. Section 2.1 introduces the concept of a coordinate frame and how it may be used to represent an object, reference, or set of resolving axes. The main coordinate frames used in navigation are described. Section 2.2 explains the different methods of representing attitude, rotation, and resolving axes transformations, and shows how to convert between them. Section 2.3 defines the angular rate, Cartesian position, velocity, and acceleration in a multiple coordinate frame environment where the reference frame or resolving axes may be rotating; it then introduces the centrifugal and Coriolis pseudo-forces. Section 2.4 shows how the Earth's surface is modeled and defines latitude, longitude, and height. It also describes projected coordinates and Earth rotation, introduces specific force, and explains the difference between gravity and gravitation. Finally, Section 2.5 presents the equations for transforming between different coordinate frame representations.

2.1 Coordinate Frames

The science of navigation describes the position, orientation, and motion of objects. An object may be a piece of navigation equipment, such as a GNSS antenna or an INS. It may be a vehicle, such as an aircraft, ship, submarine, car, train, or satellite. It may also be a person, animal, mobile computing device, or high-value asset.

To describe the position and linear motion of an object, a specific point on that object must be selected. This is known as the *origin* of that object. It may be the center of mass of that object, the geometrical center, or an arbitrarily convenient point, such as a corner. For radio positioning equipment, the phase center of the antenna is a suitable origin as this is the point at which the radio signals appear to arrive. A point at which the sensitive axes of a number of dead-reckoning sensors intersect is also a suitable origin.

To describe the orientation and angular motion of an object, a set of three axes must also be selected. These axes must be noncoplanar and should also be mutually perpendicular. Suitable axis choices include the normal direction of motion of the object, the vertical direction when the object is at rest, the sensitive axis of an inertial or other dead-reckoning sensor, and an antenna's boresight (the normal to its plane and usually also the direction of maximum sensitivity).

However, the position, orientation, and motion of an object are meaningless on their own. Some form of reference is needed, relative to which the object may be described. The reference is also defined by an origin and a set of axes. Suitable origins include the center of the Earth, the center of the solar system, and convenient local landmarks. Suitable axes include the north, east, and vertical directions; the Earth's axis of rotation and vectors within the equatorial plane; the alignment of a local road grid; the walls of a building; and a line joining two landmarks. Another object may also act as the reference.

The origin and axes of either an object or a reference collectively comprise a *coordinate frame*. When the axes are mutually perpendicular, the coordinate frame is orthogonal and has six degrees of freedom. These are the position of the origin, $\mathbf{o}$, and the orientation of the axes, x, y, and z. They must be expressed with respect to another frame to define them. Figure 2.1 illustrates this with the superscripts denoting the frames to which the origins and axes apply. A convention is adopted here of using Greek letters to denote generic coordinate frames and Roman letters to denote specifically defined frames.

In the *right-handed convention*, the x-, y-, and z-axes are always oriented such that if the thumb and first two fingers of the right hand are extended perpendicularly, the thumb is the x-axis, the first finger is the y-axis, and the second finger is the z-axis. The opposite convention is left-handed and is rarely used. All coordinate frames considered here are both orthogonal and follow the right-handed convention. In formal terms, their axes may be described as orthogonal right-handed basis sets.

A coordinate frame may be used to describe either an object or a reference. The two concepts are actually interchangeable. In a two-frame problem, defining which one is the object frame and which one is the reference frame is arbitrary and tends to be a matter of conceptual convenience. It is equally valid to describe the position and orientation of frame α with respect to frame β as it is to describe frame β with respect to frame α. This is a principle of relativity: the laws of physics appear the same for all observers. In other words, describing the position of a road with respect to a car conveys the same information as the position of the car with respect to the road.

Any navigation problem thus involves at least two coordinate frames. These are the object frame, describing the body whose position and/or orientation is desired, and the reference frame, describing a known body, such as the Earth, relative to which the object position and/or orientation is desired. However, many navigation problems involve more than one reference frame or even more than one object frame. For example, inertial sensors measure motion with respect to inertial space, whereas a typical navigation system user wants to know their position with respect to the Earth. It is not sufficient to model motion with respect to the Earth while ignoring its rotation, as is typically done in simple mechanics problems; this can cause significant errors. Reference frame rotation also impacts GNSS positioning

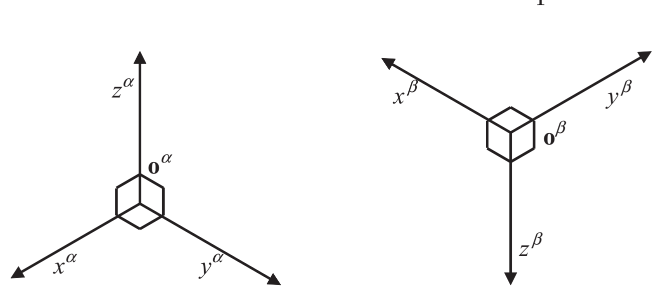

Figure 2.1 Two orthogonal coordinate frames. (From: [1]. ©2002 QinetiQ Ltd. Reprinted with permission.)

as it affects the apparent signal propagation speed. Thus, for accurate navigation, the relationship between the different coordinate frames must be properly modeled.

Any two coordinate frames may have any relative orientation, known as attitude. This may be represented in a number of different ways, as described in Section 2.2. However, within each representation, the attitude of one frame with respect to the other comprises a unique set of numbers.

A pair of coordinate frames may also have any relative position, velocity, acceleration, angular rate, and so forth. However, these quantities comprise vectors which may be resolved into components along any set of three mutually-perpendicular axes. For example, the position of frame α with respect to frame β may be described using the α-frame axes, the β-frame axes, or the axes of a third frame, γ. In practical terms, the position of a car with respect to a local road grid could be resolved about the axes of the car body frame; the road grid frame; or north, east, and down. Here, a superscript is used to denote the axes in which a quantity is expressed, known as the resolving frame. Note that it is not necessary to define the origin of the resolving frame. The position, velocity, acceleration, and angular rate in a multiple coordinate frame problem are defined in Section 2.3.

A coordinate frame definition comprises a set of rules, known as a *coordinate system*, and a set of measurements that enable known objects to be described with respect to that frame using the coordinate system. A coordinate frame may be considered a realization of the corresponding coordinate system using the measurements. Frames that are different realizations of the same coordinate system will differ slightly. Historically, nations performed their own realizations. However, international realizations, coordinated by the International Earth Rotation and Reference Systems Service (IERS), are increasingly being adopted. For more information on frame realization, the reader is directed to geodesy texts (see Selected Bibliography).

The remainder of this section defines the main coordinate systems used in navigation: Earth-centered inertial (ECI), Earth-centered Earth-fixed (ECEF), local navigation, local tangent-plane, and body frames. A brief summary of some other types of coordinate frame completes the section.

2.1.1 Earth-Centered Inertial Frame

In physics, any coordinate frame that does not accelerate or rotate with respect to the rest of the Universe is an *inertial frame*. An *Earth-centered inertial frame*, denoted by the symbol i, is nominally centered at the Earth's center of mass and oriented with respect to the Earth's spin axis and the stars. This is not strictly an inertial frame as the Earth experiences acceleration in its orbit around the Sun, its spin axis slowly moves, and the galaxy rotates. However, these effects are smaller than the measurement noise exhibited by navigation sensors, so an ECI frame may be treated as a true inertial frame for all practical purposes.

Figure 2.2 shows the origin and axes of an ECI frame and the rotation of the Earth with respect to space. The z-axis always points along the Earth's axis of rotation from the frame's origin at the center of mass to the true north pole (not the magnetic pole). The x- and y-axes lie within the equatorial plane, but do not rotate with the Earth. The y-axis points 90° ahead of the x-axis in the direction of the Earth's rotation. Note that a few authors define these axes differently.

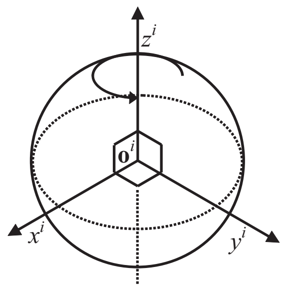

Figure 2.2 Origin and axes of an Earth-centered inertial frame. (*From:* [1]. ©2002 QinetiQ Ltd. Reprinted with permission.)

To complete the definition of the coordinate system, it is also necessary to specify the time at which the inertial frame axes coincide with those of the corresponding Earth-centered Earth-fixed frame. There are three common solutions. The first solution is simply to align the two coordinate frames when the navigation solution is initialized. The second solution is to align the coordinate frames at midnight, noting that a number of different time bases may be used, such as local time, Coordinated Universal Time (UTC), International Atomic Time (TAI), or GPS time. The final solution, used within the scientific community, is to define the x-axis as the direction from the Earth to the Sun at the vernal equinox, which is the spring equinox in the northern hemisphere. This is the same as the direction from the center of the Earth to the intersection of the Earth's equatorial plane with the Earth-Sun orbital plane (ecliptic). This version of an ECI frame is sometimes known as celestial coordinates.

A problem with realizing an ECI frame in practice is determining where the center of the Earth is with respect to known points on the surface. Instead, the origin of an ECI frame is taken as the center of an ellipsoidal representation of the Earth's surface (Section 2.4.1), which is close to the true center of mass.

A further problem, in which a precise realization of the coordinate frame is needed, is polar motion. The spin axis actually moves with respect to the solid Earth, with the poles roughly following a circular path of radius 15m. One solution is to adopt the IERS Reference Pole (IRP) or Conventional Terrestrial Pole (CTP), which is the average position of the pole surveyed between 1900 and 1905. The inertial coordinate system that adopts the center of an ellipsoidal representation of the Earth's surface as its origin, the IRP/CTP as its z-axis, and the x-axis based on the Earth-Sun axis at vernal equinox is known as the Conventional Inertial Reference System (CIRS).

Inertial frames are important in navigation because inertial sensors measure motion with respect to a generic inertial frame. An inertial reference frame and resolving axes also enables the simplest form of navigation equations to be used, as shown in later chapters.

2.1.2 Earth-Centered Earth-Fixed Frame

An *Earth-centered Earth-fixed frame*, commonly abbreviated to Earth frame, is similar to an Earth-centered inertial frame, except that all axes remain fixed with respect to the Earth. The two coordinate systems share a common origin, the center of the ellipsoid modeling the Earth's surface (Section 2.4.1), which is roughly at the center of mass. An ECEF frame is denoted by the symbol e.

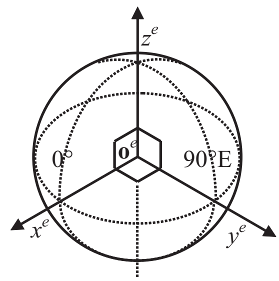

Figure 2.3 Origin and axes of an Earth-centered Earth-fixed frame. (*From:* [1]. ©2002 QinetiQ Ltd. Reprinted with permission.)

Figure 2.3 shows the origin and axes of an ECEF frame. The z-axis is the same as that of the corresponding ECI frame. It always points along the Earth's axis of rotation from the center to the north pole (true not magnetic). The x-axis points from the center to the intersection of the equator with the IERS Reference Meridian (IRM) or Conventional Zero Meridian (CZM), which defines 0° longitude. The y-axis completes the right-handed orthogonal set, pointing from the center to the intersection of the equator with the 90° east meridian. Again, note that a few authors define these axes differently. The ECEF coordinate system using the IRP/CTP and the IRM/CZM is also known as the Conventional Terrestrial Reference System (CTRS), and some authors use the symbol t to denote it.

The Earth-centered Earth-fixed coordinate system is important in navigation because the user wants to know his or her position relative to the Earth, so its realizations are commonly used as both a reference frame and a resolving frame.

2.1.3 Local Navigation Frame

A *local navigation frame*, local level navigation frame, or geodetic, geographic, or topocentric frame is denoted by the symbol n (some authors use g or l). Its origin is the object described by the navigation solution. This could be part of the navigation system itself or the center of mass of the host vehicle or user.

Figure 2.4 shows the origin and axes of a local navigation frame. The axes are aligned with the topographic directions: north, east, and vertical. In the convention used here, the z-axis, also known as the down (D) axis, is defined as the normal to

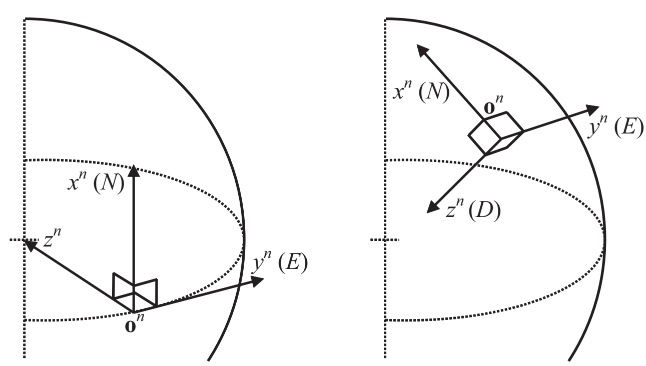

Figure 2.4 Origin and axes of a local navigation frame.

the surface of the reference ellipsoid (Section 2.4.1) in the direction pointing towards the Earth. Simple gravity models (Section 2.4.7) assume that the gravity vector is coincident with the z-axis of the corresponding local navigation frame. True gravity deviates from this slightly due to local anomalies. The x-axis, or north (N) axis, is the projection in the plane orthogonal to the z-axis of the line from the user to the north pole. The y-axis completes the orthogonal set by pointing east and is known as the east (E) axis.

North, east, down is the most common order of the axes in a local navigation coordinate system and will always be used here. However, there are other forms in use. The combination x = east, y = north, z = up is common, while x = north, y = west, z = up and x = south, y = west, z = down are also used, noting that the axes must form a right-handed set.

The local navigation coordinate system is important in navigation because the user wants to know his or her attitude relative to the north, east, and down directions. For position and velocity, it provides a convenient set of resolving axes, but is not used as a reference frame.

A major drawback of local navigation frames is that there is a singularity at each pole because the north and east axes are undefined there. Thus, navigation equations mechanized using this frame are unsuitable for use near the poles. Instead, an alternative frame should be used with conversion of the navigation solution to the local navigation frame at the end of the processing chain.

In a multibody problem, each body will have its local navigation frame. However, only one is typically of interest in practice. Furthermore, the differences in orientation between the local navigation frames of objects in close proximity are usually negligible.

2.1.4 Local Tangent-Plane Frame

A *local tangent-plane frame*, denoted by l (some authors use t), has a fixed origin with respect to the Earth, usually a point on the surface. Like the local navigation frame, its z-axis is aligned with the vertical (pointing either up or down). Its x- and y-axes may be also aligned with the topographic directions (i.e., north and east), in which case it may be known as a local geodetic frame or topocentric frame. However, the x- and y-axes may be also aligned with an environmental feature, such as a road or building. As with the other frames, the axes form a right-handed orthogonal set. Thus, this frame is Earth-fixed, but not Earth-centered. This type of frame is used for navigation within a localized area. Examples include aircraft landing and urban and indoor positioning.

A planar frame, denoted by p, is used for two-dimensional positioning; its third dimension is neglected. It may comprise the horizontal components of the local tangent-plane frame or may be used to express projected coordinates (Section 2.4.5).

2.1.5 Body Frame

A *body frame*, sometimes known as a vehicle frame, comprises the origin and orientation of the object described by the navigation solution. The origin is thus coincident with that of the corresponding local navigation frame. However, the axes remain fixed

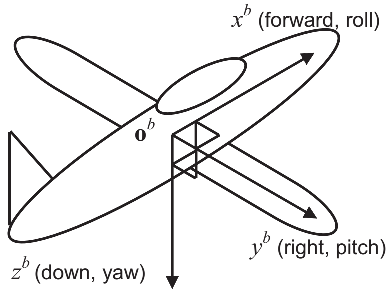

Figure 2.5 Body frame axes. (*From:* [1]. ©2002 QinetiQ Ltd. Reprinted with permission.)

with respect to the body. Here, the most common convention is adopted, whereby x is the forward axis, pointing in the usual direction of travel; z is the down axis, pointing in the usual direction of gravity; and y is the right axis, completing the orthogonal set. For angular motion, the body-frame axes are also known as roll, pitch, and yaw. Roll motion is about the x-axis, pitch motion is about the y-axis, and yaw motion is about the z-axis. Figure 2.5 illustrates this. A right-handed corkscrew rule applies, whereby if the axis is pointing away, then positive rotation about that axis is clockwise.

A body frame is essential in navigation because it describes the object that the navigation solution refers to. Inertial sensors and other dead-reckoning sensors measure the motion of a body frame and most have a fixed orientation with respect to that frame.

The symbol b is used to denote the body frame of the primary object of interest. The body frame origin may be within a navigation sensor or it may be the center of mass of the host vehicle as this simplifies the kinematics in a control system. Many navigation problems involve multiple objects, each with their own body frame, for which alternative symbols must be used. Examples include a for an antenna; c for a camera's imaging sensor; f for front wheels or an environmental feature; r for rear wheels, a reference station, or a radar transponder; s for a satellite; and t for a transmitter. For multiple satellites, transmitters, or environmental features, frames can be denoted by numbers.

2.1.6 Other Frames

A *wander-azimuth frame*, w (some authors use n), is a variant of a local navigation frame and shares the same origin and z-axis. However, the x- and y-axes are displaced from north and east by an angle, ψ_{nw} (some authors use α), known as the wander angle. Figure 2.6 illustrates this. The wander angle varies as the frame moves with respect to the Earth and is always known, so transformation of the navigation solution to a local navigation frame is straightforward. A wander-azimuth frame avoids the polar singularity of a local navigation frame, so is commonly used to mechanize inertial navigation equations. It is discussed further in Section 5.3.5.

Another variant of a local navigation frame is a *geocentric frame*. This differs in that the z-axis points from the origin to the center of the Earth instead of along the normal to the ellipsoid. The x-axis is defined in the same way as the projection of the line to the north pole in the plane orthogonal to the z-axis.

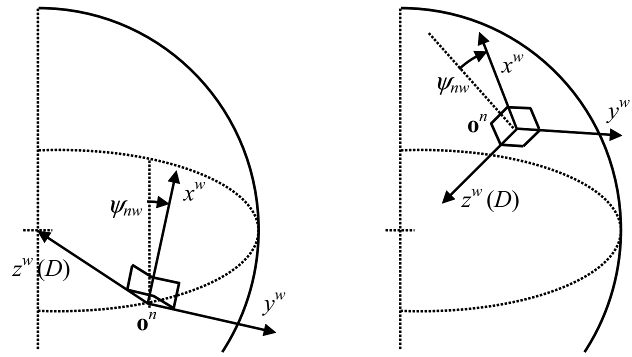

Figure 2.6 Axes of the wander-azimuth frame.

In navigation systems with directional sensors, such as an IMU, odometer, radar, Doppler sonar, and imaging sensors, the sensitive axis of each sensor may be considered to have its own body frame, known as a *sensor frame* or *instrument frame*. Thus, an IMU could be considered as having a coordinate frame for each accelerometer and gyro. However, it is generally simpler to assume that each sensor has a known orientation with respect to the navigation system body frame, particularly in cases, such as most IMUs, where the sensitive axes of the instruments are nominally aligned with the body frame. Departures from this (i.e., the instrument mounting misalignments) are then treated as a set of perturbations that must be accounted for when modeling the errors of the system.

For some sensors, such as accelerometers and odometers, a lever arm transformation (Section 2.5.5) must be performed to translate measurements from the sensor frame origin to the system body frame origin. For inertial navigation, this transformation is usually performed within the IMU (see Section 4.3).

In calculating the motion of satellites, orbital coordinate frames, denoted by o, are used. An orbital frame is an inertial frame with its origin at the Earth's center of mass, but its axes tilted with respect to the ECI frame so that satellite moves in the xy plane. More details may be found in Section 8.5.2.

A line-of-sight (LOS) frame is essentially a body frame with a zero-bank constraint (see Section 2.2.1). It is defined with its x-axis along the boresight from the sensor to the target, its y-axis in the horizontal plane, pointing to the right when looking along boresight, and its z-axis completing the orthogonal set, such that it points down when the boresight is in the horizontal plane.

2.2 Attitude, Rotation, and Resolving Axes Transformations

Attitude describes the orientation of the axes of one coordinate frame with respect to those of another. One way of representing attitude is the rotation required to align one set of axes with another. Figure 2.7 illustrates this in two dimensions; a clockwise rotation of frame γ through angle ψ, with respect to frame β, is required to align the axes of frame γ with those of frame β. Alternatively, frame β could be rotated through an angle of $-\psi$ with respect to frame γ to achieve the same axis alignment. Unless a third frame is introduced, the two rotations are indistinguishable. It is not necessary for the frame origins to coincide.

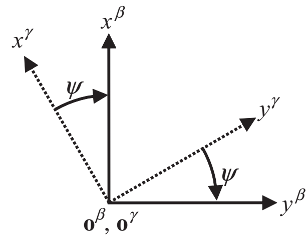

Figure 2.7 Rotation of the axes of frame γ to align with those of frame β.

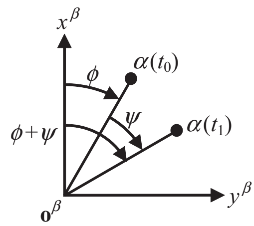

Figure 2.8 Rotation of the line $\beta\alpha$ with respect to the axes of frame β.

Consider now a line of fixed length, $r_{\beta\alpha}$, from the origin of frame β to a point, α, that is free to rotate about the origin of frame β. Figure 2.8 shows the position of the line at times t_0 and t_1.

At time t_0, the position of α with respect to the origin of frame β and resolved about the axes of that frame may be described by

$$\begin{aligned} x^\beta_{\beta\alpha}(t_0) &= r_{\beta\alpha}\cos\phi \\ y^\beta_{\beta\alpha}(t_0) &= r_{\beta\alpha}\sin\phi \end{aligned}, \tag{2.1}$$

where the superscript β denotes the frame of the resolving axes.

At time t_1, the line has rotated through an angle ψ, so the position of α is described by

$$\begin{aligned} x^\beta_{\beta\alpha}(t_1) &= r_{\beta\alpha}\cos(\phi+\psi) \\ y^\beta_{\beta\alpha}(t_1) &= r_{\beta\alpha}\sin(\phi+\psi) \end{aligned}. \tag{2.2}$$

Using trigonometric identities, it may be shown that the coordinates describing the position of α at the two times are related by

$$\begin{pmatrix} x^\beta_{\beta\alpha}(t_1) \\ y^\beta_{\beta\alpha}(t_1) \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} x^\beta_{\beta\alpha}(t_0) \\ y^\beta_{\beta\alpha}(t_0) \end{pmatrix}. \tag{2.3}$$

Note that the matrix describing the rotation is a function only of the angle of rotation, not the original orientation of the line.

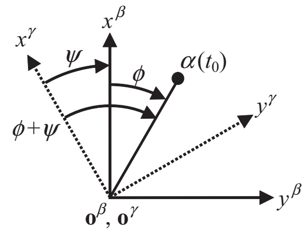

Figure 2.9 Orientation of the line $\beta\alpha$ with respect to the axes of frames β and γ.

Figure 2.9 depicts the orientation of the line $\beta\alpha$ at time t_0 with respect to frames β and γ. The position of α with respect to the origin of frame β, but resolved about the axes of frame γ is thus

$$\begin{aligned} x_{\beta\alpha}^{\gamma}(t_0) &= r_{\beta\alpha}\cos(\phi+\psi) \\ y_{\beta\alpha}^{\gamma}(t_0) &= r_{\beta\alpha}\sin(\phi+\psi) \end{aligned}. \tag{2.4}$$

Applying trigonometric identities again, it may be shown that the coordinates describing the position of α resolved about the two sets of axes are related by

$$\begin{pmatrix} x_{\beta\alpha}^{\gamma} \\ y_{\beta\alpha}^{\gamma} \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} x_{\beta\alpha}^{\beta} \\ y_{\beta\alpha}^{\beta} \end{pmatrix}. \tag{2.5}$$

Note that the matrix describing the coordinate transformation is a function only of the angle of rotation required to align one set of resolving axes with the other. Comparing this with (2.3), it can be seen that the rotation matrix of (2.3) is identical to the coordinate transformation matrix of (2.5). This is because the rotation of an object with respect to a set of resolving axes is indistinguishable from an equal and opposite rotation of the resolving axes with respect to the object.

Consequently, a coordinate transformation matrix may be used to describe a rotation and is thus a valid way of representing attitude. Conversely, a coordinate transformation (without a change in reference frame) may be represented as rotation. As the magnitude of the vector does not change, transforming a vector from one set of resolving axes to another may be thought of as applying a rotation in space to that vector.

Extending this to three dimensions, the coordinate transformation matrix is simply expanded from a 2×2 matrix to a 3×3 matrix. The 3-D extension of the rotation angle is more complex. It may be expressed as three successive scalar rotations, known as Euler angles, about defined axes. Alternatively, it may be expressed as a single scalar rotation about a particular axis that must be defined; this is represented either as a set of quaternions or as a rotation vector.

This section presents detailed descriptions of Euler angles and the coordinate transformation matrix, basic descriptions of quaternions and the rotation vector, and the equations for converting between these different attitude representations.

When combining successive rotations or axes transformations, it is essential that they are applied in the correct order, regardless of the method used to represent

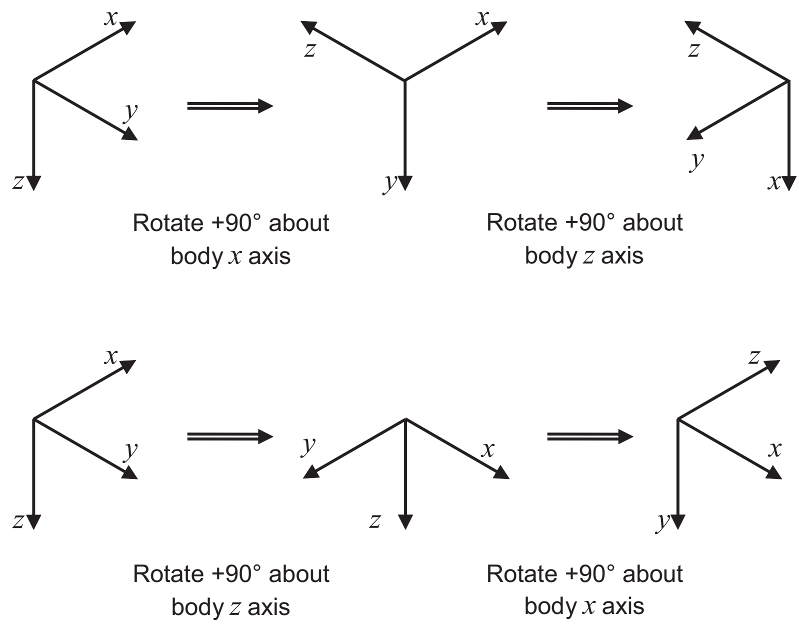

Figure 2.10 Noncommutivity of rotations.

them. This is because the order of rotations or transformations determines the final outcome. In formal terms, they do not commute. For example, a 90° rotation about the x-axis followed by a 90° rotation about the z-axis leads to a different orientation from a 90° z-axis rotation followed by a 90° x-axis rotation. This applies regardless of whether the rotations are made about the axes of the object's body frame or of a reference frame. Figure 2.10 illustrates this.

2.2.1 Euler Attitude

Euler angles (pronounced as "oiler") are the most intuitive way of describing an attitude, particularly that of a body frame with respect to the corresponding local navigation frame. The attitude is broken down into three successive rotations, with each rotation about an axis orthogonal to that of its predecessor and/or successor. Figure 2.11 illustrates this for the rotation of the axes of a coordinate frame from alignment with frame β to alignment with frame α, via alignments with two intermediate frames, ψ and θ.

The first rotation, through the angle $\psi_{\beta\alpha}$, is the *yaw* rotation. This is performed about the common z-axis of the β frame and the first intermediate frame. Thus, the x- and y-axes are rotated but the z-axis is not. Next, the *pitch rotation*, through $\theta_{\beta\alpha}$, is performed about the common y-axis of the first and second intermediate frames. This rotates the x- and z-axes. Finally, the *roll rotation*, through $\phi_{\beta\alpha}$, is performed about the common x-axis of the second intermediate frame and the α frame. This rotates the y- and z-axes.

It is convenient to represent the orientation of an object frame with respect to a reference frame using the Euler angles describing the rotation from the reference frame resolving axes to those of the object frame. Thus, the roll, pitch, and yaw Euler rotations, $\phi_{\beta\alpha}$, $\theta_{\beta\alpha}$, and $\psi_{\beta\alpha}$, describe the orientation of the object frame, α,

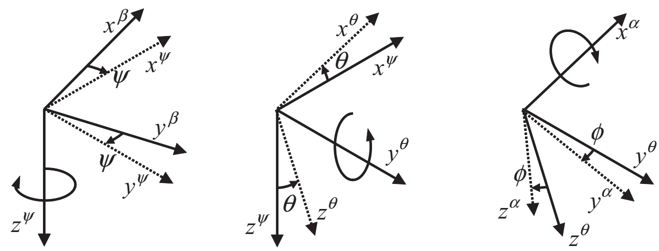

Figure 2.11 Euler angle rotations.

with respect to the reference frame, β. In the specific case in which the Euler angles describe the attitude of the body frame with respect to the local navigation frame, the roll rotation, ϕ_{nb}, is known as *bank*, the pitch rotation, θ_{nb}, is known as *elevation*, and the yaw rotation, ψ_{nb}, is known as *heading* or *azimuth*. Some authors use the term attitude to describe only the bank and elevation, excluding heading. The bank and elevation are also collectively known as *tilts*. Here, attitude always describes all three components of orientation.

Euler angles can also be used to transform a vector, $\mathbf{x} = (x, y, z)$, from one set of resolving axes, β, to a second set, α. As with rotation, the transformation occurs in three stages. First, the yaw step transforms the x and y components of the vector by performing a rotation through the angle $\psi_{\beta\alpha}$, but leaves the z component unchanged. The resulting vector is resolved about the axes of the first intermediate frame, denoted by ψ:

$$\begin{aligned} x^\psi &= x^\beta \cos\psi_{\beta\alpha} + y^\beta \sin\psi_{\beta\alpha} \\ y^\psi &= -x^\beta \sin\psi_{\beta\alpha} + y^\beta \cos\psi_{\beta\alpha} \; . \\ z^\psi &= z^\beta \end{aligned} \tag{2.6}$$

Note that this resolving axes rotation is in the opposite direction to that in the earlier example described by (2.4).

Next, the pitch step transforms the x and z components of the vector by performing a rotation through $\theta_{\beta\alpha}$. This results in a vector resolved about the axes of the second intermediate frame, denoted by θ:

$$\begin{aligned} x^\theta &= x^\psi \cos\theta_{\beta\alpha} - z^\psi \sin\theta_{\beta\alpha} \\ y^\theta &= y^\psi \qquad\qquad\qquad\qquad . \\ z^\theta &= x^\psi \sin\theta_{\beta\alpha} + z^\psi \cos\theta_{\beta\alpha} \end{aligned} \tag{2.7}$$

Finally, the roll step transforms the y and z components by performing a rotation through $\phi_{\beta\alpha}$.This produces a vector resolved about the axes of the α frame as required:

$$\begin{aligned} x^\alpha &= x^\theta \\ y^\alpha &= y^\theta \cos\phi_{\beta\alpha} + z^\theta \sin\phi_{\beta\alpha} \; . \\ z^\alpha &= -y^\theta \sin\phi_{\beta\alpha} + z^\theta \cos\phi_{\beta\alpha} \end{aligned} \tag{2.8}$$

The Euler rotation from frame β to frame α may be denoted by the vector

$$\boldsymbol{\psi}_{\beta\alpha} = \begin{pmatrix} \phi_{\beta\alpha} \\ \theta_{\beta\alpha} \\ \psi_{\beta\alpha} \end{pmatrix}, \tag{2.9}$$

noting that the Euler angles are listed in the reverse order to that in which they are applied. The order in which the three rotations are carried out is critical as each is performed in a different coordinate frame. If they are performed in a different order (e.g., with the roll first), the orientation of the axes at the end of the transformation is generally different. In formal terms, the three Euler rotations do not commute.

The Euler rotation $(\phi_{\beta\alpha} + \pi, \pi - \theta_{\beta\alpha}, \psi_{\beta\alpha} + \pi)$ gives the same result as the Euler rotation $(\phi_{\beta\alpha}, \theta_{\beta\alpha}, \psi_{\beta\alpha})$. Consequently, to avoid duplicate sets of Euler angles representing the same attitude, a convention is adopted of limiting the pitch rotation, θ, to the range $(-90° \le \theta \le 90°)$. Another property of Euler angles is that the axes about which the roll and yaw rotations are made are usually not orthogonal, although both are orthogonal to the axis about which the pitch rotation is made.

To reverse an Euler rotation, either the original operation must be reversed, beginning with the roll, or a different transformation must be applied. Simply reversing the sign of the Euler angles does not return to the original orientation, thus*

$$\begin{pmatrix} \phi_{\alpha\beta} \\ \theta_{\alpha\beta} \\ \psi_{\alpha\beta} \end{pmatrix} \ne \begin{pmatrix} -\phi_{\beta\alpha} \\ -\theta_{\beta\alpha} \\ -\psi_{\beta\alpha} \end{pmatrix}. \tag{2.10}$$

Similarly, successive rotations cannot be expressed simply by adding the Euler angles:

$$\begin{pmatrix} \phi_{\beta\gamma} \\ \theta_{\beta\gamma} \\ \psi_{\beta\gamma} \end{pmatrix} \ne \begin{pmatrix} \phi_{\beta\alpha} + \phi_{\alpha\gamma} \\ \theta_{\beta\alpha} + \theta_{\alpha\gamma} \\ \psi_{\beta\alpha} + \psi_{\alpha\gamma} \end{pmatrix}. \tag{2.11}$$

A further difficulty is that the Euler angles exhibit a singularity at ±90° pitch, where the roll and yaw become indistinguishable. Because of these difficulties, Euler angles are rarely used for attitude computation.†

2.2.2 Coordinate Transformation Matrix

The *coordinate transformation matrix*, or rotation matrix, is a 3×3 matrix, denoted as $\mathbf{C}_\alpha^\beta$ (some authors use $\mathbf{R}$ or $\mathbf{T}$). A vector may be transformed in one step from one

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set of resolving axes to another by premultiplying it by the appropriate coordinate transformation matrix. Thus, for an arbitrary vector, $\mathbf{x}$,

$$\mathbf{x}^{\beta} = \mathbf{C}_{\alpha}^{\beta}\mathbf{x}^{\alpha}, \tag{2.12}$$

where the superscript of $\mathbf{x}$ denotes the resolving axes. The lower index of the matrix represents the "from" coordinate frame and the upper index the "to" frame. The rows of a coordinate transformation matrix are in the "to" frame, whereas the columns are in the "from" frame.

When the matrix is used to represent attitude, it is more common to use the upper index (the "to" frame) to represent the reference frame, β, and the lower index (the "from" frame) to represent the object frame, α. This is because the rotation of an object's orientation with respect to a set of resolving axes is equal and opposite to the rotation of the resolving axes with respect to the object. The first case corresponds to attitude and the second to coordinate transformation. However, many authors represent attitude as a reference frame to object frame transformation, $\mathbf{C}_{\beta}^{\alpha}$.

Figure 2.12 shows the role of each element of the coordinate transformation matrix in transforming the resolving axes of a vector from frame α to frame β.

It can be shown [2] that the coordinate transformation matrix elements are the product of the unit vectors describing the axes of the two frames, which, in turn, are equal to the cosines of the angles between the axes:

$$\mathbf{C}_{\alpha}^{\beta} = \begin{pmatrix} \mathbf{u}_{\beta x}\cdot\mathbf{u}_{\alpha x} & \mathbf{u}_{\beta x}\cdot\mathbf{u}_{\alpha y} & \mathbf{u}_{\beta x}\cdot\mathbf{u}_{\alpha z} \\ \mathbf{u}_{\beta y}\cdot\mathbf{u}_{\alpha x} & \mathbf{u}_{\beta y}\cdot\mathbf{u}_{\alpha y} & \mathbf{u}_{\beta y}\cdot\mathbf{u}_{\alpha z} \\ \mathbf{u}_{\beta z}\cdot\mathbf{u}_{\alpha x} & \mathbf{u}_{\beta z}\cdot\mathbf{u}_{\alpha y} & \mathbf{u}_{\beta z}\cdot\mathbf{u}_{\alpha z} \end{pmatrix} = \begin{pmatrix} \cos\mu_{\beta x,\alpha x} & \cos\mu_{\beta x,\alpha y} & \cos\mu_{\beta x,\alpha z} \\ \cos\mu_{\beta y,\alpha x} & \cos\mu_{\beta y,\alpha y} & \cos\mu_{\beta y,\alpha z} \\ \cos\mu_{\beta z,\alpha x} & \cos\mu_{\beta z,\alpha y} & \cos\mu_{\beta z,\alpha z} \end{pmatrix}, \tag{2.13}$$

where $\mathbf{u}_i$ is a unit vector describing axis i and $\mu_{i,j}$ is the resultant angle between axes i and j. Hence, the term direction cosine matrix (DCM) is often used to describe these matrices.

Coordinate transformation matrices are easy to manipulate. As (2.13) shows, to reverse a rotation or coordinate transformation, the transpose of the matrix, denoted by the superscript, T (see Section A.2 in Appendix A on the CD), is used. Thus,

$$\mathbf{C}_{\beta}^{\alpha} = \left(\mathbf{C}_{\alpha}^{\beta}\right)^{\mathrm{T}}. \tag{2.14}$$

$$\mathbf{C}_{\alpha}^{\beta} = \begin{array}{|c|c|c|} \hline \alpha_x \rightarrow \beta_x & \alpha_y \rightarrow \beta_x & \alpha_z \rightarrow \beta_x \\ \hline \alpha_x \rightarrow \beta_y & \alpha_y \rightarrow \beta_y & \alpha_z \rightarrow \beta_y \\ \hline \alpha_x \rightarrow \beta_z & \alpha_y \rightarrow \beta_z & \alpha_z \rightarrow \beta_z \\ \hline \end{array}$$

Figure 2.12 The coordinate transformation matrix component functions.

To perform successive transformations or rotations, the coordinate transformation matrices are simply multiplied:

$$\mathbf{C}_{\alpha}^{\gamma} = \mathbf{C}_{\beta}^{\gamma}\mathbf{C}_{\alpha}^{\beta}. \tag{2.15}$$

However, as with any matrix multiplication, the order is critical, so

$$\mathbf{C}_{\alpha}^{\gamma} \neq \mathbf{C}_{\alpha}^{\beta}\mathbf{C}_{\beta}^{\gamma}. \tag{2.16}$$

This reflects the fact that rotations themselves do not commute as shown in Figure 2.10.

Performing a transformation and then reversing the process must return the original vector or matrix, so

$$\mathbf{C}_{\alpha}^{\beta}\mathbf{C}_{\beta}^{\alpha} = \mathbf{I}_3, \tag{2.17}$$

where $\mathbf{I}_n$ is the $n \times n$ identity or unit matrix. Thus, coordinate transformation matrices are orthonormal (see Section A.3 in Appendix A on the CD).

A coordinate transformation matrix can also be used to transform a matrix to which specific resolving axes apply. Consider a matrix, $\mathbf{M}$, used to transform a vector $\mathbf{a}$ into a vector $\mathbf{b}$. If $\mathbf{a}$ and $\mathbf{b}$ may be resolved about axes α or β, the transformation may be written as

$$\mathbf{b}^{\alpha} = \mathbf{M}^{\alpha}\mathbf{a}^{\alpha} \tag{2.18}$$

or

$$\mathbf{b}^{\beta} = \mathbf{M}^{\beta}\mathbf{a}^{\beta}. \tag{2.19}$$

Thus, the rows and columns of $\mathbf{M}$ must be resolved about the same axes as $\mathbf{a}$ and $\mathbf{b}$. Applying (2.12) to (2.18) gives

$$\mathbf{C}_{\beta}^{\alpha}\mathbf{b}^{\beta} = \mathbf{M}^{\alpha}\mathbf{C}_{\beta}^{\alpha}\mathbf{a}^{\beta}. \tag{2.20}$$

Premultiplying by $\mathbf{C}_{\alpha}^{\beta}$, applying (2.17), and substituting the result into (2.19) give

$$\mathbf{M}^{\beta} = \mathbf{C}_{\alpha}^{\beta}\mathbf{M}^{\alpha}\mathbf{C}_{\beta}^{\alpha}. \tag{2.21}$$

where the left-hand coordinate transformation matrix transforms the rows of $\mathbf{M}$ and the right-hand matrix transforms the columns. When the resolving frame of only the rows or only the columns of a matrix are to be transformed, respectively, only the left-hand or the right-hand coordinate transformation matrix is applied.

Although a coordinate transformation matrix has nine components, the requirement to meet (2.17) means that only three of these are independent. Thus, it has the same number of independent components as Euler attitude. A set of Euler angles is converted to a coordinate transformation matrix by first representing each of the rotations of (2.6)–(2.8) as a matrix and then multiplying, noting that with matrices, the first operation is placed on the right. Thus, for coordinate transformations, Euler angles are converted to a coordinate transformation matrix using

$$\mathbf{C}_\beta^\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_{\beta\alpha} & \sin\phi_{\beta\alpha} \\ 0 & -\sin\phi_{\beta\alpha} & \cos\phi_{\beta\alpha} \end{pmatrix} \begin{pmatrix} \cos\theta_{\beta\alpha} & 0 & -\sin\theta_{\beta\alpha} \\ 0 & 1 & 0 \\ \sin\theta_{\beta\alpha} & 0 & \cos\theta_{\beta\alpha} \end{pmatrix} \begin{pmatrix} \cos\psi_{\beta\alpha} & \sin\psi_{\beta\alpha} & 0 \\ -\sin\psi_{\beta\alpha} & \cos\psi_{\beta\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} \cos\theta_{\beta\alpha}\cos\psi_{\beta\alpha} & \cos\theta_{\beta\alpha}\sin\psi_{\beta\alpha} & -\sin\theta_{\beta\alpha} \\ \begin{pmatrix} -\cos\phi_{\beta\alpha}\sin\psi_{\beta\alpha} \\ +\sin\phi_{\beta\alpha}\sin\theta_{\beta\alpha}\cos\psi_{\beta\alpha} \end{pmatrix} & \begin{pmatrix} \cos\phi_{\beta\alpha}\cos\psi_{\beta\alpha} \\ +\sin\phi_{\beta\alpha}\sin\theta_{\beta\alpha}\sin\psi_{\beta\alpha} \end{pmatrix} & \sin\phi_{\beta\alpha}\cos\theta_{\beta\alpha} \\ \begin{pmatrix} \sin\phi_{\beta\alpha}\sin\psi_{\beta\alpha} \\ +\cos\phi_{\beta\alpha}\sin\theta_{\beta\alpha}\cos\psi_{\beta\alpha} \end{pmatrix} & \begin{pmatrix} -\sin\phi_{\beta\alpha}\cos\psi_{\beta\alpha} \\ +\cos\phi_{\beta\alpha}\sin\theta_{\beta\alpha}\sin\psi_{\beta\alpha} \end{pmatrix} & \cos\phi_{\beta\alpha}\cos\theta_{\beta\alpha} \end{bmatrix}, \tag{2.22}$$

while the reverse conversion is

$$\begin{aligned} \phi_{\beta\alpha} &= \arctan_2\left(C^\alpha_{\beta 2,3}, C^\alpha_{\beta 3,3}\right) \\ \theta_{\beta\alpha} &= -\arcsin C^\alpha_{\beta 1,3} \\ \psi_{\beta\alpha} &= \arctan_2\left(C^\alpha_{\beta 1,2}, C^\alpha_{\beta 1,1}\right) \end{aligned}, \tag{2.23}$$

noting that four-quadrant (360°) arctangent functions must be used where $\arctan_2(a, b)$ is equivalent to $\arctan(a/b)$. These conversions are used in the MATLAB functions, Euler_to_CTM and CTM_to_Euler, included on the accompanying CD.

For converting between attitude representations (e.g., between $\boldsymbol{\psi}_{nb}$ and $\mathbf{C}_b^n$), the following is normally used

$$\mathbf{C}_\alpha^\beta = \begin{bmatrix} \cos\theta_{\beta\alpha}\cos\psi_{\beta\alpha} & \begin{pmatrix} -\cos\phi_{\beta\alpha}\sin\psi_{\beta\alpha} \\ +\sin\phi_{\beta\alpha}\sin\theta_{\beta\alpha}\cos\psi_{\beta\alpha} \end{pmatrix} & \begin{pmatrix} \sin\phi_{\beta\alpha}\sin\psi_{\beta\alpha} \\ +\cos\phi_{\beta\alpha}\sin\theta_{\beta\alpha}\cos\psi_{\beta\alpha} \end{pmatrix} \\ \cos\theta_{\beta\alpha}\sin\psi_{\beta\alpha} & \begin{pmatrix} \cos\phi_{\beta\alpha}\cos\psi_{\beta\alpha} \\ +\sin\phi_{\beta\alpha}\sin\theta_{\beta\alpha}\sin\psi_{\beta\alpha} \end{pmatrix} & \begin{pmatrix} -\sin\phi_{\beta\alpha}\cos\psi_{\beta\alpha} \\ +\cos\phi_{\beta\alpha}\sin\theta_{\beta\alpha}\sin\psi_{\beta\alpha} \end{pmatrix} \\ -\sin\theta_{\beta\alpha} & \sin\phi_{\beta\alpha}\cos\theta_{\beta\alpha} & \cos\phi_{\beta\alpha}\cos\theta_{\beta\alpha} \end{bmatrix} \tag{2.24}$$

and

$$\begin{aligned} \phi_{\beta\alpha} &= \arctan_2\left(C^\beta_{\alpha 3,2}, C^\beta_{\alpha 3,3}\right) \\ \theta_{\beta\alpha} &= -\arcsin C^\beta_{\alpha 3,1} \\ \psi_{\beta\alpha} &= \arctan_2\left(C^\beta_{\alpha 2,1}, C^\beta_{\alpha 1,1}\right) \end{aligned}. \tag{2.25}$$

Again, four-quadrant arctangent functions must be used. Example 2.1 on the CD illustrates the conversion of the coordinate transformation matrix to and from Euler angles and is editable using Microsoft Excel.

When the coordinate transformation matrix and Euler angles represent a small angular perturbation for which the small angle approximation is valid, (2.22) becomes

$$\mathbf{C}_\beta^\alpha \approx \begin{pmatrix} 1 & \psi_{\beta\alpha} & -\theta_{\beta\alpha} \\ -\psi_{\beta\alpha} & 1 & \phi_{\beta\alpha} \\ \theta_{\beta\alpha} & -\phi_{\beta\alpha} & 1 \end{pmatrix} = \mathbf{I}_3 - \left[\boldsymbol{\psi}_{\beta\alpha} \wedge\right] \tag{2.26}$$

and (2.24) becomes

$$\mathbf{C}_\alpha^\beta \approx \mathbf{I}_3 + \left[\boldsymbol{\psi}_{\beta\alpha} \wedge\right] \tag{2.27}$$

where $[\boldsymbol{\psi}_{\beta\alpha} \wedge]$ denotes the skew-symmetric matrix of the Euler angles (see Section A.3 in Appendix A on the CD). Note that under the small angle approximation, $\boldsymbol{\psi}_{\alpha\beta} \approx -\boldsymbol{\psi}_{\beta\alpha}$.

One of the eigenvalues (see Section A.6 in Appendix A on the CD) of a coordinate transformation matrix is 1 (the other two are complex and have a magnitude of 1). Consequently, there exist vectors that remain unchanged following the application of a coordinate transformation matrix, or its transpose. These vectors are of the form $k\mathbf{e}_{\beta\alpha}^{\alpha/\beta}$, where k is any scalar and $\mathbf{e}_{\beta\alpha}^{\alpha/\beta}$ is the unit vector describing the axis of the rotation that the coordinate transformation matrix can be used to represent. As this vector is unchanged by (2.12), the axis of rotation is the same when resolved in the axes of the two frames transformed between. Thus,

$$\mathbf{e}_{\beta\alpha}^\alpha = \mathbf{e}_{\beta\alpha}^\beta = \mathbf{e}_{\beta\alpha}^{\alpha/\beta}. \tag{2.28}$$

Note that this rotation-axis vector takes a different value when resolved in the axes of any other frame. It may be obtained by solving

$$\begin{aligned} \mathbf{e}_{\beta\alpha}^{\alpha/\beta} &= \mathbf{C}_\beta^\alpha \mathbf{e}_{\beta\alpha}^{\alpha/\beta} \\ \mathbf{e}_{\beta\alpha}^{\alpha/\beta\mathrm{T}} \mathbf{e}_{\beta\alpha}^{\alpha/\beta} &= 1 \end{aligned}. \tag{2.29}$$

This has two solutions with opposite signs. It is conventional to select the solution

$$\mathbf{e}_{\beta\alpha}^{\alpha/\beta} = \frac{1}{2\sin\mu_{\beta\alpha}} \begin{pmatrix} C_{\beta 2,3}^\alpha - C_{\beta 3,2}^\alpha \\ C_{\beta 3,1}^\alpha - C_{\beta 1,3}^\alpha \\ C_{\beta 1,2}^\alpha - C_{\beta 2,1}^\alpha \end{pmatrix}, \tag{2.30}$$

where $\mu_{\beta\alpha}$ is the magnitude of the rotation. This is given by

$$\begin{aligned} \mu_{\beta\alpha} &= \arcsin\left(\tfrac{1}{2}\sqrt{\left(C_{\beta 2,3}^\alpha - C_{\beta 3,2}^\alpha\right)^2 + \left(C_{\beta 3,1}^\alpha - C_{\beta 1,3}^\alpha\right)^2 + \left(C_{\beta 1,2}^\alpha - C_{\beta 2,1}^\alpha\right)^2}\right) \\ &= \arccos\left[\tfrac{1}{2}\left(C_{\beta 1,1}^\alpha + C_{\beta 2,2}^\alpha + C_{\beta 3,3}^\alpha - 1\right)\right] \end{aligned}. \tag{2.31}$$

Note that $\mathbf{e}_{\alpha\beta}^{\beta/\alpha} = -\mathbf{e}_{\beta\alpha}^{\alpha/\beta}$. The axis of rotation and scalar multiples thereof are the only vectors which are invariant to a coordinate transformation (except where $\mathbf{C}_{\beta}^{\alpha} = \mathbf{I}_3$).

2.2.3 Quaternion Attitude

A rotation may be represented using a *quaternion*, which is a hyper-complex number with four components:

$$\mathbf{q} = (q_0, q_1, q_2, q_3),$$

where q_0 is a function only of the magnitude of the rotation and the other three components are functions of both the magnitude and the axis of rotation. Some authors number the components 1 to 4, with the magnitude component either at the beginning as q_1 or at the end as q_4. Thus, care must be taken to ensure that a quaternion is interpreted correctly.

As with coordinate transformation matrices, the axis of rotation is the same in both the "to" and the "from" coordinate frames of the rotation. As with the other attitude representations, only three components of the attitude quaternion are independent. It is defined as

$$\mathbf{q}_{\beta}^{\alpha} = \begin{pmatrix} \cos\left(\mu_{\beta\alpha}/2\right) \\ e_{\beta\alpha,1}^{\alpha/\beta}\sin\left(\mu_{\beta\alpha}/2\right) \\ e_{\beta\alpha,2}^{\alpha/\beta}\sin\left(\mu_{\beta\alpha}/2\right) \\ e_{\beta\alpha,3}^{\alpha/\beta}\sin\left(\mu_{\beta\alpha}/2\right) \end{pmatrix}, \tag{2.32}$$

where $\mu_{\beta\alpha}$ and $\mathbf{e}_{\beta\alpha}^{\alpha/\beta}$ are the magnitude and axis of rotation as defined in Section 2.2.2. Conversely,

$$\mu_{\beta\alpha} = 2\arccos\left(q_{\beta\alpha 0}\right), \qquad \mathbf{e}_{\beta\alpha}^{\alpha/\beta} = \frac{\mathbf{q}_{\beta 1:3}^{\alpha}}{\left|\mathbf{q}_{\beta 1:3}^{\alpha}\right|}, \tag{2.33}$$

where $(\mathbf{q}_{1:3} = q_1, q_2, q_3)$.

With only four components, the quaternion attitude representation is more computationally efficient for some processes than the coordinate transformation matrix. It also avoids the singularities inherent in Euler angles. However, manipulation of quaternions is not intuitive, so their use makes navigation equations more difficult to follow, increasing the chances of mistakes being made. Consequently, discussion of quaternions in the main body of the book is limited to their transformations to and from the other attitude representations. More details on quaternion properties and methods may be found in Section E.6 of Appendix E on the CD.

A quaternion attitude is converted to and from the corresponding coordinate transformation matrix using [3]

$$\mathbf{C}_\beta^\alpha = \begin{pmatrix} {q_{\beta 0}^\alpha}^2 + {q_{\beta 1}^\alpha}^2 - {q_{\beta 2}^\alpha}^2 - {q_{\beta 3}^\alpha}^2 & 2(q_{\beta 1}^\alpha q_{\beta 2}^\alpha + q_{\beta 3}^\alpha q_{\beta 0}^\alpha) & 2(q_{\beta 1}^\alpha q_{\beta 3}^\alpha - q_{\beta 2}^\alpha q_{\beta 0}^\alpha) \\ 2(q_{\beta 1}^\alpha q_{\beta 2}^\alpha - q_{\beta 3}^\alpha q_{\beta 0}^\alpha) & {q_{\beta 0}^\alpha}^2 - {q_{\beta 1}^\alpha}^2 + {q_{\beta 2}^\alpha}^2 - {q_{\beta 3}^\alpha}^2 & 2(q_{\beta 2}^\alpha q_{\beta 3}^\alpha + q_{\beta 1}^\alpha q_{\beta 0}^\alpha) \\ 2(q_{\beta 1}^\alpha q_{\beta 3}^\alpha + q_{\beta 2}^\alpha q_{\beta 0}^\alpha) & 2(q_{\beta 2}^\alpha q_{\beta 3}^\alpha - q_{\beta 1}^\alpha q_{\beta 0}^\alpha) & {q_{\beta 0}^\alpha}^2 - {q_{\beta 1}^\alpha}^2 - {q_{\beta 2}^\alpha}^2 + {q_{\beta 3}^\alpha}^2 \end{pmatrix}, \tag{2.34}$$

$$\begin{aligned} q_{\beta 0}^\alpha &= \tfrac{1}{2}\sqrt{1 + C_{\beta 1,1}^\alpha + C_{\beta 2,2}^\alpha + C_{\beta 3,3}^\alpha} &&= \tfrac{1}{2}\sqrt{1 + C_{\alpha 1,1}^\beta + C_{\alpha 2,2}^\beta + C_{\alpha 3,3}^\beta} \\ q_{\beta 1}^\alpha &= \frac{C_{\beta 2,3}^\alpha - C_{\beta 3,2}^\alpha}{4q_{\beta 0}^\alpha} &&= \frac{C_{\alpha 3,2}^\beta - C_{\alpha 2,3}^\beta}{4q_{\beta 0}^\alpha} \\ q_{\beta 2}^\alpha &= \frac{C_{\beta 3,1}^\alpha - C_{\beta 1,3}^\alpha}{4q_{\beta 0}^\alpha} &&= \frac{C_{\alpha 1,3}^\beta - C_{\alpha 3,1}^\beta}{4q_{\beta 0}^\alpha} \\ q_{\beta 3}^\alpha &= \frac{C_{\beta 1,2}^\alpha - C_{\beta 2,1}^\alpha}{4q_{\beta 0}^\alpha} &&= \frac{C_{\alpha 2,1}^\beta - C_{\alpha 1,2}^\beta}{4q_{\beta 0}^\alpha} \end{aligned} . \tag{2.35}$$

In cases where $q_{\beta 0}^\alpha$ is close to zero, (2.35) should be replaced by

$$\begin{aligned} q_{\beta 1}^\alpha &= \tfrac{1}{2}\sqrt{1 + C_{\beta 1,1}^\alpha - C_{\beta 2,2}^\alpha - C_{\beta 3,3}^\alpha} &&= \tfrac{1}{2}\sqrt{1 + C_{\alpha 1,1}^\beta - C_{\alpha 2,2}^\beta - C_{\alpha 3,3}^\beta} \\ q_{\beta 0}^\alpha &= \frac{C_{\beta 2,3}^\alpha - C_{\beta 3,2}^\alpha}{4q_{\beta 1}^\alpha} &&= \frac{C_{\alpha 3,2}^\beta - C_{\alpha 2,3}^\beta}{4q_{\beta 1}^\alpha} \\ q_{\beta 2}^\alpha &= \frac{C_{\beta 2,1}^\alpha + C_{\beta 1,2}^\alpha}{4q_{\beta 1}^\alpha} &&= \frac{C_{\alpha 1,2}^\beta + C_{\alpha 2,1}^\beta}{4q_{\beta 1}^\alpha} \\ q_{\beta 3}^\alpha &= \frac{C_{\beta 3,1}^\alpha + C_{\beta 1,3}^\alpha}{4q_{\beta 1}^\alpha} &&= \frac{C_{\alpha 1,3}^\beta + C_{\alpha 3,1}^\beta}{4q_{\beta 1}^\alpha} \end{aligned} . \tag{2.36}$$

The transformation between quaternion and Euler attitude is [3]

$$\begin{aligned} \phi_{\beta\alpha} &= \arctan_2\left[2\left(q_{\beta 0}^\alpha q_{\beta 1}^\alpha + q_{\beta 2}^\alpha q_{\beta 3}^\alpha\right), \left(1 - 2{q_{\beta 1}^\alpha}^2 - 2{q_{\beta 2}^\alpha}^2\right)\right] \\ \theta_{\beta\alpha} &= \arcsin\left[2\left(q_{\beta 0}^\alpha q_{\beta 2}^\alpha - q_{\beta 1}^\alpha q_{\beta 3}^\alpha\right)\right] \\ \psi_{\beta\alpha} &= \arctan_2\left[2\left(q_{\beta 0}^\alpha q_{\beta 3}^\alpha + q_{\beta 1}^\alpha q_{\beta 2}^\alpha\right), \left(1 - 2{q_{\beta 2}^\alpha}^2 - 2{q_{\beta 3}^\alpha}^2\right)\right] \end{aligned} , \tag{2.37}$$

where four-quadrant arctangent functions must be used, and

$$
\begin{aligned}
q_{\beta 0}^{\alpha} &= \cos\left(\frac{\phi_{\beta\alpha}}{2}\right)\cos\left(\frac{\theta_{\beta\alpha}}{2}\right)\cos\left(\frac{\psi_{\beta\alpha}}{2}\right) + \sin\left(\frac{\phi_{\beta\alpha}}{2}\right)\sin\left(\frac{\theta_{\beta\alpha}}{2}\right)\sin\left(\frac{\psi_{\beta\alpha}}{2}\right) \\
q_{\beta 1}^{\alpha} &= \sin\left(\frac{\phi_{\beta\alpha}}{2}\right)\cos\left(\frac{\theta_{\beta\alpha}}{2}\right)\cos\left(\frac{\psi_{\beta\alpha}}{2}\right) - \cos\left(\frac{\phi_{\beta\alpha}}{2}\right)\sin\left(\frac{\theta_{\beta\alpha}}{2}\right)\sin\left(\frac{\psi_{\beta\alpha}}{2}\right) \\
q_{\beta 2}^{\alpha} &= \cos\left(\frac{\phi_{\beta\alpha}}{2}\right)\sin\left(\frac{\theta_{\beta\alpha}}{2}\right)\cos\left(\frac{\psi_{\beta\alpha}}{2}\right) + \sin\left(\frac{\phi_{\beta\alpha}}{2}\right)\cos\left(\frac{\theta_{\beta\alpha}}{2}\right)\sin\left(\frac{\psi_{\beta\alpha}}{2}\right) \\
q_{\beta 3}^{\alpha} &= \cos\left(\frac{\phi_{\beta\alpha}}{2}\right)\cos\left(\frac{\theta_{\beta\alpha}}{2}\right)\sin\left(\frac{\psi_{\beta\alpha}}{2}\right) - \sin\left(\frac{\phi_{\beta\alpha}}{2}\right)\sin\left(\frac{\theta_{\beta\alpha}}{2}\right)\cos\left(\frac{\psi_{\beta\alpha}}{2}\right)
\end{aligned}
\qquad (2.38)
$$

Example 2.1 on the CD also illustrates the conversion of quaternion attitude to and from the coordinate transformation matrix and Euler forms.

2.2.4 Rotation Vector

The final method of representing attitude discussed here is the rotation vector [4]. This is a three-component vector, $\boldsymbol{\rho}$ (some authors use $\boldsymbol{\sigma}$), and is simply the product of the axis-of-rotation unit vector and the magnitude of the rotation. Thus,

$$\boldsymbol{\rho}_{\beta\alpha} = \mu_{\beta\alpha}\mathbf{e}_{\beta\alpha}^{\alpha/\beta}. \qquad (2.39)$$

Conversely,

$$\mu_{\beta\alpha} = \left|\boldsymbol{\rho}_{\beta\alpha}\right|, \qquad \mathbf{e}_{\beta\alpha}^{\alpha/\beta} = \frac{\boldsymbol{\rho}_{\beta\alpha}}{\left|\boldsymbol{\rho}_{\beta\alpha}\right|}. \qquad (2.40)$$

Like quaternion attitude, manipulation of rotation vectors is not intuitive, so coverage in this book is limited. More details on rotation vector methods in navigation may be found in [2].

The transformation between a rotation vector and quaternion attitude is:

$$q_{\beta 0}^{\alpha} = \cos\left(\frac{\left|\boldsymbol{\rho}_{\beta\alpha}\right|}{2}\right), \qquad \mathbf{q}_{\beta 1:3}^{\alpha} = \sin\left(\frac{\left|\boldsymbol{\rho}_{\beta\alpha}\right|}{2}\right)\frac{\boldsymbol{\rho}_{\beta\alpha}}{\left|\boldsymbol{\rho}_{\beta\alpha}\right|}, \qquad (2.41)$$

$$\boldsymbol{\rho}_{\beta\alpha} = \frac{2\arccos\left(q_{\beta 0}^{\alpha}\right)}{\sqrt{1 - {q_{\beta 0}^{\alpha}}^{2}}}\mathbf{q}_{\beta 1:3}^{\alpha}. \qquad (2.42)$$

A rotation vector is converted to a coordinate transformation matrix using

$$
\begin{aligned}
\mathbf{C}_{\beta}^{\alpha} &= \exp\left[-\boldsymbol{\rho}_{\beta\alpha} \wedge\right] \\
&= \mathbf{I}_3 - \frac{\sin\left|\boldsymbol{\rho}_{\beta\alpha}\right|}{\left|\boldsymbol{\rho}_{\beta\alpha}\right|}\left[\boldsymbol{\rho}_{\beta\alpha} \wedge\right] + \frac{1-\cos\left|\boldsymbol{\rho}_{\beta\alpha}\right|}{\left|\boldsymbol{\rho}_{\beta\alpha}\right|^{2}}\left[\boldsymbol{\rho}_{\beta\alpha} \wedge\right]^{2}
\end{aligned}
\qquad (2.43)
$$

From (2.30) and (2.39), the reverse transformation is

$$\boldsymbol{\rho}_{\beta\alpha} = \frac{\mu_{\beta\alpha}}{2\sin\mu_{\beta\alpha}} \begin{pmatrix} C^{\alpha}_{\beta 2,3} - C^{\alpha}_{\beta 3,2} \\ C^{\alpha}_{\beta 3,1} - C^{\alpha}_{\beta 1,3} \\ C^{\alpha}_{\beta 1,2} - C^{\alpha}_{\beta 2,1} \end{pmatrix}, \tag{2.44}$$

where $\mu_{\beta\alpha}$ is defined in terms of C^{α}_{β} by (2.31).

The transformation from a rotation vector to the corresponding Euler attitude is

$$\begin{aligned}
\phi_{\beta\alpha} &= \arctan_2\left[\left(\frac{\sin\left|\boldsymbol{\rho}_{\beta\alpha}\right|}{\left|\boldsymbol{\rho}_{\beta\alpha}\right|}\rho_{\beta\alpha 1} + \frac{1-\cos\left|\boldsymbol{\rho}_{\beta\alpha}\right|}{\left|\boldsymbol{\rho}_{\beta\alpha}\right|^2}\rho_{\beta\alpha 2}\rho_{\beta\alpha 3}\right), \frac{\rho^2_{\beta\alpha 3} - \cos\left|\boldsymbol{\rho}_{\beta\alpha}\right|\left(\rho^2_{\beta\alpha 1} + \rho^2_{\beta\alpha 2}\right)}{\left|\boldsymbol{\rho}_{\beta\alpha}\right|^2}\right] \\
\theta_{\beta\alpha} &= \arcsin\left[\frac{\sin\left|\boldsymbol{\rho}_{\beta\alpha}\right|}{\left|\boldsymbol{\rho}_{\beta\alpha}\right|}\rho_{\beta\alpha 2} - \frac{1-\cos\left|\boldsymbol{\rho}_{\beta\alpha}\right|}{\left|\boldsymbol{\rho}_{\beta\alpha}\right|^2}\rho_{\beta\alpha 1}\rho_{\beta\alpha 3}\right] \qquad , \\
\psi_{\beta\alpha} &= \arctan_2\left[\left(\frac{\sin\left|\boldsymbol{\rho}_{\beta\alpha}\right|}{\left|\boldsymbol{\rho}_{\beta\alpha}\right|}\rho_{\beta\alpha 3} + \frac{1-\cos\left|\boldsymbol{\rho}_{\beta\alpha}\right|}{\left|\boldsymbol{\rho}_{\beta\alpha}\right|^2}\rho_{\beta\alpha 1}\rho_{\beta\alpha 2}\right), \frac{\rho^2_{\beta\alpha 1} - \cos\left|\boldsymbol{\rho}_{\beta\alpha}\right|\left(\rho^2_{\beta\alpha 2} + \rho^2_{\beta\alpha 3}\right)}{\left|\boldsymbol{\rho}_{\beta\alpha}\right|^2}\right]
\end{aligned} \tag{2.45}$$

noting that the rotation vector and Euler angles are the same in the small angle approximation. In general, the reverse transformation is complicated, so it is better performed via the quaternion attitude or coordinate transformation matrix.

Rotation vectors are useful for interpolating attitudes as they are the only form of attitude that enables rotations to be linearly interpolated. For example, if the frame γ is at the orientation where a proportion k of the rotation from frame β to frame α has completed, the rotation vectors describing the relative attitudes of the three frames are related by

$$\begin{aligned} \boldsymbol{\rho}_{\beta\gamma} &= k\boldsymbol{\rho}_{\beta\alpha} \\ \boldsymbol{\rho}_{\gamma\alpha} &= (1-k)\boldsymbol{\rho}_{\beta\alpha} \end{aligned}. \tag{2.46}$$

Note that noncolinear rotation vectors neither commute nor combine additively.

2.3 Kinematics

In navigation, the linear and angular motion of one coordinate frame must be described with respect to another. Kinematics is the study of the motion of objects without consideration of the causes of that motion. This is in contrast to dynamics, which studies the relationship between the motion of objects and its causes.

Most kinematic quantities, such as position, velocity, acceleration, and angular rate, involve three coordinate frames:

- The frame whose motion is described, known as the *object frame*, α;
- The frame with which that motion is respect to, known as the *reference frame*, β;
- The set of axes in which that motion is represented, known as the *resolving frame*, γ.

The object frame, α, and the reference frame, β, must be different; otherwise, there is no motion. The resolving frame, γ, may be either the object frame, the reference frame, or a third frame. Its origin need not be defined; only the orientation of its axes is required. Note also that the choice of resolving frame does not affect the magnitude of a vector.

To describe these kinematic quantities fully, all three frames must be explicitly stated. Most authors do not do this, potentially causing confusion. Here, the following notation is used for Cartesian position, velocity, acceleration, and angular rate:

$$\mathbf{x}_{\beta\alpha}^{\gamma}$$

where the vector, $\mathbf{x}$, describes a kinematic property of frame α with respect to frame β, expressed in the frame γ axes. Note that, for attitude, only the object frame, α, and reference frame, β, are involved; there is no resolving frame.

In this section, the angular rate, Cartesian (as opposed to curvilinear) position, velocity, and acceleration are described in turn, correctly accounting for any rotation of the reference frame and resolving frame. Motion with respect to a rotating reference frame and the ensuing centrifugal and Coriolis pseudo-forces are then described.

2.3.1 Angular Rate

The *angular rate vector*, $\boldsymbol{\omega}_{\beta\alpha}^{\gamma}$, is the rate of rotation of the α-frame axes with respect to the β-frame axes, resolved about the γ-frame axes. Figure 2.13 illustrates the directions of the angular rate vector and the corresponding rotation that it represents. The rotation is within the plane perpendicular to the angular rate vector. Some authors use the notation p, q, and r to denote the components of angular rate about, respectively, the x-, y-, and z-axes of the resolving frame, so $\boldsymbol{\omega}_{\beta\alpha}^{\gamma} = (p_{\beta\alpha}^{\gamma}, q_{\beta\alpha}^{\gamma}, r_{\beta\alpha}^{\gamma})$.

The object and reference frames of an angular rate may be transposed simply by reversing the sign:

$$\boldsymbol{\omega}_{\beta\alpha}^{\gamma} = -\boldsymbol{\omega}_{\alpha\beta}^{\gamma}. \quad (2.47)$$

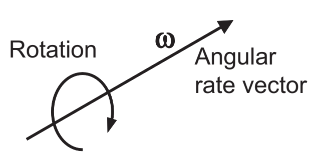

Figure 2.13 Angular rate rotation and vector directions.

Angular rates resolved about the same axes may simply be added, provided the object frame of one angular rate is the same as the reference frame of the other; thus

$$\boldsymbol{\omega}_{\beta\alpha}^{\gamma} = \boldsymbol{\omega}_{\beta\delta}^{\gamma} + \boldsymbol{\omega}_{\delta\alpha}^{\gamma}. \tag{2.48}$$

The resolving axes may be changed simply by premultiplying by the relevant coordinate transformation matrix:

$$\boldsymbol{\omega}_{\beta\alpha}^{\delta} = \mathbf{C}_{\gamma}^{\delta}\boldsymbol{\omega}_{\beta\alpha}^{\gamma}. \tag{2.49}$$

Note that the magnitude of the angular rate, $\left|\boldsymbol{\omega}_{\beta\alpha}^{\gamma}\right|$, is independent of the resolving axes, so may be written simply as $\omega_{\beta\alpha}$. However, the magnitude of the angular acceleration, $\left|\dot{\boldsymbol{\omega}}_{\beta\alpha}^{\gamma}\right|$, does depend on the choice of resolving frame.

The skew-symmetric matrix of the angular rate vector is also commonly used:

$$\boldsymbol{\Omega}_{\beta\alpha}^{\gamma} = \left[\boldsymbol{\omega}_{\beta\alpha}^{\gamma} \wedge\right] = \begin{pmatrix} 0 & -\omega_{\beta\alpha 3}^{\gamma} & \omega_{\beta\alpha 2}^{\gamma} \\ \omega_{\beta\alpha 3}^{\gamma} & 0 & -\omega_{\beta\alpha 1}^{\gamma} \\ -\omega_{\beta\alpha 2}^{\gamma} & \omega_{\beta\alpha 1}^{\gamma} & 0 \end{pmatrix}, \tag{2.50}$$

where the resolving frame, γ, of the vector $\boldsymbol{\omega}_{\beta\alpha}^{\gamma}$ applies to both the rows and the columns of its skew-symmetric matrix $\boldsymbol{\Omega}_{\beta\alpha}^{\gamma}$. Therefore, from (2.21), skew-symmetric matrices transform as

$$\boldsymbol{\Omega}_{\beta\alpha}^{\delta} = \mathbf{C}_{\gamma}^{\delta}\boldsymbol{\Omega}_{\beta\alpha}^{\gamma}\mathbf{C}_{\delta}^{\gamma}. \tag{2.51}$$

The time derivative of a coordinate transformation matrix is defined as [5, 6]

$$\dot{\mathbf{C}}_{\beta}^{\alpha}(t) = \lim_{\delta t \to 0}\left(\frac{\mathbf{C}_{\beta}^{\alpha}(t+\delta t) - \mathbf{C}_{\beta}^{\alpha}(t)}{\delta t}\right). \tag{2.52}$$

If the object frame, α, is considered to be rotating with respect to a stationary reference frame, β, the coordinate transformation matrix at time $t+\delta t$ may be written as

$$\mathbf{C}_{\beta}^{\alpha}(t+\delta t) = \mathbf{C}_{\alpha(t)}^{\alpha(t+\delta t)}\mathbf{C}_{\beta}^{\alpha}(t). \tag{2.53}$$

The rotation of the object frame over the interval t to $t+\delta t$ is infinitesimal, so may be represented by the small angle $\boldsymbol{\psi}_{\alpha(t)\alpha(t+\delta t)}$. Therefore, from (2.26),

$$\begin{aligned} \mathbf{C}_{\beta}^{\alpha}(t+\delta t) &= \left(\mathbf{I}_3 - \left[\boldsymbol{\psi}_{\alpha(t)\alpha(t+\delta t)} \wedge\right]\right)\mathbf{C}_{\beta}^{\alpha}(t) \\ &= \left(\mathbf{I}_3 - \delta t\left[\boldsymbol{\omega}_{\beta\alpha}^{\alpha} \wedge\right]\right)\mathbf{C}_{\beta}^{\alpha}(t) \quad . \\ &= \left(\mathbf{I}_3 - \delta t\boldsymbol{\Omega}_{\beta\alpha}^{\alpha}\right)\mathbf{C}_{\beta}^{\alpha}(t) \end{aligned} \tag{2.54}$$

Substituting this into (2.52) gives

$$\dot{\mathbf{C}}_{\beta}^{\alpha} = -\mathbf{\Omega}_{\beta\alpha}^{\alpha}\mathbf{C}_{\beta}^{\alpha}. \tag{2.55}$$

If the above steps are repeated under the assumption that the β frame is rotating and the α frame is stationary, the result $\dot{\mathbf{C}}_{\beta}^{\alpha} = -\mathbf{C}_{\beta}^{\alpha}\mathbf{\Omega}_{\beta\alpha}^{\beta}$ is obtained. However, applying (2.51) and (2.17) shows that these results are equivalent. From (2.47), the general result is

$$\begin{aligned}\dot{\mathbf{C}}_{\beta}^{\alpha} &= -\mathbf{C}_{\beta}^{\alpha}\mathbf{\Omega}_{\beta\alpha}^{\beta} &&= \mathbf{C}_{\beta}^{\alpha}\mathbf{\Omega}_{\alpha\beta}^{\beta} \\ &= -\mathbf{\Omega}_{\beta\alpha}^{\alpha}\mathbf{C}_{\beta}^{\alpha} &&= \mathbf{\Omega}_{\alpha\beta}^{\alpha}\mathbf{C}_{\beta}^{\alpha}\end{aligned}. \tag{2.56}$$

The inverse relationship is

$$\begin{aligned}\mathbf{\Omega}_{\alpha\beta}^{\alpha} &= \dot{\mathbf{C}}_{\beta}^{\alpha}\mathbf{C}_{\alpha}^{\beta}, \qquad & \mathbf{\Omega}_{\beta\alpha}^{\alpha} &= -\dot{\mathbf{C}}_{\beta}^{\alpha}\mathbf{C}_{\alpha}^{\beta} \\ \mathbf{\Omega}_{\alpha\beta}^{\beta} &= \mathbf{C}_{\alpha}^{\beta}\dot{\mathbf{C}}_{\beta}^{\alpha}, \qquad & \mathbf{\Omega}_{\beta\alpha}^{\beta} &= -\mathbf{C}_{\alpha}^{\beta}\dot{\mathbf{C}}_{\beta}^{\alpha}\end{aligned}. \tag{2.57}$$

The time derivative of the Euler attitude may be expressed in terms of the angular rate using [5]

$$\begin{pmatrix}\dot{\phi}_{\beta\alpha} \\ \dot{\theta}_{\beta\alpha} \\ \dot{\psi}_{\beta\alpha}\end{pmatrix} = \begin{pmatrix}1 & \sin\phi_{\beta\alpha}\tan\theta_{\beta\alpha} & \cos\phi_{\beta\alpha}\tan\theta_{\beta\alpha} \\ 0 & \cos\phi_{\beta\alpha} & -\sin\phi_{\beta\alpha} \\ 0 & \sin\phi_{\beta\alpha}/\cos\theta_{\beta\alpha} & \cos\phi_{\beta\alpha}/\cos\theta_{\beta\alpha}\end{pmatrix}\boldsymbol{\omega}_{\beta\alpha}^{\alpha}. \tag{2.58}$$

The inverse relationship is

$$\boldsymbol{\omega}_{\beta\alpha}^{\alpha} = \begin{pmatrix}1 & 0 & -\sin\theta_{\beta\alpha} \\ 0 & \cos\phi_{\beta\alpha} & \sin\phi_{\beta\alpha}\cos\theta_{\beta\alpha} \\ 0 & -\sin\phi_{\beta\alpha} & \cos\phi_{\beta\alpha}\cos\theta_{\beta\alpha}\end{pmatrix}\begin{pmatrix}\dot{\phi}_{\beta\alpha} \\ \dot{\theta}_{\beta\alpha} \\ \dot{\psi}_{\beta\alpha}\end{pmatrix}. \tag{2.59}$$

2.3.2 Cartesian Position

As Figure 2.14 shows, the *Cartesian position* of the origin of frame α with respect to the origin of frame β, resolved about the axes of frame γ, is $\mathbf{r}_{\beta\alpha}^{\gamma} = (x_{\beta\alpha}^{\gamma}, y_{\beta\alpha}^{\gamma}, z_{\beta\alpha}^{\gamma})$, where x, y, and z are the components of position in the x, y, and z axes of the γ frame. Cartesian position differs from curvilinear position (Section 2.4.2) in that the resolving axes are independent of the position vector. It is also known as the Euclidean position.

The object and reference frames of a Cartesian position may be transposed simply by reversing the sign:

$$\mathbf{r}_{\beta\alpha}^{\gamma} = -\mathbf{r}_{\alpha\beta}^{\gamma}. \tag{2.60}$$

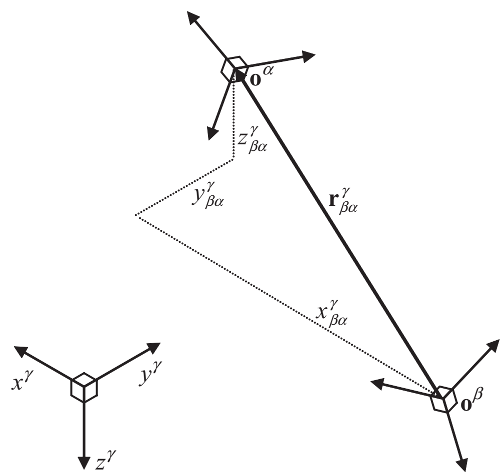

Figure 2.14 Position of the origin of frame α with respect to the origin of frame β in frame γ axes.

Similarly, two positions with common resolving axes may be subtracted if the reference frames are common or added provided the object frame of one matches the reference frame of the other:

$$\begin{aligned} \mathbf{r}_{\beta\alpha}^{\gamma} &= \mathbf{r}_{\delta\alpha}^{\gamma} - \mathbf{r}_{\delta\beta}^{\gamma} \\ &= \mathbf{r}_{\beta\delta}^{\gamma} + \mathbf{r}_{\delta\alpha}^{\gamma} \end{aligned}. \tag{2.61}$$

This may also be used to transform a position from one reference frame to another and holds for time derivatives.

Position may be resolved in a different frame by applying a coordinate transformation matrix:*

$$\mathbf{r}_{\beta\alpha}^{\delta} = \mathbf{C}_{\gamma}^{\delta}\mathbf{r}_{\beta\alpha}^{\gamma}. \tag{2.62}$$

Note that†

$$\mathbf{r}_{\alpha\beta}^{\alpha} = -\mathbf{C}_{\beta}^{\alpha}\mathbf{r}_{\beta\alpha}^{\beta}. \tag{2.63}$$

and

$$\begin{aligned} \mathbf{r}_{\beta\alpha}^{\beta} &= \mathbf{C}_{\delta}^{\beta}\left(\mathbf{r}_{\beta\delta}^{\delta} + \mathbf{r}_{\delta\alpha}^{\delta}\right) \\ &= \mathbf{r}_{\beta\delta}^{\beta} + \mathbf{C}_{\delta}^{\beta}\mathbf{r}_{\delta\alpha}^{\delta} \end{aligned}. \tag{2.64}$$

The magnitude of the Cartesian position, $\left|\mathbf{r}_{\beta\alpha}^{\gamma}\right|$, is independent of the resolving axes, so it may be written simply as $r_{\beta\alpha}$. However, the magnitude of its time derivative,

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$\left|\dot{\mathbf{r}}^{\gamma}_{\beta\alpha}\right|$, depends on the rate of rotation of the resolving frame with respect to the reference and object frames (see Section 2.3.3).

Considering specific frames, the origins of commonly realized ECI and ECEF frames coincide, as do those of local navigation and body frames for the same object. Therefore,

$$\mathbf{r}^{\gamma}_{ie} = \mathbf{r}^{\gamma}_{nb} = 0, \tag{2.65}$$

and

$$\mathbf{r}^{\gamma}_{ib} = \mathbf{r}^{\gamma}_{eb} = \mathbf{r}^{\gamma}_{in} = \mathbf{r}^{\gamma}_{en}, \tag{2.66}$$

which also holds for the time derivatives.

2.3.3 Velocity

Velocity is defined as the rate of change of the position of the origin of an object frame with respect to the origin and axes of a reference frame. This may, in turn, be resolved about the axes of a third frame. Thus, the velocity of frame α with respect to frame β, resolved about the axes of frame γ, is‡

$$\mathbf{v}^{\gamma}_{\beta\alpha} = \mathbf{C}^{\gamma}_{\beta}\dot{\mathbf{r}}^{\beta}_{\beta\alpha} \tag{2.67}$$

A velocity is thus registered if the object frame, α, moves with respect to the β-frame origin, or the reference frame, β, moves with respect to the α-frame origin. However, the velocity is defined not only with respect to the origin of the reference frame, but with respect to the axes as well. Therefore, a velocity is also registered if the reference frame, β, rotates with respect to the α-frame origin. For example, if an observer is spinning on an office chair, surrounding objects will be moving with respect to the axes of a chair-fixed reference frame. This is important in navigation as many of the commonly used reference frames rotate with respect to each other.

Figure 2.15 illustrates the three types of motion that register a velocity. No velocity is registered if the object frame rotates. Rotation of the resolving axes, γ, with respect to the reference frame, β, has no impact on the magnitude of the velocity.

It should be noted that the velocity, $\mathbf{v}^{\gamma}_{\beta\alpha}$, is not equal to the time derivative of the Cartesian position, $\mathbf{r}^{\gamma}_{\beta\alpha}$, where there is rotation of the resolving frame, γ, with respect to the reference frame, β. From (2.62) and (2.67),

$$\begin{aligned} \dot{\mathbf{r}}^{\gamma}_{\beta\alpha} &= \dot{\mathbf{C}}^{\gamma}_{\beta}\mathbf{r}^{\beta}_{\beta\alpha} + \mathbf{C}^{\gamma}_{\beta}\dot{\mathbf{r}}^{\beta}_{\beta\alpha} \\ &= \dot{\mathbf{C}}^{\gamma}_{\beta}\mathbf{r}^{\beta}_{\beta\alpha} + \mathbf{v}^{\gamma}_{\beta\alpha} \end{aligned}. \tag{2.68}$$

Rotation between the resolving axes and the reference frame is important in navigation because a local navigation frame rotates with respect to an ECEF frame as the origin of the former moves with respect to the Earth.

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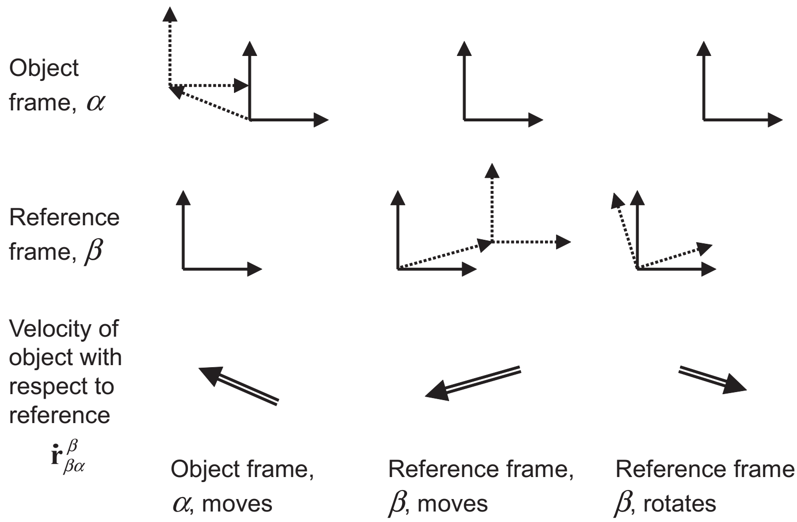

Figure 2.15 Motion causing a velocity to register.

Unlike with Cartesian position, the reference and object frames cannot be interchanged by reversing the sign unless there is no angular motion between them. The correct relationship is

$$\mathbf{v}^{\gamma}_{\alpha\beta} = -\mathbf{v}^{\gamma}_{\beta\alpha} - \mathbf{C}^{\gamma}_{\alpha}\dot{\mathbf{C}}^{\alpha}_{\beta}\mathbf{r}^{\beta}_{\beta\alpha}, \tag{2.69}$$

although

$$\mathbf{v}^{\gamma}_{\alpha\beta}\Big|_{\dot{\mathbf{C}}^{\alpha}_{\beta}=0} = -\mathbf{v}^{\gamma}_{\beta\alpha}. \tag{2.70}$$

Similarly, addition of velocities is not valid if the reference frames are rotating with respect to each other. Thus,

$$\mathbf{v}^{\gamma}_{\beta\alpha} \neq \mathbf{v}^{\gamma}_{\beta\delta} + \mathbf{v}^{\gamma}_{\delta\alpha}, \tag{2.71}$$

although

$$\mathbf{v}^{\gamma}_{\beta\alpha}\Big|_{\dot{\mathbf{C}}^{\delta}_{\beta}=0} = \mathbf{v}^{\gamma}_{\beta\delta} + \mathbf{v}^{\gamma}_{\delta\alpha}. \tag{2.72}$$

Velocity may be transformed from one resolving frame to another using the appropriate coordinate transformation matrix:

$$\mathbf{v}^{\delta}_{\beta\alpha} = \mathbf{C}^{\delta}_{\gamma}\mathbf{v}^{\gamma}_{\beta\alpha}. \tag{2.73}$$

Commonly-realized ECI and ECEF frames have a common origin, as do body and local navigation frames of the same object. Therefore,

$$\mathbf{v}^{\gamma}_{ie} = \mathbf{v}^{\gamma}_{nb} = 0, \quad \mathbf{v}^{\gamma}_{ib} = \mathbf{v}^{\gamma}_{in}, \quad \mathbf{v}^{\gamma}_{eb} = \mathbf{v}^{\gamma}_{en}. \tag{2.74}$$

However, because an ECEF frame rotates with respect to an inertial frame,

$$\mathbf{v}_{ib}^{\gamma} \neq \mathbf{v}_{eb}^{\gamma}, \quad \mathbf{v}_{in}^{\gamma} \neq \mathbf{v}_{en}^{\gamma}, \tag{2.75}$$

regardless of the resolving axes.

The Earth-referenced velocity resolved in local navigation frame axes, $\mathbf{v}_{eb}^{n}$ or $\mathbf{v}_{en}^{n}$, is often abbreviated in the literature to $\mathbf{v}^{n}$. Its counterpart resolved in ECEF frame axes, $\mathbf{v}_{eb}^{e}$, is commonly abbreviated to $\mathbf{v}^{e}$, and the inertial-referenced velocity, $\mathbf{v}_{ib}^{i}$, is abbreviated to $\mathbf{v}^{i}$.

Speed is simply the magnitude of the velocity and is independent of the resolving axes, so $v_{\beta\alpha} = \left|\mathbf{v}_{\beta\alpha}^{\gamma}\right|$. However, the magnitude of the time derivative of velocity, $\left|\dot{\mathbf{v}}_{\beta\alpha}^{\gamma}\right|$, is dependent on the choice of resolving frame.

2.3.4 Acceleration

Acceleration is defined as the second time derivative of the position of the origin of one frame with respect to the origin and axes of another frame. Thus, the acceleration of frame α with respect to frame β, resolved about the axes of frame γ, is[‡]

$$\mathbf{a}_{\beta\alpha}^{\gamma} = \mathbf{C}_{\beta}^{\gamma}\ddot{\mathbf{r}}_{\beta\alpha}^{\beta}. \tag{2.76}$$

The acceleration is the force per unit mass on the object applied from the reference frame. Its magnitude is necessarily independent of the resolving frame. It is not the same as the time derivative of $\mathbf{v}_{\beta\alpha}^{\gamma}$ or the second time derivative of $\mathbf{r}_{\beta\alpha}^{\gamma}$. These depend on the rotation of the resolving frame, γ, with respect to the reference frame, β:

$$\dot{\mathbf{v}}_{\beta\alpha}^{\gamma} = \dot{\mathbf{C}}_{\beta}^{\gamma}\dot{\mathbf{r}}_{\beta\alpha}^{\beta} + \mathbf{a}_{\beta\alpha}^{\gamma}, \tag{2.77}$$

$$\begin{aligned} \ddot{\mathbf{r}}_{\beta\alpha}^{\gamma} &= \ddot{\mathbf{C}}_{\beta}^{\gamma}\mathbf{r}_{\beta\alpha}^{\beta} + \dot{\mathbf{C}}_{\beta}^{\gamma}\dot{\mathbf{r}}_{\beta\alpha}^{\beta} + \dot{\mathbf{v}}_{\beta\alpha}^{\gamma} \\ &= \ddot{\mathbf{C}}_{\beta}^{\gamma}\mathbf{r}_{\beta\alpha}^{\beta} + 2\dot{\mathbf{C}}_{\beta}^{\gamma}\dot{\mathbf{r}}_{\beta\alpha}^{\beta} + \mathbf{a}_{\beta\alpha}^{\gamma} \end{aligned}. \tag{2.78}$$

From (2.56) and (2.62),

$$\ddot{\mathbf{C}}_{\beta}^{\gamma}\mathbf{r}_{\beta\alpha}^{\beta} = \left(\mathbf{\Omega}_{\beta\gamma}^{\gamma}\mathbf{\Omega}_{\beta\gamma}^{\gamma} - \dot{\mathbf{\Omega}}_{\beta\gamma}^{\gamma}\right)\mathbf{r}_{\beta\alpha}^{\gamma}, \tag{2.79}$$

while from (2.68), (2.56), and (2.62),

$$\begin{aligned} \dot{\mathbf{C}}_{\beta}^{\gamma}\dot{\mathbf{r}}_{\beta\alpha}^{\beta} = -\mathbf{\Omega}_{\beta\gamma}^{\gamma}\mathbf{C}_{\beta}^{\gamma}\dot{\mathbf{r}}_{\beta\alpha}^{\beta} &= \mathbf{\Omega}_{\beta\gamma}^{\gamma}\left(\dot{\mathbf{C}}_{\beta}^{\gamma}\mathbf{r}_{\beta\alpha}^{\beta} - \dot{\mathbf{r}}_{\beta\alpha}^{\gamma}\right) \\ &= -\mathbf{\Omega}_{\beta\gamma}^{\gamma}\mathbf{\Omega}_{\beta\gamma}^{\gamma}\mathbf{r}_{\beta\alpha}^{\gamma} - \mathbf{\Omega}_{\beta\gamma}^{\gamma}\dot{\mathbf{r}}_{\beta\alpha}^{\gamma} \end{aligned}. \tag{2.80}$$

[‡]This paragraph, up to this point, is based on material written by the author for QinetiQ, so comprises QinetiQ copyright material.

Substituting these into (2.78) gives

$$\ddot{\mathbf{r}}^{\gamma}_{\beta\alpha} = -\mathbf{\Omega}^{\gamma}_{\beta\gamma}\mathbf{\Omega}^{\gamma}_{\beta\gamma}\mathbf{r}^{\gamma}_{\beta\alpha} - 2\mathbf{\Omega}^{\gamma}_{\beta\gamma}\dot{\mathbf{r}}^{\gamma}_{\beta\alpha} - \dot{\mathbf{\Omega}}^{\gamma}_{\beta\gamma}\mathbf{r}^{\gamma}_{\beta\alpha} + \mathbf{a}^{\gamma}_{\beta\alpha}. \tag{2.81}$$

The first three terms of this are related to the centrifugal, Coriolis, and Euler pseudo-forces described in Section 2.3.5 [7].

As with velocity, addition of accelerations is not valid if the reference frames are rotating with respect to each other:

$$\mathbf{a}^{\gamma}_{\beta\alpha} \neq \mathbf{a}^{\gamma}_{\beta\delta} + \mathbf{a}^{\gamma}_{\delta\alpha}, \tag{2.82}$$

Similarly, an acceleration may be resolved about a different set of axes by applying the appropriate coordinate transformation matrix:

$$\mathbf{a}^{\delta}_{\beta\alpha} = \mathbf{C}^{\delta}_{\gamma}\mathbf{a}^{\gamma}_{\beta\alpha}. \tag{2.83}$$

2.3.5 Motion with Respect to a Rotating Reference Frame

In navigation, it is convenient to describe the motion of objects with respect to a rotating reference frame, such as an ECEF frame. Newton's laws of motion state that, with respect to an inertial reference frame, an object will move at constant velocity unless acted upon by a force. This does not apply with respect to a rotating frame.

Consider an object that is stationary with respect to a reference frame that is rotating at a constant rate. With respect to an inertial frame, the same object is moving in a circle centered about the axis of rotation of the rotating frame (assuming that axis is fixed with respect to the inertial frame). As the object is moving in a circle with respect to inertial space, it must be subject to a force. If the position of the object, α, with respect to inertial frame, i, is described by

$$\begin{aligned} x^{i}_{i\alpha} &= r\cos\omega_{i\beta}t \\ y^{i}_{i\alpha} &= r\sin\omega_{i\beta}t \end{aligned}, \tag{2.84}$$

where $\omega_{i\beta}$ is the angular rate and t is time, then the acceleration is

$$\begin{pmatrix} \ddot{x}^{i}_{i\alpha} \\ \ddot{y}^{i}_{i\alpha} \end{pmatrix} = -\omega^{2}_{i\beta}r\begin{pmatrix} \cos\omega_{i\beta}t \\ \sin\omega_{i\beta}t \end{pmatrix} = -\omega^{2}_{i\beta}\begin{pmatrix} x^{i}_{i\alpha} \\ y^{i}_{i\alpha} \end{pmatrix}. \tag{2.85}$$

Thus, the acceleration is towards the axis of rotation. This is *centripetal acceleration* and the corresponding force is *centripetal force*. A person on a carousel (roundabout) must be subject to a centripetal force in order to remain on the carousel.

With respect to the rotating reference frame, however, the acceleration of the object is zero. The centripetal force is still present. Therefore, from the perspective of the rotating frame, there must be another force that is equal and opposite to the centripetal force. This is the *centrifugal force* and is an example of a pseudo-force,

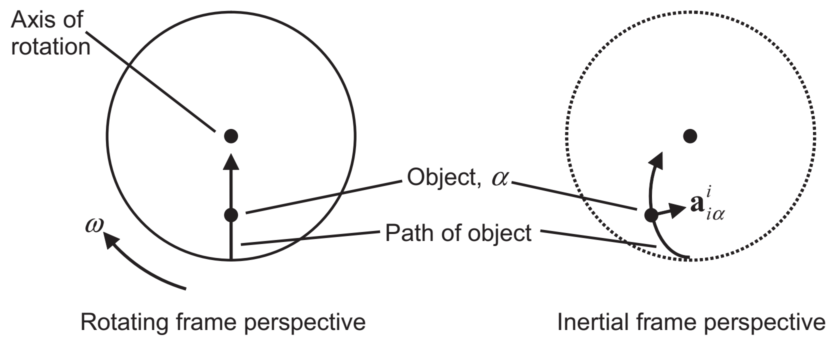

Figure 2.16 Object moving at constant velocity with respect to a rotating frame.

also known as a virtual force or a fictitious force. It arises from the use of a rotating reference frame rather than from a physical process. However, it can behave like a real force. A person on a carousel who is subject to insufficient centripetal force will appear to be pulled off the carousel by the centrifugal force.

Consider now an object that, with respect to the rotating reference frame, is moving towards the axis of rotation at a constant velocity. With respect to the inertial frame, the object is moving in a curved path and must therefore be accelerating. Figure 2.16 illustrates this. The object's velocity, with respect to inertial space, along the direction of rotation reduces as it approaches the axis of rotation. Therefore, it must be subject to a retarding force along the direction opposing rotation as well as to the centripetal force. With respect to the rotating frame, the object is moving at constant velocity, so it must have zero acceleration. Therefore, there must be a second pseudo-force that opposes the retarding force. This is the *Coriolis force* [7].

The Coriolis acceleration is always in a direction perpendicular to the object's velocity with respect to the rotating reference frame. Figure 2.17 presents some examples. If an object is set in motion, but no force is applied to maintain a constant velocity with respect to a rotating reference frame, its velocity will be constant with respect to an inertial frame. Therefore, with respect to the rotating frame, its path will appear to be curved by the action of the Coriolis force. This may be demonstrated experimentally by throwing or rolling a ball on a carousel.

Consider an object that is stationary with respect to an inertial frame. With respect to a rotating frame, the object is ascribing a circle centered at the rotation axis and is thus subject to centripetal acceleration. However, all objects described with respect to a rotating reference frame are subject to a centrifugal acceleration, in this case equal and opposite to the centripetal acceleration. However, there is no contradiction because the object is moving with respect to the rotating frame and is thus subject to a Coriolis acceleration. The centrifugal and Coriolis pseudo-accelerations sum to the centripetal acceleration required to describe the object's motion with respect to the rotating frame.

Consider the motion of an object frame, α, with respect to a rotating reference frame, β. The pseudo-acceleration, $\mathbf{a}_{\beta\alpha}^{P\beta}$, is obtained by subtracting the difference in inertially referenced accelerations of the object and reference from the total acceleration. Thus,

$$\mathbf{a}_{\beta\alpha}^{P\beta} = \mathbf{a}_{\beta\alpha}^{\beta} - \mathbf{a}_{i\alpha}^{\beta} + \mathbf{a}_{i\beta}^{\beta}, \tag{2.86}$$

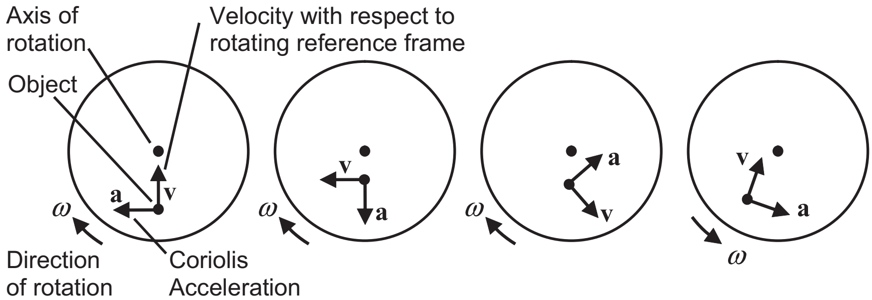

Figure 2.17 Examples of Coriolis acceleration.

where i is an inertial frame. Applying (2.76) and then (2.61),

$$\begin{aligned}\mathbf{a}_{\beta\alpha}^{P\beta} &= \ddot{\mathbf{r}}_{\beta\alpha}^{\beta} - \mathbf{C}_i^{\beta}\left(\ddot{\mathbf{r}}_{i\alpha}^{i} - \ddot{\mathbf{r}}_{i\beta}^{i}\right) \\ &= \ddot{\mathbf{r}}_{\beta\alpha}^{\beta} - \mathbf{C}_i^{\beta}\ddot{\mathbf{r}}_{\beta\alpha}^{i}\end{aligned}. \quad (2.87)$$

From (2.29),

$$\mathbf{r}_{\beta\alpha}^{\beta} = \mathbf{C}_i^{\beta}\mathbf{r}_{\beta\alpha}^{i}. \quad (2.88)$$

Differentiating this twice,

$$\ddot{\mathbf{r}}_{\beta\alpha}^{\beta} = \ddot{\mathbf{C}}_i^{\beta}\mathbf{r}_{\beta\alpha}^{i} + 2\dot{\mathbf{C}}_i^{\beta}\dot{\mathbf{r}}_{\beta\alpha}^{i} + \mathbf{C}_i^{\beta}\ddot{\mathbf{r}}_{\beta\alpha}^{i}. \quad (2.89)$$

Substituting this into (2.87),

$$\mathbf{a}_{\beta\alpha}^{P\beta} = \ddot{\mathbf{C}}_i^{\beta}\mathbf{r}_{\beta\alpha}^{i} + 2\dot{\mathbf{C}}_i^{\beta}\dot{\mathbf{r}}_{\beta\alpha}^{i}. \quad (2.90)$$

Applying (2.79) and (2.80) and rearranging,

$$\mathbf{a}_{\beta\alpha}^{P\beta} = -\mathbf{\Omega}_{i\beta}^{\beta}\mathbf{\Omega}_{i\beta}^{\beta}\mathbf{r}_{\beta\alpha}^{\beta} - 2\mathbf{\Omega}_{i\beta}^{\beta}\dot{\mathbf{r}}_{\beta\alpha}^{\beta} - \dot{\mathbf{\Omega}}_{i\beta}^{\beta}\mathbf{r}_{\beta\alpha}^{\beta}, \quad (2.91)$$

where the first term is the centrifugal acceleration, the second term is the Coriolis acceleration, and the final term is the Euler acceleration. The Euler force is the third pseudo-force and arises when the reference frame undergoes angular acceleration with respect to inertial space.

2.4 Earth Surface and Gravity Models

For most applications, a position solution with respect to the Earth's surface is required. Obtaining this requires a reference surface to be defined with respect to the center and axes of the Earth. A set of coordinates for expressing position with respect to that surface, the latitude, longitude, and height, must then be defined. For mapping, a method of projecting these coordinates onto a flat surface is required. To transform inertially referenced measurements to Earth referenced, the Earth's rotation must also be defined. This section addresses each of these issues in turn, showing how the science of geodesy is applied to navigation. It then explains the distinctions between specific force and acceleration and between gravity and gravitation, which are key concepts in inertial navigation. Finally, a selection of gravity models is presented. Appendix C on the CD presents additional information on position representations, datum transformations, and coordinate conversions.

2.4.1 The Ellipsoid Model of the Earth's Surface

An Earth-centered Earth-fixed coordinate frame enables the user to navigate with respect to the center of the Earth. However, for most practical navigation problems, the user wants to know his or her position relative to the Earth's surface. The first step is to define that surface in an ECEF frame. The Earth's surface is an oblate spheroid. *Oblate* means that it is wider at its equatorial plane than along its axis of rotational symmetry, while *spheroid* means that it is close to a sphere. Unfortunately, the Earth's surface is irregular. Modeling it accurately within a navigation system is not practical, requiring a large amount of data storage and more complex navigation algorithms. Therefore, the Earth's surface is approximated to a regular shape, which is then fitted to the true surface of the Earth at mean sea level.

The model of the Earth's surface used in most navigation systems is an oblate ellipsoid of revolution. Figure 2.18 depicts a cross-section of this reference ellipsoid, noting that this and subsequent diagrams exaggerate the flattening of the Earth. The ellipsoid exhibits rotational symmetry about the north-south (z^e) axis and mirror symmetry over the equatorial plane. It is defined by two radii. The equatorial radius, R_0, or the length of the semi-major axis, a, is the distance from the center to any point on the equator, which is the furthest part of the surface from the center. The polar radius, R_P, or the length of the semi-minor axis, b, is the distance from the center to either pole, which are the nearest points on the surface to the center.

The ellipsoid is commonly defined in terms of the equatorial radius and either the (primary or major) eccentricity of the ellipsoid, e, or the flattening of the ellipsoid, f. These are defined by

$$e = \sqrt{1 - \frac{R_P^2}{R_0^2}}, \qquad f = \frac{R_0 - R_P}{R_0}, \tag{2.92}$$

and are related by

$$e = \sqrt{2f - f^2}, \qquad f = 1 - \sqrt{1 - e^2}. \tag{2.93}$$

The Cartesian position of a point, S, on the ellipsoid surface is $\mathbf{r}_{eS}^e = \left(x_{eS}^e, y_{eS}^e, z_{eS}^e\right)$. The distance of that point from the center of the Earth is known as the geocentric radius and is simply

$$r_{eS}^e = \left|\mathbf{r}_{eS}^e\right| = \sqrt{{x_{eS}^e}^2 + {y_{eS}^e}^2 + {z_{eS}^e}^2}. \tag{2.94}$$

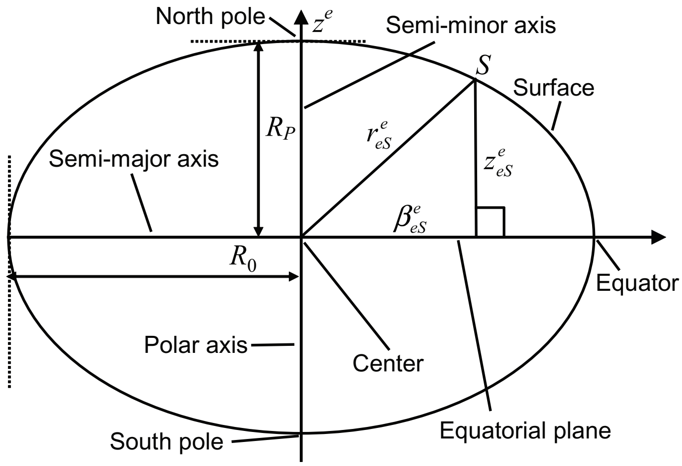

Figure 2.18 Cross-section of the ellipsoid representing the Earth's surface.

It is useful to define the magnitude of the projection of $\mathbf{r}^e_{eS}$ into the equatorial plane as β^e_{eS}. Thus,

$$\beta^e_{eS} = \sqrt{x^{e\,2}_{eS} + y^{e\,2}_{eS}}\,. \tag{2.95}$$

The cross-section of the ellipsoid shown in Figure 2.18 is the vertical plane containing the vector $\mathbf{r}^e_{eS}$. Thus, z^e_{eS} and β^e_{eS} are constrained by the ellipse equation:

$$\left(\frac{\beta^e_{eS}}{R_0}\right)^2 + \left(\frac{z^e_{eS}}{R_P}\right)^2 = 1. \tag{2.96}$$

Substituting in (2.95) defines the surface of the ellipsoid:

$$\left(\frac{x^e_{eS}}{R_0}\right)^2 + \left(\frac{y^e_{eS}}{R_0}\right)^2 + \left(\frac{z^e_{eS}}{R_P}\right)^2 = 1. \tag{2.97}$$

As well as providing a reference for determining position, the ellipsoid model is also crucial in defining a local navigation frame (Section 2.1.3), as the down direction of this frame is defined as the normal to the ellipsoid, pointing to the equatorial plane. Note that the normal to an ellipsoid does not intersect the ellipsoid center unless it passes through the poles or the equator.

Realizing the ellipsoid model in practice requires the positions of a large number of points on the Earth's surface to be measured. There is no practical method of measuring position with respect to the center of the Earth, noting that the center of an ellipsoid is not necessarily the center of mass. Consequently, position has been measured by surveying the relative positions of a number of points, a process known as triangulation. This has been done on national, regional, and continental bases, providing a host of different ellipsoid models, or geodetic datums, that provide a good fit to the Earth's surface across the area of interest, but a poor fit elsewhere in the world [8].

The advent of satellite navigation has enabled the position of points across the whole of the Earth's surface to be measured with respect to a common reference, the satellite constellation, leading to the development of global ellipsoid models. The two main standards are the World Geodetic System 1984 (WGS 84) [9] and the International Terrestrial Reference Frame (ITRF) [10]. Both of these datums have their origin at the Earth's center of mass and define rotation using the IRP/CTP.

WGS 84 was developed by the Defense Mapping Agency, now the National Geospatial-Intelligence Agency (NGA), as a standard for the U.S. military and is a refinement of predecessors WGS 60, WGS 66, and WGS 72. Its use for GPS and in most INSs led to its adoption as a global standard for navigation systems. WGS 84 was originally realized with 1691 Transit position fixes, each accurate to 1–2m, and was revised in the 1990s using GPS measurements and ITRF data [11]. As well as defining an ECEF coordinate frame and an ellipsoid, WGS 84 provides models of the Earth's geoid (Section 2.4.4) and gravity field (Section 2.4.7) and a set of fundamental constants. WGS 84 defines the ellipsoid in terms of the equatorial radius and the flattening. The polar radius and eccentricity may be derived from this. The values are listed in Table 2.1.

The ITRF is maintained by the IERS and is the datum of choice for the scientific community, particularly geodesists. It is based on a mixture of measurements from satellite laser ranging, lunar laser ranging, very long baseline interferometry (VLBI) and GPS. It is used in association with the Geodetic Reference System 1980 (GRS80) ellipsoid, also described in Table 2.1, which differs by less than a millimeter from the WGS84 ellipsoid.

ITRF is more precise than WGS 84, although the revision of the latter in the 1990s brought the two into closer alignment and WGS 84 is now considered to be a realization of the ITRF. Galileo uses a realization of the ITRF known as the Galileo Terrestrial Reference Frame (GTRF). GLONASS uses the PZ-90.02 datum, which has an origin offset from that of the ITRF by about 0.4 m. Similarly, Beidou uses the China Geodetic Coordinate System 2000 (CGCS 2000), also nominally aligned with the ITRF.

All datums must be regularly updated to account for plate tectonic motion, which causes the position of all points on the surface to move by a few centimeters each year with respect to the center of the Earth. Section C.1 of Appendix C on the CD presents more information on datums, including the transformation of coordinates between datums.

Table 2.1 Parameters of the WGS84 and GRS80 Ellipsoids

Parameter	*WGS84 Value*	*GRS80 Value*
Equatorial radius, R_0	6,378,137.0m	6,378,137.0m
Polar radius, R_P	6,356,752.31425m	6,356,752.31414m
Flattening, f	1/298.257223563	1/298.257222101
Eccentricity, e	0.0818191908425	0.0818191910428

2.4.2 Curvilinear Position

Position with respect to the Earth's surface is described using three mutually orthogonal coordinates, aligned with the axes of a local navigation frame. The distance from the body described to the surface along the normal to that surface is the *height* or *altitude*. The north-south axis coordinate of the point on the surface where that normal intersects is the *latitude*, and the coordinate of that point in the east-west axis is the *longitude*. Each of these is defined in detail later. Because the orientation of all three axes with respect to the Earth varies with location, the latitude, longitude, and height are collectively known as *curvilinear* or *ellipsoidal position*.

Connecting all points on the ellipsoid surface of the same latitude produces a circle centered about the polar (north-south) axis; this is known as a *parallel* and has radius β_{eS}^e. Similarly, the points of constant longitude on the ellipsoid surface define a semi-ellipse, running from pole to pole, known as a *meridian*. A parallel and a meridian always intersect at 90°. Planes containing a parallel or a meridian are known as parallel sections and meridian sections, respectively.

Traditionally, latitude was measured by determining the local vertical with a plumb bob and the Earth's axis of rotation from the motion of the stars. However, this *astronomical latitude* has two drawbacks. First, due to local gravity variation, multiple points along a meridian can have the same astronomical latitude [8]. Second, as a result of polar motion, the astronomical latitude of any point on the Earth varies slightly with time.

The *geocentric latitude*, Φ, illustrated in Figure 2.19, is the angle of intersection of the line from the center to a point on the surface of the ellipsoid with the equatorial plane. For all types of latitude, the convention is that latitude is positive in the northern hemisphere and negative in the southern hemisphere. By trigonometry, the geocentric latitude of a point S on the surface is given by

$$\tan\Phi_S = \frac{z_{eS}^e}{\beta_{eS}^e} = \frac{z_{eS}^e}{\sqrt{x_{eS}^{e\;2} + y_{eS}^{e\;2}}}, \qquad \sin\Phi_S = \frac{z_{eS}^e}{r_{eS}^e}. \tag{2.98}$$

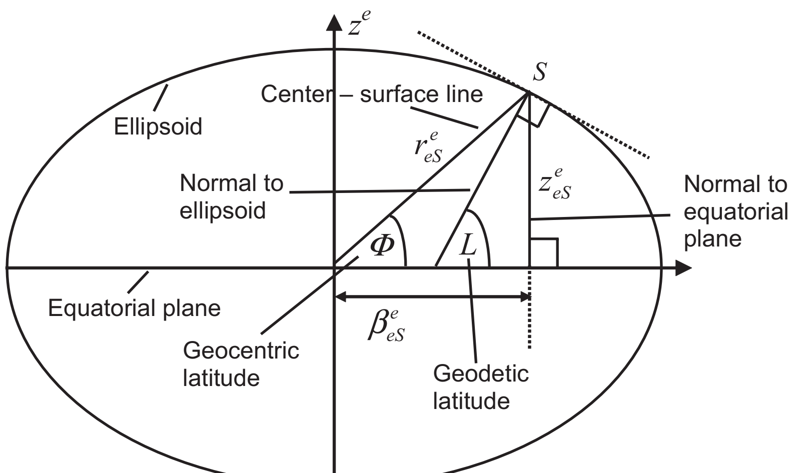

Figure 2.19 Geocentric and geodetic latitude.

The *geodetic latitude*, L, also shown in Figure 2.19, is the angle of intersection of the normal to the ellipsoid with the equatorial plane. This is sometimes known as the ellipsoidal latitude. The symbol ϕ is also commonly used. Geodetic latitude is a rationalization of astronomical latitude, retaining the basic principle, but removing the ambiguity. It is the standard form of latitude used in terrestrial navigation. As the geodetic latitude is defined by the normal to the surface, it can be obtained from the gradient of that surface. Thus, for a point S on the surface of the ellipsoid,

$$\tan L_S = -\frac{\partial \beta_{eS}^e}{\partial z_{eS}^e}. \tag{2.99}$$

Differentiating (2.96) and then substituting (2.92) and (2.95),

$$\frac{\partial \beta_{eS}^e}{\partial z_{eS}^e} = -\frac{z_{eS}^e R_0^2}{\beta_{eS}^e R_P^2} = -\frac{z_{eS}^e}{\left(1-e^2\right)\beta_{eS}^e}. \tag{2.100}$$

Thus,

$$\tan L_S = \frac{z_{eS}^e}{\left(1-e^2\right)\beta_{eS}^e} = \frac{z_{eS}^e}{\left(1-e^2\right)\sqrt{x_{eS}^{e\ 2} + y_{eS}^{e\ 2}}}. \tag{2.101}$$

Substituting in (2.98) gives the relationship between the geodetic and geocentric latitudes:

$$\tan \Phi_S = \left(1-e^2\right)\tan L_S. \tag{2.102}$$

For a body, b, which is not on the surface of the ellipsoid, the geodetic latitude is given by the coordinates of the point, $S(b)$, where the normal to the surface from that body intersects the surface. Thus,

$$\begin{aligned} \tan L_b &= \frac{z_{eS(b)}^e}{\left(1-e^2\right)\sqrt{x_{eS(b)}^{e\ \ 2} + y_{eS(b)}^{e\ \ 2}}} \\ \tan L_b &\neq \frac{z_{eb}^e}{\left(1-e^2\right)\sqrt{x_{eb}^{e\ 2} + y_{eb}^{e\ 2}}} \end{aligned}. \tag{2.103}$$

The *longitude*, λ, illustrated in Figure 2.20, is the angle subtended in the equatorial plane between the meridian plane containing the point of interest and the IERS Reference Meridian/Conventional Zero Meridian, also known as the prime meridian. The IRM is defined as the mean value of the zero longitude determinations from the adopted longitudes of a number of observatories around the world. It is approximately, but not exactly, equal to the original British zero meridian at Greenwich, London. The convention is that longitude is positive for meridians to the east of the IRM, so longitudes are positive in the eastern hemisphere and negative in the western hemisphere. Alternatively, they may be expressed between 0° and 360° or 0 and 2π rad. Note that some authors use the symbol λ for latitude and l,

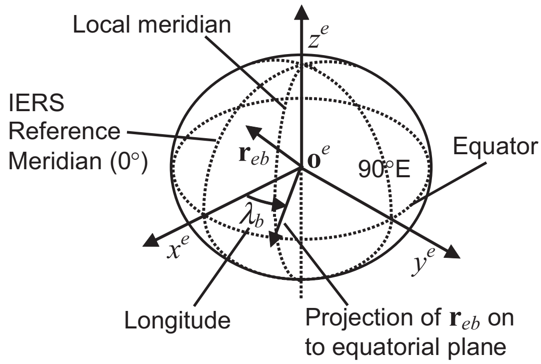

Figure 2.20 Illustration of longitude.

L, or ϕ for longitude. By trigonometry, the longitude of a point S on the surface and of any body, b, is given by

$$\tan \lambda_S = \frac{y_{eS}^e}{x_{eS}^e}, \qquad \tan \lambda_b = \frac{y_{eb}^e}{x_{eb}^e}. \tag{2.104}$$

Note that longitude is undefined at the poles, exhibiting a singularity similar to that of Euler angles at ±90° pitch. Significant numerical computation errors can occur when attempting to compute longitude very close to the north or south pole.

At this point, it is useful to define the radii of curvature of the ellipsoid. The radius of curvature for north-south motion, R_N (some authors use M or ρ), is known as the meridian radius of curvature. It is the radius of curvature of a meridian, a cross-section of the ellipsoid surface in the north-down plane, at the point of interest. This is the same as the radius of the best-fitting circle to the meridian ellipse at the point of interest. The meridian radius of curvature varies with latitude and is smallest at the equator, where the geocentric radius is largest, and largest at the poles. It is given by

$$R_N(L) = \frac{R_0\left(1 - e^2\right)}{\left(1 - e^2 \sin^2 L\right)^{3/2}} \tag{2.105}$$

The rate of change of geodetic latitude for a body traveling at unit velocity along a meridian is $1/R_N$.

The radius of curvature for east-west motion, R_E (some authors use N or v), is known as the transverse radius of curvature, normal radius of curvature, or prime vertical radius of curvature. It is the radius of curvature of a cross-section of the ellipsoid surface in the east-down plane at the point of interest. This is the vertical plane perpendicular to the meridian plane and is not the plane of constant latitude. The transverse radius of curvature varies with latitude and is smallest at the equator. It is also equal to the length of the normal from a point on the surface to the polar axis. It is given by

$$R_E(L) = \frac{R_0}{\sqrt{1 - e^2 \sin^2 L}}. \tag{2.106}$$

The rate of change of the angle subtended at the rotation axis for a body traveling at unit velocity along the surface normal to a meridian (which is not the same as a parallel) is $1/R_E$.

The transverse radius of curvature is also useful in defining the parallels on the ellipsoid surface. From (2.92), (2.96), (2.101), and (2.106), the radius of the circle of constant latitude, β_{eS}^e, and its distance from the equatorial plane, z_{eS}^e, are given by

$$\begin{aligned} \beta_{eS}^e &= R_E(L_S)\cos L_S \\ z_{eS}^e &= \left(1 - e^2\right) R_E(L_S)\sin L_S \end{aligned} . \tag{2.107}$$

The rate of change of longitude for a body traveling at unit velocity along a parallel is $1/\beta_{eS}^e$.

Figure 2.21 shows the meridian and transverse radii of curvature and the geocentric radius as a function of latitude and compares them with the equatorial and polar radii. Note that the two radii of curvature are the same at the poles, where the north-south and east-west directions are undefined. Both radii of curvature are calculated by the MATLAB function, Radii_of_curvature, on the CD.

The radius of curvature in an arbitrary direction described by the azimuth ψ_{nu} is

$$R = \left(\frac{\cos^2 \psi_{nu}}{R_N} + \frac{\sin^2 \psi_{nu}}{R_E} \right)^{-1} . \tag{2.108}$$

The *geodetic height* or altitude, h, sometimes known as the ellipsoidal height or altitude, is the distance from a body to the ellipsoid surface along the normal to that ellipsoid, with positive height denoting that the body is outside the ellipsoid. This is illustrated in Figure 2.22. By trigonometry, the height of a body, b, is given by

$$h_b = \frac{z_{eb}^e - z_{eS(b)}^e}{\sin L_b} . \tag{2.109}$$

Substituting in (2.107),

$$h_b = \frac{z_{eb}^e}{\sin L_b} - \left(1 - e^2\right) R_E(L_b) . \tag{2.110}$$

The curvilinear position of a body, b, may be expressed in vector form as $\mathbf{p}_b = (L_b, \lambda_b, h_b)$. Note that only the object frame is specified as an ECEF reference frame and local navigation frame resolving axes are implicit in the definition of curvilinear position.

At a height h_b above the ellipsoid, the meridian and transverse radii of curvature are, respectively, $R_N(L_b) + h_b$ and $R_E(L_b) + h_b$. Similarly, the radius of curvature within the parallel plane is $(R_E(L_b) + h_b)\cos L_b$. The velocity along a curve divided by the radius of curvature of that curve is equal to the time derivative of the angle

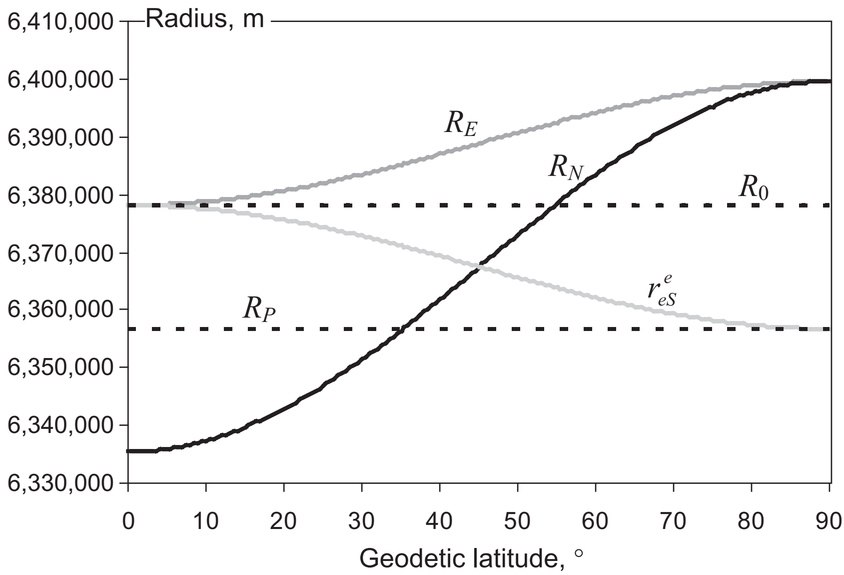

Figure 2.21 Variation of meridian and transverse radii of curvature and geocentric radius with latitude.

subtended. Therefore, the time derivative of curvilinear position is the following linear function of the Earth-referenced velocity in local navigation frame axes:

$$\begin{aligned}
\dot{L}_b &= \frac{v_{eb,N}^n}{R_N(L_b) + h_b} \\
\dot{\lambda}_b &= \frac{v_{eb,E}^n}{\left(R_E(L_b) + h_b\right)\cos L_b} \\
\dot{h}_b &= -v_{eb,D}^n
\end{aligned} \quad . \tag{2.111}$$

This enables curvilinear position to be integrated directly from velocity without having to use the Cartesian position as an intermediary.

2.4.3 Position Conversion

Using (2.95), (2.104), (2.107), and (2.110), the Cartesian ECEF position may be obtained from the curvilinear position using

$$\begin{aligned}
x_{eb}^e &= \left(R_E(L_b) + h_b\right)\cos L_b \cos\lambda_b \\
y_{eb}^e &= \left(R_E(L_b) + h_b\right)\cos L_b \sin\lambda_b \\
z_{eb}^e &= \left[\left(1 - e^2\right)R_E(L_b) + h_b\right]\sin L_b
\end{aligned} \quad . \tag{2.112}$$

Example 2.2 on the CD illustrates this and is editable using Microsoft Excel. It is also included in the MATLAB function, NED_to_ECEF, also on the CD.

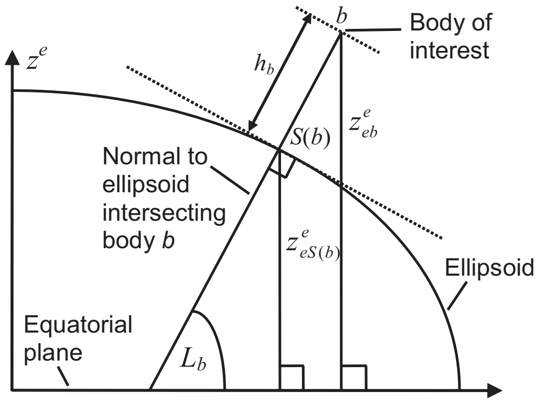

Figure 2.22 Height and geodetic latitude of a body, *b*.

The curvilinear position is obtained from the Cartesian ECEF position by implementing the inverse of the previous [12]:

$$\begin{aligned}
\tan L_b &= \frac{z^e_{eb}\left[R_E(L_b) + h_b\right]}{\sqrt{{x^e_{eb}}^2 + {y^e_{eb}}^2}\left[\left(1 - e^2\right)R_E(L_b) + h_b\right]} \\
\tan \lambda_b &= \frac{y^e_{eb}}{x^e_{eb}} \\
h_b &= \frac{\sqrt{{x^e_{eb}}^2 + {y^e_{eb}}^2}}{\cos L_b} - R_E(L_b)
\end{aligned}, \qquad (2.113)$$

where a four-quadrant arctangent function must be used for longitude. Note that, because R_E is a function of latitude, the latitude and height must be solved iteratively as Section C.2.1 of Appendix C on the CD explains in more detail. When a previous curvilinear position solution is available, it should be used to initialize the calculation. Otherwise, the convergence of the iterative process may be sped up by initializing the geodetic latitude with the geocentric latitude, Φ_b, given by

$$\Phi_b = \arctan\left(\frac{z^e_{eb}}{\sqrt{{x^e_{eb}}^2 + {y^e_{eb}}^2}}\right). \qquad (2.114)$$

In polar regions, (2.113) is replaced by

$$\begin{aligned}
\tan\left(\frac{\pi}{2} - L_b\right) &= \frac{\sqrt{x^{e2}_{eb} + y^{e2}_{eb}}\left[\left(1 - e^2\right)R_E(L_b) + h_b\right]}{z^e_{eb}\left[R_E(L_b) + h_b\right]} \\
\tan \lambda_b &= \frac{y^e_{eb}}{x^e_{eb}} \\
h_b &= \frac{z^e_{eb}}{\sin L_b} - \left(1 - e^2\right)R_E(L_b)
\end{aligned}. \qquad (2.115)$$

The following approximate closed-form latitude solution is accurate to within 1 cm for positions close to the Earth's surface [13]:

$$\tan L_b \approx \frac{z_{eb}^e \sqrt{1-e^2} + e^2 R_0 \sin^3 \zeta_b}{\sqrt{1-e^2}\left(\sqrt{x_{eb}^{e\,2} + y_{eb}^{e\,2}} - e^2 R_0 \cos^3 \zeta_b\right)}, \tag{2.116}$$

where

$$\tan \zeta_b = \frac{z_{eb}^e}{\sqrt{1-e^2}\sqrt{x_{eb}^{e\,2} + y_{eb}^{e\,2}}}. \tag{2.117}$$

Section C.2 of Appendix C on the CD presents an iterative version of this. It also describes further iterated solutions and two closed-form exact solutions. All methods are included in Example 2.2 on the CD, while the Borkowski closed-form exact solution is included in the MATLAB function, ECEF_to_NED, on the CD.

Great care should be taken when Cartesian and curvilinear positions are mixed within a set of navigation equations to ensure that the curvilinear position computation is performed with sufficient precision. Otherwise, a divergent position solution could result.

Small perturbations to the position may be converted between Cartesian and curvilinear representation using

$$\delta \mathbf{r}_{eb}^e \approx \mathbf{C}_n^e \mathbf{T}_p^{r(n)} \delta \mathbf{p}_b, \qquad \delta \mathbf{p}_b \approx \mathbf{T}_{r(n)}^p \mathbf{C}_e^n \delta \mathbf{r}_{eb}^e \tag{2.118}$$

where

$$\mathbf{T}_{r(n)}^p = \frac{\partial \mathbf{p}_b}{\partial \mathbf{r}_{eb}^n} = \begin{pmatrix} \dfrac{1}{R_N(L_b) + h_b} & 0 & 0 \\ 0 & \dfrac{1}{\left(R_E(L_b) + h_b\right)\cos L_b} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{2.119}$$

$$\mathbf{T}_p^{r(n)} = \frac{\partial \mathbf{r}_{eb}^n}{\partial \mathbf{p}_b} = \begin{pmatrix} R_N(L_b) + h_b & 0 & 0 \\ 0 & \left(R_E(L_b) + h_b\right)\cos L_b & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.120}$$

These are particularly useful for converting error standard deviations.

Section C.3 of Appendix C on the CD describes the normal vector representation of curvilinear position, which avoids the longitude singularity at the poles.

Finally, although most navigation systems now use the WGS 84 datum, many maps are based on national and regional datums. This is partly for historical reasons and partly because it is convenient to map features using datums that move with the tectonic plates. Consequently, it may be necessary to transform curvilinear or Cartesian position from one datum to another. The datums may use different origins, axis alignments, and scalings as well as different radii of curvature. Datum transformations are described in Section C.1 of Appendix C on the CD. No conversion between WGS 84 and ITRF position is needed as the differences between the two datums are less than the uncertainty bounds.

2.4.4 The Geoid, Orthometric Height, and Earth Tides

The gravity potential is the potential energy required to overcome gravity (see Section 2.4.7). As water will always flow from an area of higher gravity potential to an area of lower gravity potential, mean sea level, which is averaged over the tide cycle, maintains a surface of approximately equal gravity potential (differences arise due to permanent ocean currents). The geoid is a model of the Earth's surface that has a constant gravity potential; it is an example of an equipotential surface. The geoid is generally within 1m of mean sea level [13]. Note that, over land, the physical surface of the Earth, known as the terrain, is generally above the geoid. The gravity vector at any point on the Earth's surface is thus perpendicular to the geoid, not the ellipsoid or the terrain, although, in practice, the difference is small.

As the Earth's gravity field varies with location, the geoid can differ from the ellipsoid by up to 100m. The height of the geoid with respect to the ellipsoid is denoted N; this is known as the *geoid–ellipsoid separation*. The current WGS 84 geoid model is known as the Earth Gravitational Model 2008 (EGM 08) and has 4,730,400 (= 2,160 × 2,190) coefficients defining the geoid height, N, and gravitational potential as a spherical harmonic function of geodetic latitude and longitude [14]. A geoid model is also known as a vertical datum.

The height of a body above the geoid is known as the *orthometric height* or *orthometric altitude* and is denoted as H. The height or altitude above mean sea level (AMSL) is also commonly used. The orthometric height of the terrain is known as *elevation*. The orthometric height is related to the geodetic height by

$$H_b \approx h_b - N(L_b, \lambda_b). \tag{2.121}$$

This is not exact because the geodetic height is measured normal to the ellipsoid, whereas the orthometric height is measured normal to the geoid. Figure 2.23 illustrates the two heights, the geoid, ellipsoid, and terrain.

For many applications, orthometric height is more useful than geodetic height. Maps tend to express the height of the terrain and features with respect to the geoid, making orthometric height critical for aircraft approach, landing, and low-level flight. It is also important in civil engineering, for example, to determine the direction of flow of water. Thus, a navigation system will often need to incorporate a geoid model to convert between geodetic and orthometric height.

It is well known that lunar gravitation causes ocean tides. However, it also causes tidal movement of the Earth's crust and there are tidal effects due to solar gravitation. Together, these are known as solid Earth tides and cause the positions of the terrain and features thereon to vary with respect to the geoid and ellipsoid with an amplitude of about half a meter. The vertical displacement is largest, but there is also horizontal displacement. There are multiple oscillations with varying

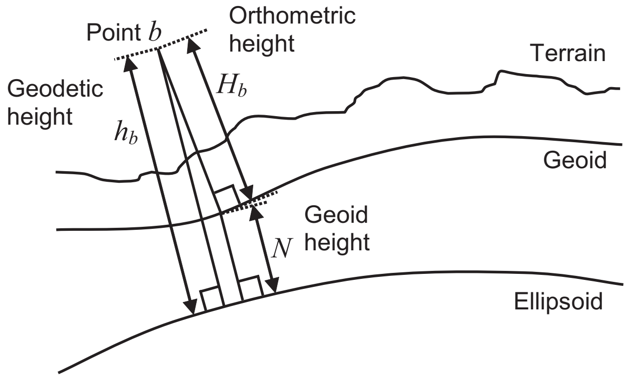

Figure 2.23 Height, geoid, ellipsoid, and terrain. (*After*: [12].)

periods that contribute to the solid Earth tides, with the largest components having approximately diurnal (~24 hour) and semidiurnal (~12 hour) periods.

Solid Earth tides predominantly affect positioning using GNSS and other satellite-based techniques. An appropriate correction is applied to obtain a time-invariant position solution [15]. However, for most navigation applications, solid Earth tides are neglected.

2.4.5 Projected Coordinates

Projected coordinates provide a way of representing the ellipsoid as a flat surface. This is essential for printing maps on paper or displaying them on a flat screen. A projection converts geodetic latitude and longitude to and from planar Cartesian coordinates. The projection may be arbitrary, but more commonly represents a straight line from a focal point or line, through the surface of the ellipsoid, to the corresponding point on the 2-D surface. The 2-D surface may be represented in 3-D space as a plane. Alternatively, it may be wrapped into a cylinder or cone. Projections are thus categorized as cylindrical, conical, or planar. Figure 2.24 illustrates some examples [13, 16].

The aspect denotes the orientation of the 2-D surface. A cylindrical or conical projection has a normal aspect if its axis of rotational symmetry is aligned with the north-south axis of the ellipsoid, a transverse aspect if its axis is within the equatorial plane, and an oblique aspect otherwise. A planar projection has a normal aspect if it is perpendicular to the equatorial plane, a polar aspect if it is parallel, and an oblique aspect otherwise [16]. Aspects are indicated in Figure 2.24.

All projections distort the shape of large-scale features as geometry on a flat surface is fundamentally different from that on a curved surface. Different classes of projections preserve some features of geometry and distort others. A conformal projection preserves the shape of small-scale features, an equal-area projection preserves areas, an equidistant projection preserves distances along at least one line, and an azimuthal projection preserves the angular relationship of all features with respect to the center of the projection [16].

A transverse Mercator projection is a conformal transverse cylindrical projection, commonly used by national mapping agencies. Examples include the Universal

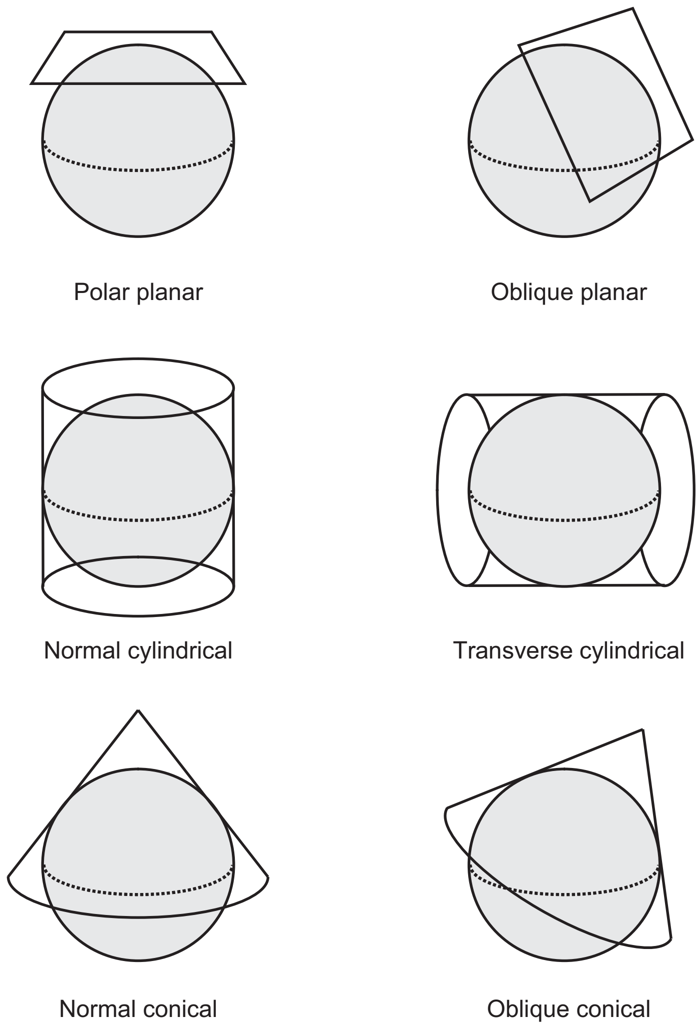

Figure 2.24 Example projections.

Transverse Mercator (UTM) system, Gauss-Krueger zoned system, many U.S. state planes, and the U.K. National Grid. Section C.4 of Appendix C on the CD describes the projection in more detail and presents formulae for converting between latitude and longitude and transverse Mercator projected coordinates.

2.4.6 Earth Rotation

The ECI and ECEF coordinate systems are defined such that the Earth rotates, with respect to space, clockwise about their common z-axis, shown in Figure 2.25. Thus, the Earth-rotation vector resolved in an ECI or ECEF frame is given by

$$\boldsymbol{\omega}_{ie}^{i} = \boldsymbol{\omega}_{ie}^{e} = \begin{pmatrix} 0 \\ 0 \\ \omega_{ie} \end{pmatrix}. \tag{2.122}$$

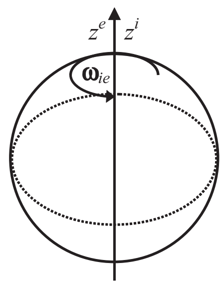

Figure 2.25 Earth rotation in an ECI or ECEF frame. (*From:* [1]. © 2002 QinetiQ Ltd. Reprinted with permission.)

The Earth-rotation vector resolved into local navigation frame axes is a function of geodetic latitude:

$$\boldsymbol{\omega}_{ie}^n = \begin{pmatrix} \omega_{ie}\cos L_b \\ 0 \\ -\omega_{ie}\sin L_b \end{pmatrix}. \tag{2.123}$$

The period of rotation of the Earth with respect to space is known as the sidereal day and is about 23 hours, 56 minutes, 4 seconds. This differs from the 24-hour mean solar day as the Earth's orbital motion causes the Earth–Sun direction with respect to space to vary, resulting in one more rotation than solar day each year (note that 1/365 of a day is about 4 minutes). The rate of rotation is not constant and the sidereal day can vary by several milliseconds from day to day. There are random changes due to wind and seasonal changes as ice forming and melting alters the Earth's moment of inertia. There is also a long term reduction of the Earth rotation rate due to tidal friction [12].

For navigation purposes, a constant rotation rate is assumed, based on the mean sidereal day. The WGS 84 value of the Earth's angular rate is $\omega_{ie} = 7.292115 \times 10^{-5}$ rad s^{-1} [9].

The velocity of the Earth's surface due to Earth rotation is given by

$$\mathbf{v}_{iS}^e = \boldsymbol{\omega}_{ie}^e \wedge \mathbf{r}_{eS}^e, \qquad \mathbf{v}_{iS}^n = \mathbf{C}_e^n\left(\boldsymbol{\omega}_{ie}^e \wedge \mathbf{r}_{eS}^e\right). \tag{2.124}$$

See Section 2.5.5 for how this is obtained. The maximum speed is 465 m s^{-1} at the equator.

2.4.7 Specific Force, Gravitation, and Gravity

Specific force is the nongravitational force per unit mass on a body, sensed with respect to an inertial frame. It has no meaning with respect to any other frame, although it can be resolved in any axes. *Gravitation* is the fundamental mass attraction force; it does not incorporate any centripetal components.*

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Specific force is what people and instruments sense. Gravitation is not sensed because it acts equally on all points, causing them to move together. Other forces are sensed as they are transmitted from point to point. The sensation of weight is caused by the forces opposing gravity.† This reaction to gravity is known as the restoring force on land, buoyancy at sea, and lift in the air.

During freefall, the specific force is zero so there is no sensation of weight. Conversely, under zero acceleration when the specific force is equal and opposite to the acceleration due to gravitation, the reaction to gravitation is sensed as weight. Figure 2.26 illustrates this with a mass in freefall and a mass suspended by a spring. In both cases, the gravitational force on the mass is the same. However, in the suspended case, the spring exerts an equal and opposite force.

A further example is provided by the upward motion of an elevator, illustrated by Figure 2.27. As the elevator accelerates upward, the specific force is higher and the occupants appear to weigh more. As the elevator decelerates, the specific force is lower than normal and the occupants feel lighter. In a windowless elevator, this can create the illusion that the elevator has overshot the destination floor and is dropping down to correct for it.*

Thus, specific force, $\mathbf{f}$, varies with acceleration, $\mathbf{a}$, and the acceleration due to the gravitational force, $\boldsymbol{\gamma}$, as

$$\mathbf{f}_{ib}^{\gamma} = \mathbf{a}_{ib}^{\gamma} - \boldsymbol{\gamma}_{ib}^{\gamma} \tag{2.125}$$

Specific force is the quantity measured by accelerometers. The measurements are made in the body frame of the accelerometer triad; thus, the sensed specific force is $\mathbf{f}_{ib}^{b}$.

As a prelude to defining gravity, it is useful to consider an object that is stationary with respect to a rotating frame, such as an ECEF frame. This has the properties

$$\mathbf{v}_{eb}^{e} = 0, \qquad \mathbf{a}_{eb}^{e} = 0. \tag{2.126}$$

From (2.67) and (2.76), and applying (2.66),

$$\dot{\mathbf{r}}_{ib}^{e} = \dot{\mathbf{r}}_{eb}^{e} = 0, \qquad \ddot{\mathbf{r}}_{ib}^{e} = \ddot{\mathbf{r}}_{eb}^{e} = 0. \tag{2.127}$$

The inertially referenced acceleration in ECEF frame axes is given by (2.81), noting that $\dot{\boldsymbol{\Omega}}_{ie}^{e} = 0$ as the Earth rate is assumed constant:

$$\mathbf{a}_{ib}^{e} = \boldsymbol{\Omega}_{ie}^{e}\boldsymbol{\Omega}_{ie}^{e}\mathbf{r}_{ib}^{e} + 2\boldsymbol{\Omega}_{ie}^{e}\dot{\mathbf{r}}_{ib}^{e} + \ddot{\mathbf{r}}_{ib}^{e}. \tag{2.128}$$

Applying (2.127),

$$\mathbf{a}_{ib}^{e} = \boldsymbol{\Omega}_{ie}^{e}\boldsymbol{\Omega}_{ie}^{e}\mathbf{r}_{eb}^{e}. \tag{2.129}$$

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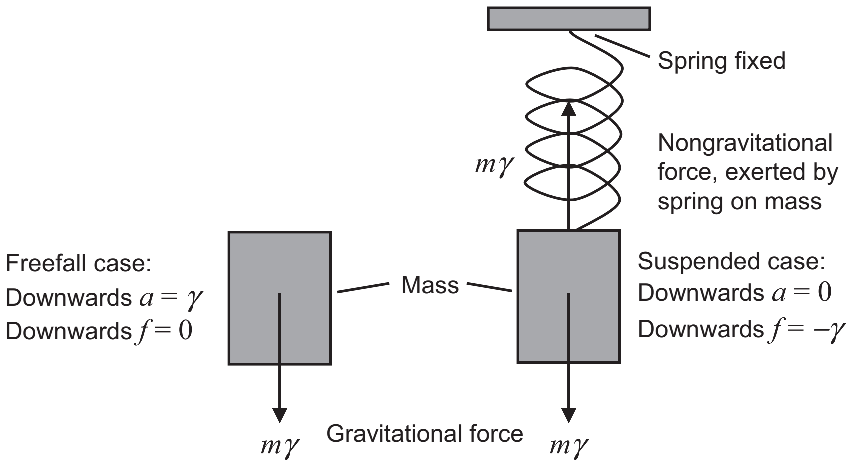

Figure 2.26 Forces on a mass in freefall and a mass suspended by a spring.

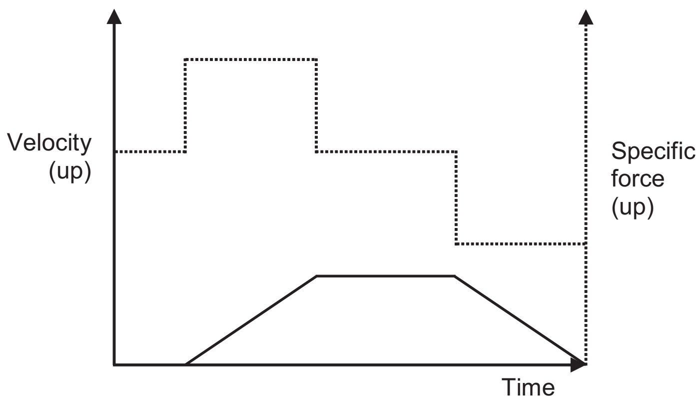

Figure 2.27 Velocity and specific force of an elevator moving up (*From:* [1]. © 2002 QinetiQ Ltd. Reprinted with permission.)

Substituting this into the specific force definition, (2.125), gives

$$\mathbf{f}_{ib}^{e} = \boldsymbol{\Omega}_{ie}^{e}\boldsymbol{\Omega}_{ie}^{e}\mathbf{r}_{eb}^{e} - \boldsymbol{\gamma}_{ib}^{e}. \tag{2.130}$$

The specific force sensed when stationary with respect to an Earth frame is the reaction to what is known as the acceleration due to *gravity*, which is thus defined by[†]

$$\mathbf{g}_{b}^{\gamma} = -\mathbf{f}_{ib}^{\gamma}\Big|_{\mathbf{a}_{eb}^{\gamma}=0,\mathbf{v}_{eb}^{\gamma}=0}. \tag{2.131}$$

Therefore, from (2.130), the acceleration due to gravity is

$$\mathbf{g}_{b}^{\gamma} = \boldsymbol{\gamma}_{ib}^{\gamma} - \boldsymbol{\Omega}_{ie}^{\gamma}\boldsymbol{\Omega}_{ie}^{\gamma}\mathbf{r}_{eb}^{\gamma}, \tag{2.132}$$

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noting from (2.122) and (2.123) that

$$\begin{aligned}
\mathbf{g}_b^e &= \boldsymbol{\gamma}_{ib}^e + \omega_{ie}^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{r}_{eb}^e \\
\mathbf{g}_b^n &= \boldsymbol{\gamma}_{ib}^n + \omega_{ie}^2 \begin{pmatrix} \sin^2 L_b & 0 & \cos L_b \sin L_b \\ 0 & 1 & 0 \\ \cos L_b \sin L_b & 0 & \cos^2 L_b \end{pmatrix} \mathbf{r}_{eb}^n
\end{aligned}. \tag{2.133}$$

The first term in (2.132) and (2.133) is the gravitational acceleration. The second term is the outward centrifugal acceleration due to the Earth's rotation; this is a pseudo-acceleration arising from the use of a rotating reference frame as discussed in Section 2.3.5. Figure 2.28 illustrates the two components of gravity. From an inertial frame perspective, a centripetal acceleration, (2.129), is applied to maintain an object stationary with respect to the rotating Earth. It is important not to confuse gravity, $\mathbf{g}$, with gravitation, $\boldsymbol{\gamma}$. At the Earth's surface, the total acceleration due to gravity is about 9.8 m s^{-2}, with the centrifugal component contributing up to 0.034 m s^{-2}. In orbit, the gravitational component is smaller and the centrifugal component is larger. However, an inertial reference frame is normally used for orbital applications.

The centrifugal component of gravity can be calculated exactly at all locations, but calculation of the gravitational component is more complex. For air applications, it is standard practice to use an empirical model of the surface gravity, $\mathbf{g}_0$, and apply a simple scaling law to calculate the variation with height.‡

The WGS 84 datum [9] provides a simple model of the acceleration due to gravity at the ellipsoid as a function of latitude:

$$g_0(L) \approx 9.7803253359 \frac{(1 + 0.001931853 \sin^2 L)}{\sqrt{1 - e^2 \sin^2 L}} \text{ m s}^{-2}. \tag{2.134}$$

This is known as the Somigliana model. Note that it is a gravity model, not a gravitational model. The geoid (Section 2.4.4) defines a surface of constant gravity potential. However, the acceleration due to gravity is obtained from the gradient of the gravity potential, so it is not constant across the geoid. Although the true gravity vector is perpendicular to the geoid (not the terrain), it is a reasonable approximation for most navigation applications to treat it as perpendicular to the ellipsoid. Thus,

$$\mathbf{g}_0^\gamma(L) \approx g_0(L)\mathbf{u}_{nD}^\gamma, \tag{2.135}$$

where $\mathbf{u}_{nD}^\gamma$ is the down unit vector of a local navigation frame.

The gravitational acceleration at the ellipsoid can be obtained from the acceleration due to gravity by subtracting the centrifugal acceleration. Thus,

$$\boldsymbol{\gamma}_0^\gamma(L) = \mathbf{g}_0^\gamma(L) + \boldsymbol{\Omega}_{ie}^\gamma \boldsymbol{\Omega}_{ie}^\gamma \mathbf{r}_{eS}^\gamma(L). \tag{2.136}$$

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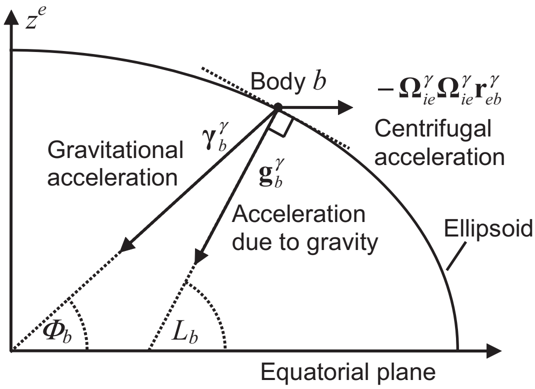

Figure 2.28 Gravity, gravitation and centrifugal acceleration.

From (2.107), the geocentric radius at the surface is given by

$$r_{eS}^{e}(L) = R_E(L)\sqrt{\cos^2 L + \left(1 - e^2\right)^2 \sin^2 L}. \tag{2.137}$$

The gravitational field varies roughly as that for a point mass, so gravitational acceleration can be scaled with height as

$$\boldsymbol{\gamma}_{ib}^{\gamma} \approx \frac{\left(r_{eS}^{e}(L_b)\right)^2}{\left(r_{eS}^{e}(L_b) + h_b\right)^2}\boldsymbol{\gamma}_0^{\gamma}(L_b). \tag{2.138}$$

For heights less than about 10 km, the scaling can be further approximated to $(1 - 2h_b/\mathbf{r}_{eS}^{e}(L_b))$. The acceleration due to gravity, $\mathbf{g}$, may then be recombined using (2.132). As the centrifugal component of gravity is small, it is reasonable to apply the height scaling to $\mathbf{g}$ where the height is small and/or poor quality accelerometers are used. Alternatively, a more accurate set of formulae for calculating gravity as a function of latitude and height is given in [9]. An approximation for the variation of the down component with height is

$$g_{b,D}^{n}(L_b, h_b) \approx g_0(L_b)\left\{1 - \frac{2}{R_0}\left[1 + f\left(1 - 2\sin^2 L_b\right) + \frac{\omega_{ie}^2 R_0^2 R_P}{\mu}\right]h_b + \frac{3}{R_0^2}h_b^2\right\}, \tag{2.139}$$

where μ is the Earth's gravitational constant and its WGS 84 value [9] is 3.986004418 $\times 10^{14}$ m^{3}s^{-2}. The north component of gravity varies with height as [17]

$$g_{b,N}^{n}(L_b, h_b) \approx -8.08 \times 10^{-9} h_b \sin 2L_b \text{ m s}^{-2}. \tag{2.140}$$

This model is used in the MATLAB function, Gravity_NED, on the accompanying CD. Example 2.3 on the CD comprises calculations of the acceleration due to gravity at different latitudes and heights.

When working in an inertial reference frame, only the gravitational acceleration is required. This can be calculated directly at varying height using [18]

$$\boldsymbol{\gamma}_{ib}^{i} = -\frac{\mu}{\left|\mathbf{r}_{ib}^{i}\right|^{3}}\left\{\mathbf{r}_{ib}^{i} + \frac{3}{2}J_{2}\frac{R_{0}^{2}}{\left|\mathbf{r}_{ib}^{i}\right|^{2}}\begin{pmatrix}\left[1-5\left(r_{ib,z}^{i}/\left|\mathbf{r}_{ib}^{i}\right|\right)^{2}\right]r_{ib,x}^{i}\\ \left[1-5\left(r_{ib,z}^{i}/\left|\mathbf{r}_{ib}^{i}\right|\right)^{2}\right]r_{ib,y}^{i}\\ \left[3-5\left(r_{ib,z}^{i}/\left|\mathbf{r}_{ib}^{i}\right|\right)^{2}\right]r_{ib,z}^{i}\end{pmatrix}\right\}, \tag{2.141}$$

where J_2 is the Earth's second gravitational constant and takes the value 1.082627 $\times 10^{-3}$ [9]. Resolved about ECEF-frame axes, it becomes

$$\boldsymbol{\gamma}_{ib}^{e} = -\frac{\mu}{\left|\mathbf{r}_{eb}^{e}\right|^{3}}\left\{\mathbf{r}_{eb}^{e} + \frac{3}{2}J_{2}\frac{R_{0}^{2}}{\left|\mathbf{r}_{eb}^{e}\right|^{2}}\begin{pmatrix}\left[1-5\left(r_{eb,z}^{e}/\left|\mathbf{r}_{eb}^{e}\right|\right)^{2}\right]r_{eb,x}^{e}\\ \left[1-5\left(r_{eb,z}^{e}/\left|\mathbf{r}_{eb}^{e}\right|\right)^{2}\right]r_{eb,y}^{e}\\ \left[3-5\left(r_{eb,z}^{e}/\left|\mathbf{r}_{eb}^{e}\right|\right)^{2}\right]r_{eb,z}^{e}\end{pmatrix}\right\}. \tag{2.142}$$

These models are used in the MATLAB functions, Gravitation_ECI and Gravity_ECEF, on the CD.

Much higher precision may be obtained using a spherical harmonic model, such as the 4,730,400-coefficient EGM 2008 gravity model [14]. Further precision is given by a gravity anomaly database, which comprises the difference between the measured and modeled gravity fields over a grid of locations. Gravity anomalies tend to be largest over major mountain ranges and ocean trenches.

2.5 Frame Transformations

An essential feature of navigation mathematics is the capability to transform quantities between different coordinate frames. This section summarizes the equations for expressing the attitude of one frame with respect to another and transforming Cartesian position, velocity, acceleration, and angular rate between references to inertial, Earth, and local navigation frames, and between ECEF and local tangent-plane frames. The section concludes with the equations for transposing a navigation solution between different objects.

Cartesian position, velocity, acceleration, and angular rate referenced to the same frame transform between resolving axes simply by applying the coordinate transformation matrix (2.12): *

$$\mathbf{x}_{\beta\alpha}^{\gamma} = \mathbf{C}_{\delta}^{\gamma}\mathbf{x}_{\beta\alpha}^{\delta} \quad \mathbf{x} \in \mathbf{r},\mathbf{v},\mathbf{a},\boldsymbol{\omega} \quad \gamma,\delta \in i,e,n,l,b. \tag{2.143}$$

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Therefore, these transforms are not presented explicitly for each pair of frames.†

The coordinate transformation matrices involving the body frame—that is,

$$\mathbf{C}_b^\beta, \mathbf{C}_\beta^b \quad \beta \in i, e, n, l$$

describe the attitude of that body with respect to a reference frame. The body attitude with respect to a new reference frame may be obtained simply by multiplying by the coordinate transformation matrix between the two reference frames:

$$\mathbf{C}_b^\delta = \mathbf{C}_\beta^\delta \mathbf{C}_b^\beta \qquad \mathbf{C}_\delta^b = \mathbf{C}_\beta^b \mathbf{C}_\delta^\beta \qquad \beta, \delta \in i, e, n, l. \tag{2.144}$$

Transforming Euler, quaternion, or rotation vector attitude to a new reference frame is more complex. One solution is to convert to the coordinate transformation matrix representation, transform the reference, and then convert back.

2.5.1 Inertial and Earth Frames

The center and z-axes of commonly-realized Earth-centered inertial and Earth-centered Earth-fixed coordinate frames are coincident. The x- and y-axes are coincident at time t_0 and the frames rotate about the z-axes at ω_{ie} (see Section 2.4.6). Thus,*

$$\begin{aligned}
\mathbf{C}_i^e &= \begin{pmatrix} \cos\omega_{ie}(t - t_0) & \sin\omega_{ie}(t - t_0) & 0 \\ -\sin\omega_{ie}(t - t_0) & \cos\omega_{ie}(t - t_0) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\mathbf{C}_e^i &= \begin{pmatrix} \cos\omega_{ie}(t - t_0) & -\sin\omega_{ie}(t - t_0) & 0 \\ \sin\omega_{ie}(t - t_0) & \cos\omega_{ie}(t - t_0) & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}. \tag{2.145}$$

Positions referenced to the two frames are the same, so only the resolving axes need to be transformed:

$$\mathbf{r}_{eb}^e = \mathbf{C}_i^e \mathbf{r}_{ib}^i, \quad \mathbf{r}_{ib}^i = \mathbf{C}_e^i \mathbf{r}_{eb}^e. \tag{2.146}$$

Velocity and acceleration transformation is more complex:

$$\begin{aligned}
\mathbf{v}_{eb}^e &= \mathbf{C}_i^e \left(\mathbf{v}_{ib}^i - \mathbf{\Omega}_{ie}^i \mathbf{r}_{ib}^i \right) \\
\mathbf{v}_{ib}^i &= \mathbf{C}_e^i \left(\mathbf{v}_{eb}^e + \mathbf{\Omega}_{ie}^e \mathbf{r}_{eb}^e \right)
\end{aligned}, \tag{2.147}$$

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$$\begin{aligned}\mathbf{a}_{eb}^{e} &= \mathbf{C}_{i}^{e}\left(\mathbf{a}_{ib}^{i} - 2\boldsymbol{\Omega}_{ie}^{i}\mathbf{v}_{ib}^{i} + \boldsymbol{\Omega}_{ie}^{i}\boldsymbol{\Omega}_{ie}^{i}\mathbf{r}_{ib}^{i}\right)\\ \mathbf{a}_{ib}^{i} &= \mathbf{C}_{e}^{i}\left(\mathbf{a}_{eb}^{e} + 2\boldsymbol{\Omega}_{ie}^{e}\mathbf{v}_{eb}^{e} + \boldsymbol{\Omega}_{ie}^{e}\boldsymbol{\Omega}_{ie}^{e}\mathbf{r}_{eb}^{e}\right)\end{aligned}. \quad (2.148)$$

Angular rates transform as†

$$\boldsymbol{\omega}_{eb}^{e} = \mathbf{C}_{i}^{e}\left(\boldsymbol{\omega}_{ib}^{i} - \begin{pmatrix}0\\0\\\omega_{ie}\end{pmatrix}\right), \qquad \boldsymbol{\omega}_{ib}^{i} = \mathbf{C}_{e}^{i}\left(\boldsymbol{\omega}_{eb}^{e} + \begin{pmatrix}0\\0\\\omega_{ie}\end{pmatrix}\right). \quad (2.149)$$

Example 2.4 on the CD illustrates the position, velocity, acceleration, and angular rate transformations, and is editable using Microsoft Excel. The MATLAB functions, ECEF_to_ECI and ECI_to_ECEF, on the CD implement the position, velocity, and attitude transformations.

Note that accurate timing is critical for conversion between ECI and ECEF frames. For example, if there is a 1-ms offset between the time bases used to specify t and t_0, a position error of up to 0.465m will occur when transforming between frames. Great caution should therefore be exercised in using the ECI frame where accurate timing is not available.

2.5.2 Earth and Local Navigation Frames

The relative orientation of commonly-realized Earth and local navigation frames is determined by the geodetic latitude, L_b, and longitude, λ_b, of the body frame whose center coincides with that of the local navigation frame:

$$\begin{aligned}\mathbf{C}_{e}^{n} &= \begin{pmatrix}-\sin L_b\cos\lambda_b & -\sin L_b\sin\lambda_b & \cos L_b\\ -\sin\lambda_b & \cos\lambda_b & 0\\ -\cos L_b\cos\lambda_b & -\cos L_b\sin\lambda_b & -\sin L_b\end{pmatrix}\\ \mathbf{C}_{n}^{e} &= \begin{pmatrix}-\sin L_b\cos\lambda_b & -\sin\lambda_b & -\cos L_b\cos\lambda_b\\ -\sin L_b\sin\lambda_b & \cos\lambda_b & -\cos L_b\sin\lambda_b\\ \cos L_b & 0 & -\sin L_b\end{pmatrix}\end{aligned}. \quad (2.150)$$

Conversely, the latitude and longitude may be obtained from the coordinate transformation matrices using

$$\begin{aligned}L_b &= \arctan\left(-C_{e3,3}^{n}/C_{e1,3}^{n}\right) &&= \arctan\left(-C_{n3,3}^{e}/C_{n3,1}^{e}\right)\\ \lambda_b &= \arctan_2\left(-C_{e2,1}^{n}, C_{e2,2}^{n}\right) &&= \arctan_2\left(-C_{n1,2}^{e}, C_{n2,2}^{e}\right)\end{aligned}. \quad (2.151)$$

Position, velocity, and acceleration referenced to a local navigation frame are meaningless as the center of the corresponding body frame coincides with the

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navigation frame center. The resolving axes of Earth-referenced position, velocity, and acceleration are simply transformed using (2.143). Thus,

$$\begin{aligned} \mathbf{r}_{eb}^{n} &= \mathbf{C}_{e}^{n}\mathbf{r}_{eb}^{e}, & \mathbf{r}_{eb}^{e} &= \mathbf{C}_{n}^{e}\mathbf{r}_{eb}^{n} \\ \mathbf{v}_{eb}^{n} &= \mathbf{C}_{e}^{n}\mathbf{v}_{eb}^{e}, & \mathbf{v}_{eb}^{e} &= \mathbf{C}_{n}^{e}\mathbf{v}_{eb}^{n}. \\ \mathbf{a}_{eb}^{n} &= \mathbf{C}_{e}^{n}\mathbf{a}_{eb}^{e}, & \mathbf{a}_{eb}^{e} &= \mathbf{C}_{n}^{e}\mathbf{a}_{eb}^{n} \end{aligned} \tag{2.152}$$

Angular rates transform as

$$\begin{aligned} \boldsymbol{\omega}_{nb}^{n} &= \mathbf{C}_{e}^{n}\left(\boldsymbol{\omega}_{eb}^{e} - \boldsymbol{\omega}_{en}^{e}\right) \\ &= \mathbf{C}_{e}^{n}\boldsymbol{\omega}_{eb}^{e} - \boldsymbol{\omega}_{en}^{n} \end{aligned}, \qquad \boldsymbol{\omega}_{eb}^{e} = \mathbf{C}_{n}^{e}\left(\boldsymbol{\omega}_{nb}^{n} + \boldsymbol{\omega}_{en}^{n}\right), \tag{2.153}$$

noting that a solution for $\boldsymbol{\omega}_{en}^{n}$ is obtained in Section 5.4.1 The velocity, acceleration, and angular rate transformations are illustrated by Example 2.5 on the CD. The MATLAB functions, ECEF_to_NED and NED_to_ECEF, on the CD implement the velocity and attitude transformations.

2.5.3 Inertial and Local Navigation Frames

The inertial-local navigation frame coordinate transformation matrices are obtained by multiplying (2.145) and (2.150):*

$$\begin{aligned} \mathbf{C}_{i}^{n} &= \begin{pmatrix} -\sin L_b \cos(\lambda_b + \omega_{ie}(t-t_0)) & -\sin L_b \sin(\lambda_b + \omega_{ie}(t-t_0)) & \cos L_b \\ -\sin(\lambda_b + \omega_{ie}(t-t_0)) & \cos(\lambda_b + \omega_{ie}(t-t_0)) & 0 \\ -\cos L_b \cos(\lambda_b + \omega_{ie}(t-t_0)) & -\cos L_b \sin(\lambda_b + \omega_{ie}(t-t_0)) & -\sin L_b \end{pmatrix}. \\ \mathbf{C}_{n}^{i} &= \begin{pmatrix} -\sin L_b \cos(\lambda_b + \omega_{ie}(t-t_0)) & -\sin(\lambda_b + \omega_{ie}(t-t_0)) & -\cos L_b \cos(\lambda_b + \omega_{ie}(t-t_0)) \\ -\sin L_b \sin(\lambda_b + \omega_{ie}(t-t_0)) & \cos(\lambda_b + \omega_{ie}(t-t_0)) & -\cos L_b \sin(\lambda_b + \omega_{ie}(t-t_0)) \\ \cos L_b & 0 & -\sin L_b \end{pmatrix} \end{aligned} \tag{2.154}$$

Earth-referenced velocity and acceleration in navigation frame axes transform to and from their inertial frame inertial reference counterparts as†

$$\begin{aligned} \mathbf{v}_{eb}^{n} &= \mathbf{C}_{i}^{n}\left(\mathbf{v}_{ib}^{i} - \boldsymbol{\Omega}_{ie}^{i}\mathbf{r}_{ib}^{i}\right) \\ \mathbf{v}_{ib}^{i} &= \mathbf{C}_{n}^{i}\mathbf{v}_{eb}^{n} + \mathbf{C}_{e}^{i}\boldsymbol{\Omega}_{ie}^{e}\mathbf{r}_{eb}^{e} \end{aligned}, \tag{2.155}$$

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$$\begin{aligned} \mathbf{a}_{eb}^{n} &= \mathbf{C}_{i}^{n}\left(\mathbf{a}_{ib}^{i} - 2\boldsymbol{\Omega}_{ie}^{i}\mathbf{v}_{ib}^{i} + \boldsymbol{\Omega}_{ie}^{i}\boldsymbol{\Omega}_{ie}^{i}\mathbf{r}_{ib}^{i}\right) \\ \mathbf{a}_{ib}^{i} &= \mathbf{C}_{n}^{i}\left(\mathbf{a}_{eb}^{n} + 2\boldsymbol{\Omega}_{ie}^{n}\mathbf{v}_{eb}^{n}\right) + \mathbf{C}_{e}^{i}\boldsymbol{\Omega}_{ie}^{e}\boldsymbol{\Omega}_{ie}^{e}\mathbf{r}_{eb}^{e} \end{aligned}. \tag{2.156}$$

Angular rates transform as

$$\begin{aligned} \boldsymbol{\omega}_{nb}^{n} &= \mathbf{C}_{i}^{n}\left(\boldsymbol{\omega}_{ib}^{i} - \boldsymbol{\omega}_{in}^{i}\right) \\ &= \mathbf{C}_{i}^{n}\left(\boldsymbol{\omega}_{ib}^{i} - \boldsymbol{\omega}_{ie}^{i}\right) - \boldsymbol{\omega}_{en}^{n} \end{aligned}, \qquad \begin{aligned} \boldsymbol{\omega}_{ib}^{i} &= \mathbf{C}_{n}^{i}\left(\boldsymbol{\omega}_{nb}^{n} + \boldsymbol{\omega}_{in}^{n}\right) \\ &= \mathbf{C}_{n}^{i}\left(\boldsymbol{\omega}_{nb}^{n} + \boldsymbol{\omega}_{en}^{n}\right) + \boldsymbol{\omega}_{ie}^{i} \end{aligned}. \tag{2.157}$$

Example 2.6 on the CD illustrates the velocity, acceleration, and angular rate transformations. Again, timing accuracy is critical for accurate frame transformations.

2.5.4 Earth and Local Tangent-Plane Frames

The orientation with respect to an ECEF frame of a local tangent-plane frame whose axes are aligned with north, east, and down may be determined using the geodetic latitude, L_l, and longitude, λ_l, of the local tangent-plane origin:

$$\begin{aligned} \mathbf{C}_{e}^{l} &= \begin{pmatrix} -\sin L_l \cos\lambda_l & -\sin L_l \sin\lambda_l & \cos L_l \\ -\sin\lambda_l & \cos\lambda_l & 0 \\ -\cos L_l \cos\lambda_l & -\cos L_l \sin\lambda_l & -\sin L_l \end{pmatrix} \\ \mathbf{C}_{l}^{e} &= \begin{pmatrix} -\sin L_l \cos\lambda_l & -\sin\lambda_l & -\cos L_l \cos\lambda_l \\ -\sin L_l \sin\lambda_l & \cos\lambda_l & -\cos L_l \sin\lambda_l \\ \cos L_l & 0 & -\sin L_l \end{pmatrix} \end{aligned}. \tag{2.158}$$

The origin and orientation of a local tangent-plane frame with respect to an ECEF frame are constant. Therefore, the velocity, acceleration, and angular rate may be transformed simply by rotating the resolving axes:

$$\begin{aligned} \mathbf{v}_{lb}^{l} &= \mathbf{C}_{e}^{l}\mathbf{v}_{eb}^{e}, & \mathbf{v}_{eb}^{e} &= \mathbf{C}_{l}^{e}\mathbf{v}_{lb}^{l} \\ \mathbf{a}_{lb}^{l} &= \mathbf{C}_{e}^{l}\mathbf{a}_{eb}^{e}, & \mathbf{a}_{eb}^{e} &= \mathbf{C}_{l}^{e}\mathbf{a}_{lb}^{l} \\ \boldsymbol{\omega}_{lb}^{l} &= \mathbf{C}_{e}^{l}\boldsymbol{\omega}_{eb}^{e}, & \boldsymbol{\omega}_{eb}^{e} &= \mathbf{C}_{l}^{e}\boldsymbol{\omega}_{lb}^{l} \end{aligned}. \tag{2.159}$$

The Cartesian position transforms as

$$\begin{aligned} \mathbf{r}_{lb}^{l} &= \mathbf{C}_{e}^{l}\left(\mathbf{r}_{eb}^{e} - \mathbf{r}_{el}^{e}\right) \\ \mathbf{r}_{eb}^{e} &= \mathbf{r}_{el}^{e} + \mathbf{C}_{l}^{e}\mathbf{r}_{lb}^{l} \end{aligned}, \tag{2.160}$$

where $\mathbf{r}_{el}^{e}$ is the Cartesian ECEF position of the l-frame origin, obtained from L_l and λ_l using (2.112).

2.5.5 Transposition of Navigation Solutions

Sometimes, there is a requirement to transpose a navigation solution from one position to another on a vehicle, such as between an INS and a GNSS antenna, between an INS and the center of gravity, or between a reference and an aligning INS. Here, the equations for transposing position, velocity, and attitude from describing the b frame to describing the B frame are presented.*

Let the orientation of frame B with respect to frame b be $\mathbf{C}_b^B$ and the position of frame B with respect to frame b in frame b axes be $\mathbf{l}_{bB}^b$, which is known as the *lever arm* or *moment arm*. Note that the lever arm is mathematically identical to the Cartesian position with B as the object frame and b as the reference and resolving frames. Figure 2.29 illustrates this.

Attitude transformation is straightforward:

$$\mathbf{C}_\beta^B = \mathbf{C}_b^B\mathbf{C}_\beta^b, \qquad \mathbf{C}_B^\beta = \mathbf{C}_b^\beta\mathbf{C}_B^b. \tag{2.161}$$

Cartesian position may be transposed using

$$\mathbf{r}_{\beta B}^\gamma = \mathbf{r}_{\beta b}^\gamma + \mathbf{C}_b^\gamma\mathbf{l}_{bB}^b. \tag{2.162}$$

Precise transformation of latitude, longitude, and height requires conversion to Cartesian position and back. However, if the small angle approximation is applied to l / R, where R is the Earth radius, a simpler form may be used:

$$\begin{pmatrix} L_B \\ \lambda_B \\ h_B \end{pmatrix} \approx \begin{pmatrix} L_b \\ \lambda_b \\ h_b \end{pmatrix} + \begin{pmatrix} 1/\left(R_N(L_b)+h_b\right) & 0 & 0 \\ 0 & 1/\left[(R_E(L_b)+h_b)\cos L_b\right] & 0 \\ 0 & 0 & -1 \end{pmatrix}\mathbf{C}_b^n\mathbf{l}_{bB}^b. \tag{2.163}$$

The velocity transposition is obtained by differentiating (2.162) and substituting it into (2.67):

$$\mathbf{v}_{\beta B}^\gamma = \mathbf{v}_{\beta b}^\gamma + \mathbf{C}_\beta^\gamma\dot{\mathbf{C}}_b^\beta\mathbf{l}_{bB}^b, \tag{2.164}$$

assuming $\mathbf{l}_{bB}^b$ is constant. Substituting (2.56),†

$$\mathbf{v}_{\beta B}^\gamma = \mathbf{v}_{\beta b}^\gamma + \mathbf{C}_b^\gamma\left(\boldsymbol{\omega}_{\beta b}^b \wedge \mathbf{l}_{bB}^b\right). \tag{2.165}$$

Similarly, the acceleration transposition is

$$\mathbf{a}_{\beta B}^\gamma = \mathbf{a}_{\beta b}^\gamma + \mathbf{C}_b^\gamma\left[\boldsymbol{\omega}_{\beta b}^b \wedge \left(\boldsymbol{\omega}_{\beta b}^b \wedge \mathbf{l}_{bB}^b\right) + \left(\dot{\boldsymbol{\omega}}_{\beta b}^b \wedge \mathbf{l}_{bB}^b\right)\right]. \tag{2.166}$$

Problems and exercises for this chapter are on the accompanying CD.

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†End of QinetiQ copyright material.

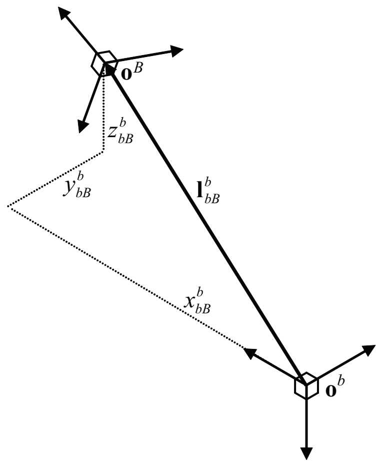

Figure 2.29 The lever arm from frame *b* to frame *B*.

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